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NEW PERSPECTIVES ON BILATERAL INEQUALITIES FOR FUSION FRAMES IN HILBERT SPACES

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Abstract. We provide a new approach to the proofs of some existing bilateral inequalities for fusion frames in Hilbert spaces from the perspective of function theory, which greatly simplifies the proving process and shows that the parameter involving in four of the results mentioned can take values from larger ranges. We also present an improvement to two results on this topic. At the end of the paper we establish several new bilateral inequalities for fusion frames in Hilbert spaces, following the approaches of which corresponding bilateral inequalities for some other generalized frames with new types of structures can be naturally obtained.

1. Introduction

Throughout the paper, the notations $\mathbb{R}$, $\mathbb{I}$ and $\mathcal{N}$ are used to denote, respectively, the set of real numbers, a countable index set and a Hilbert space. For a closed subspace $\mathcal{W}$ of $\mathcal{N}$, we denote by $\pi_{\mathcal{W}}$ the orthogonal projection onto $\mathcal{W}$.

Frames put forward in the paper [12], as a brand-new and powerful tool, were originally used to process some profound problems deriving in nonharmonic Fourier series, which were brought back to people's vision by Daubechies et al. [11] owing to their groundbreaking work on wavelets. Thanks to some of their nice properties, frames have already been applied to many research fields (see [3, 10, 19, 21] for example). We refer also to [9] for more details about frame theory.

The notion of fusion frames (called also frames of subspaces), as a generalization of frames, was introduced independently by Casazza and Kutyniok in [6] and Fornasier in [13], when dedicating their effort to examine some large systems. Because of the complicated structure, fusion frames do admit some new behaviors compared to frames, which makes the study of them interesting. The reader can consult, for example, the papers [4, 5, 7, 8] for applications of fusion frames.

Suppose $\mathcal{M}_i$ is a closed subspace of $\mathcal{N}$ for any $i \in \mathbb{I}$, and $\{\beta_i\}_{i\in\mathbb{I}}$ is a sequence of weights (each $\beta_i > 0$). The family $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is said to be a *fusion frame* for $\mathcal{N}$, if there are numbers $0 < C_{\mathcal{M}} \leqslant D_{\mathcal{M}} < \infty$ so that the bilateral inequality

$$C_{\mathcal{M}}\|x\|^2 \leqslant \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \leqslant D_{\mathcal{M}}\|x\|^2 \tag{1}$$

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holds for any $x \in \mathcal{N}$. We call $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ a *Bessel fusion sequence* if only the inequality on the right hand side of (1) is assumed to be satisfied.

Given a fusion frame $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ for $\mathcal{N}$, we can define a linear bounded operator $U_{\mathcal{M}}$, called the *analysis operator* of $\mathcal{M}$, in the following way

$$U_{\mathcal{M}} : \mathcal{N} \to \left(\sum_{i\in\mathbb{I}} \bigoplus \mathcal{M}_i\right)_{\ell^2}, \quad U_{\mathcal{M}}x = \{\beta_i \pi_{\mathcal{M}_i}(x)\}_{i\in\mathbb{I}}, \tag{2}$$

where $(\sum_{i\in\mathbb{I}} \oplus \mathcal{M}_i)_{\ell^2}$ is the Hilbert space defined by

$$\left(\sum_{i\in\mathbb{I}} \bigoplus \mathcal{M}_i\right)_{\ell^2} = \left\{ \{g_i\}_{i\in\mathbb{I}} \middle| g_i \in \mathcal{M}_i,\ \|\{g_i\}_{i\in\mathbb{I}}\|_2^2 = \sum_{i\in\mathbb{I}} \|g_i\|^2 < \infty \right\}.$$

Further, a self-adjoint and invertible operator $S_{\mathcal{M}}$, namely the *fusion frame operator* of $\mathcal{M}$, can be obtained if we take a compositional operation on $U_{\mathcal{M}}^*$ and $U_{\mathcal{M}}$:

$$S_{\mathcal{M}} : \mathcal{N} \to \mathcal{N}, \quad S_{\mathcal{M}}x = U_{\mathcal{M}}^* U_{\mathcal{M}} x = \sum_{i\in\mathbb{I}} \beta_i^2 \pi_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{N}, \tag{3}$$

which thereby leads to the so-called *reconstruction formula*:

$$x = \sum_{i\in\mathbb{I}} \beta_i^2 S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x) = \sum_{i\in\mathbb{I}} \beta_i^2 \pi_{\mathcal{M}_i}(S_{\mathcal{M}}^{-1}x), \quad \forall x \in \mathcal{N}. \tag{4}$$

Recall also that the family $\mathcal{M}' = \{(S_{\mathcal{M}}^{-1}\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is still a fusion frame for $\mathcal{N}$ (see [16]), which is called the *dual fusion frame* of $\mathcal{M}$.

For each $\tau \subset \mathbb{I}$, there are two self-adjoint operators related to τ, and τ^c, the complementary set of τ, and the fusion frame $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ for $\mathcal{N}$, given below

$$S_{\mathcal{M}}^{\tau}, S_{\mathcal{M}}^{\tau^c} : \mathcal{N} \to \mathcal{N}, \quad S_{\mathcal{M}}^{\tau}x = \sum_{i\in\tau} \beta_i^2 \pi_{\mathcal{M}_i}(x), \quad S_{\mathcal{M}}^{\tau^c}x = \sum_{i\in\tau^c} \beta_i^2 \pi_{\mathcal{M}_i}(x). \tag{5}$$

It is clear that $S_{\mathcal{M}}^{\tau} + S_{\mathcal{M}}^{\tau^c} = S_{\mathcal{M}}$.

Let $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ and $\mathcal{P} = \{(\mathcal{P}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ be respectively a fusion frame and a Bessel fusion sequence for $\mathcal{N}$. One calls $\mathcal{P}$ an *alternate dual fusion frame* of $\mathcal{M}$, if for each $x \in \mathcal{N}$ we have

$$x = \sum_{i\in\mathbb{I}} \beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x). \tag{6}$$

Suppose $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{N}$ and $\mathcal{P} = \{(\mathcal{P}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an alternate dual fusion frame of $\mathcal{M}$. Then associated with any $\tau \subset \mathbb{I}$ and the pair $(\mathcal{M}, \mathcal{P})$, there are two linear bounded operators $W^{\tau}, W^{\tau^c} : \mathcal{N} \to \mathcal{N}$ defined by

$$W^{\tau}x = \sum_{i\in\tau} \beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x), \quad W^{\tau^c}x = \sum_{i\in\tau^c} \beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x), \quad \forall x \in \mathcal{N}. \tag{7}$$

Inequalities for frames were first studied by Balan et al. in [2], where an interesting inequality for Parseval frames was presented (see [2, Proposition 4.1]). Assisted by the corresponding canonical dual frames and alternate dual frames, Găvruţa later extended the inequality to the case of general frames (see [15, Theorems 2.2 and 3.2]).

In recent years, much attention has been paid to the generalization of the frame inequalities and particulary, Poria [20] provided us with a new type of inequalities for Hilbert-Schmidt frames, which are related to a parameter taking values from intervals (see [20, Theorems 3.5 and 3.7]). In the light of the idea of Poria, Li et al. in [17] showed us several more general inequalities (bilateral inequalities associated with a parameter λ) for fusion frames (proved by means of the idea offered in [20]).

THEOREM 1. (see [17, Theorem 3]) *Suppose* $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}'=\{(S_{\mathcal{M}}^{-1}\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\mathcal{M}$. *Then for all* $\lambda\in[0,2]$, *for any* $\tau\subset\mathbb{I}$ *and any* $x\in\mathcal{N}$, *we have*

$$\begin{aligned}\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 &\geqslant \sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2+\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau}(x)\|^2\\ &=\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2+\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau^c}(x)\|^2\\ &\geqslant\left(\lambda-\frac{\lambda^2}{4}\right)\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2+\left(1-\frac{\lambda^2}{4}\right)\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2.\end{aligned}\tag{8}$$

THEOREM 2. (see [17, Theorem 5]) *Suppose* $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}'=\{(S_{\mathcal{M}}^{-1}\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\mathcal{M}$. *Then for all* $\lambda\in[1,2]$, *for any* $\tau\subset\mathbb{I}$ *and any* $x\in\mathcal{N}$, *we have*

$$\begin{aligned}0&\leqslant\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2-\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau}(x)\|^2\\ &\leqslant(\lambda-1)\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2+\left(1-\frac{\lambda}{2}\right)^2\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2.\end{aligned}\tag{9}$$

THEOREM 3. (see [17, Theorem 6]) *Suppose* $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}'=\{(S_{\mathcal{M}}^{-1}\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\mathcal{M}$. *Then for all* $\lambda\in[1,2]$, *for any* $\tau\subset\mathbb{I}$ *and any* $x\in\mathcal{N}$, *we have*

$$\begin{aligned}&\left(2\lambda-\frac{\lambda^2}{2}-1\right)\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2+\left(1-\frac{\lambda^2}{2}\right)\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2\\ &\quad\leqslant\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau}(x)\|^2+\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau^c}(x)\|^2\\ &\quad\leqslant\lambda\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2.\end{aligned}\tag{10}$$

Bilateral inequalities for some other frame versions also emerged (see for example, the papers [14, 18, 24]), which, however, share the same structures as the ones in [17]. Given this, the authors in [22] explored bilateral inequalities possessing new structures for fusion frames, and the main results obtained are as follows.

THEOREM 4. (see [22, Theorem 1]) *Suppose* $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}'=\{(S_{\mathcal{M}}^{-1}\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is the* *dual fusion frame of* $\mathcal{M}$. *Then for every* $\lambda \in [1,+\infty)$, *for any* $\tau \subset \mathbb{I}$ *and any* $x \in \mathcal{N}$, *we have*

$$\begin{aligned}
&\sum_{i\in\tau} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 - \lambda \sum_{i\in\tau^c} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \\
&\quad \leqslant \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i} S_{\mathcal{M}}^{-1} S_{\mathcal{M}}^{\tau}(x)\|^2 - \lambda \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i} S_{\mathcal{M}}^{-1} S_{\mathcal{M}}^{\tau^c}(x)\|^2 \\
&\quad \leqslant (\lambda^3 - \lambda^2 + 1) \sum_{i\in\tau} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 + (\lambda^3 - 3\lambda^2 + 2\lambda - 1) \sum_{i\in\tau^c} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2.
\end{aligned} \tag{11}$$

THEOREM 5. (see [22, Theorem 2]) *Suppose* $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}' = \{(S_{\mathcal{M}}^{-1}\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\mathcal{M}$. *Then for every* $\lambda \in [\frac{1}{2},+\infty)$, *for any* $\tau \subset \mathbb{I}$ *and any* $x \in \mathcal{N}$, *we have*

$$\begin{aligned}
&(4\lambda - 1) \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i} S_{\mathcal{M}}^{-1} S_{\mathcal{M}}^{\tau^c}(x)\|^2 + (1 - \lambda^2) \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \\
&\quad \leqslant \sum_{i\in\tau} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 + (1 + 2\lambda) \sum_{i\in\tau^c} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \\
&\quad \leqslant \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i} S_{\mathcal{M}}^{-1} S_{\mathcal{M}}^{\tau^c}(x)\|^2 + (1 + \lambda^2) \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2.
\end{aligned} \tag{12}$$

THEOREM 6. (see [22, Theorem 3]) *Suppose* $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$ *with the fusion frame operator* $S_{\mathcal{M}}$, *and* $\mathcal{M}' = \{(S_{\mathcal{M}}^{-1}\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ *is the dual fusion frame of* $\mathcal{M}$. *Then for every* $\lambda \in \mathbb{R}$, *for any* $\tau \subset \mathbb{I}$ *and any* $x \in \mathcal{N}$, *we have*

$$\begin{aligned}
&(1 + 2\lambda) \sum_{i\in\tau^c} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 - (1 + \lambda^2) \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \\
&\quad \leqslant \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i} S_{\mathcal{M}}^{-1} S_{\mathcal{M}}^{\tau^c}(x)\|^2 - \sum_{i\in\tau} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 \\
&\quad \leqslant (3 - 2\lambda) \sum_{i\in\tau^c} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2 + (\lambda^2 - 1) \sum_{i\in\mathbb{I}} \beta_i^2 \|\pi_{\mathcal{M}_i}(x)\|^2.
\end{aligned} \tag{13}$$

THEOREM 7. (see [22, Theorem 4]) *Let* $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ *be a fusion frame for* $\mathcal{N}$ *and* $\mathcal{P} = \{(\mathcal{P}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ *be an alternate dual fusion frame of* $\mathcal{M}$. *Then for each* $\lambda \in [0,1]$, *for any* $\tau \subset \mathbb{I}$ *and any* $x \in \mathcal{N}$, *we have*

$$\begin{aligned}
&(\lambda - \lambda^2) \left\| \sum_{i\in\mathbb{I}} \beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x) \right\|^2 - \lambda \mathrm{Re} \sum_{i\in\tau^c} \beta_i \upsilon_i \langle S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x), \pi_{\mathcal{P}_i}(x) \rangle \\
&\quad \leqslant \left\| \sum_{i\in\tau} \beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x) \right\|^2 - \lambda \mathrm{Re} \sum_{i\in\tau} \beta_i \upsilon_i \langle S_{\mathcal{M}}^{-1} \pi_{\mathcal{M}_i}(x), \pi_{\mathcal{P}_i}(x) \rangle \\
&\quad \leqslant \frac{\lambda (\|W^{\tau} - W^{\tau^c}\|^2 - 1) + 4(1 - \lambda) \|W^{\tau}\|^2}{4} \|x\|^2.
\end{aligned} \tag{14}$$

THEOREM 8. (see [22, Theorem 5]) *Let $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ be a fusion frame for $\mathcal{N}$ and $\mathcal{P}=\{(\mathcal{P}_i,\upsilon_i)\}_{i\in\mathbb{I}}$ be an alternate dual fusion frame of $\mathcal{M}$. Then for each $\lambda\in[0,\frac{1}{2}]$, for any $\tau\subset\mathbb{I}$ and any $x\in\mathcal{N}$, we have*

$$\begin{aligned}(2\lambda-\lambda^2)&\Big\|\sum_{i\in\mathbb{I}}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^2\\ &\leqslant\Big\|\sum_{i\in\tau}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^2+2\lambda\operatorname{Re}\sum_{i\in\tau^c}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle \\ &\leqslant\frac{3\lambda+2(1-2\lambda)\|W^\tau\|^2+\lambda\|W^\tau-W^{\tau^c}\|^2}{2}\|x\|^2.\end{aligned}\tag{15}$$

After a careful examination of above results, we found the following facts:

(1) The proofs of Theorems 1–6 are entirely dependent on results about linear bounded operators (i.e. [17, Lemmas 1 and 3] and [22, Lemma 1]), as well as the relationship of operators related to the fusion frames, making the process lengthy.

(2) As for Theorems 7 and 8, we found that the two inequalities on the left hold for any parameter belonging to $\mathbb{R}$, meaning that the involved intervals $[0,1]$ and $[0,\frac{1}{2}]$ are redundant conditions for them.

Given this, in this paper we present a new approach to the proofs of Theorems 1–6 from the perspective of function theory, which can greatly simplify the proving process. And particularly, our proof method shows that the involving parameter in Theorems 1, 2, 3 and 5 can take values from larger ranges. We also provide new expressions for Theorems 7 and 8, so that the left-hand inequalities in them can be truly determined by the intervals where the parameter is taken from, and that the intervals $[0,1]$ and $[0,\frac{1}{2}]$ in those two theorems can be extended to larger ones. Moreover, we establish several new bilateral inequalities for fusion frames in Hilbert spaces.

2. New proofs

We need to explain some symbols first. Given a fusion frame $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ for $\mathcal{N}$ with analysis operators $U_{\mathcal{M}}$ and fusion frame operator $S_{\mathcal{M}}$ respectively, let us denote $Q=U_{\mathcal{M}}S_{\mathcal{M}}^{-1}U_{\mathcal{M}}^*$. Then, as in the classical case, Q is the orthogonal projection from $(\sum_{i\in\mathbb{I}}\oplus\mathcal{M}_i)_{\ell^2}$ onto Range$(U_{\mathcal{M}})$. For any $\tau\subset\mathbb{I}$, we denote by P_τ the orthogonal projection on $(\sum_{i\in\mathbb{I}}\oplus\mathcal{M}_i)_{\ell^2}$ given by

$$P_\tau(\{y_i\}_{i\in\mathbb{I}})=\{z_i\}_{i\in\mathbb{I}},\quad\text{where}\quad\begin{cases}z_i=y_i & \text{if } i\in\tau,\\ z_i=0 & \text{if } i\in\tau^c.\end{cases}$$

With the help of P_τ, the notations $S_{\mathcal{M}}^{\tau}$ and $S_{\mathcal{M}}^{\tau^c}$ in (5) now can be expressed respectively as $S_{\mathcal{M}}^{\tau}=U_{\mathcal{M}}^*P_\tau U_{\mathcal{M}}$ and $S_{\mathcal{M}}^{\tau^c}=U_{\mathcal{M}}^*P_\tau^{\perp}U_{\mathcal{M}}$.

2.1. New proofs of Theorems 1, 2 and 3

The aim of this section is to present new proofs to Theorems 1, 2 and 3, and to show that the intervals $[0,2]$, $[1,2]$ and $[1,2]$ involved in those theorems can be respectively extended to $\mathbb{R}$, $\mathbb{R}$ and $[1,+\infty)$.

2.1.1. Proof of Theorem 1

The formula (8) in Theorem 1 can be rewritten as

$$\begin{aligned}\|U_{\mathcal{M}}x\|^2 &\geqslant \|P_\tau^\perp U_{\mathcal{M}}x\|^2 + \|QP_\tau U_{\mathcal{M}}x\|^2\\ &= \|P_\tau U_{\mathcal{M}}x\|^2 + \|QP_\tau^\perp U_{\mathcal{M}}x\|^2\\ &\geqslant \left(\lambda - \frac{\lambda^2}{4}\right)\|P_\tau U_{\mathcal{M}}x\|^2 + \left(1 - \frac{\lambda^2}{4}\right)\|P_\tau^\perp U_{\mathcal{M}}x\|^2\end{aligned} \tag{16}$$

for each $x \in \mathcal{N}$. Denote by $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$. Then $Qy = y$. The equality in (16) follows from the fact that

$$\|P_\tau y\|^2 - \|QP_\tau y\|^2 = \|Q^\perp P_\tau y\|^2 = \|Q^\perp P_\tau^\perp y\|^2 = \|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2. \tag{17}$$

To show the inequality on the right in (16), it is equivalent to show that the quadratic function

$$f(\lambda) = \|P_\tau^\perp y\|^2 + \|QP_\tau y\|^2 - \left(\lambda - \frac{\lambda^2}{4}\right)\|P_\tau y\|^2 - \left(1 - \frac{\lambda^2}{4}\right)\|P_\tau^\perp y\|^2 \geqslant 0,$$

which is really the case for any $\lambda \in \mathbb{R}$, since

$$\begin{aligned}f(\lambda) &= \|QP_\tau y\|^2 - \lambda\|P_\tau y\|^2 + \frac{\lambda^2}{4}(\|P_\tau y\|^2 + \|P_\tau^\perp y\|^2)\\ &= \frac{\lambda^2}{4} - \lambda\|P_\tau y\|^2 + \|QP_\tau y\|^2\\ &= \left(\frac{\lambda}{2} - \|P_\tau y\|^2\right)^2 + (\|QP_\tau y\|^2 - \|P_\tau y\|^4),\end{aligned}$$

and $\|P_\tau y\|^2 = \langle QP_\tau y, y\rangle \leqslant \|QP_\tau y\|$. For the left-hand inequality in (16), we have

$$\|P_\tau^\perp U_{\mathcal{M}}x\|^2 + \|QP_\tau U_{\mathcal{M}}x\|^2 \leqslant \|P_\tau^\perp U_{\mathcal{M}}x\|^2 + \|P_\tau U_{\mathcal{M}}x\|^2 = \|U_{\mathcal{M}}x\|^2$$

for any $x \in \mathcal{N}$. This concludes the proof of the theorem.

2.1.2. Proof of Theorem 2

Taking $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$ for each $x \in \mathcal{N}$. Then we can rewrite the inequalities in (9) as

$$0 \leqslant \|P_\tau y\|^2 - \|QP_\tau y\|^2 \leqslant (\lambda - 1)\|P_\tau^\perp y\|^2 + \left(1 - \frac{\lambda}{2}\right)^2. \tag{18}$$

The inequality on the left is obvious. As for the right-hand inequality in (18), it is equivalent to show that

$$\begin{aligned}0 \leqslant g(\lambda) &= (\lambda-1)\|P_\tau^\perp y\|^2 + \left(1-\frac{\lambda}{2}\right)^2 - \|P_\tau y\|^2 + \|QP_\tau y\|^2 \\ &= \lambda\|P_\tau^\perp y\|^2 - (\|P_\tau^\perp y\|^2 + \|P_\tau y\|^2) + \frac{\lambda^2}{4} - \lambda + 1 + \|QP_\tau y\|^2 \\ &= \frac{\lambda^2}{4} - \lambda(1-\|P_\tau^\perp y\|^2) + \|QP_\tau y\|^2 \\ &= \frac{\lambda^2}{4} - \lambda\|P_\tau y\|^2 + \|QP_\tau y\|^2 = \left(\frac{\lambda}{2} - \|P_\tau y\|^2\right)^2 + (\|QP_\tau y\|^2 - \|P_\tau y\|^4),\end{aligned}$$

which is true for any $\lambda \in \mathbb{R}$, since $\|QP_\tau y\| \geqslant \|P_\tau y\|^2$ as shown in Section 2.1.1, and we are done.

2.1.3. Proof of Theorem 3

Letting $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$ for all $x \in \mathcal{N}$, then the inequalities in (10) can be rewritten as

$$\left(2\lambda - \frac{\lambda^2}{2} - 1\right)\|P_\tau y\|^2 + \left(1-\frac{\lambda^2}{2}\right)\|P_\tau^\perp y\|^2 \leqslant \|QP_\tau y\|^2 + \|QP_\tau^\perp y\|^2 \leqslant \lambda. \tag{19}$$

Obviously, the inequality on the right holds for each $\lambda \geqslant 1$, since

$$\|QP_\tau y\|^2 + \|QP_\tau^\perp y\|^2 \leqslant \|P_\tau y\|^2 + \|P_\tau^\perp y\|^2 = \|y\|^2 = 1.$$

For the left-hand inequality in (19), it is equivalent to prove that

$$h(\lambda) = \|QP_\tau y\|^2 + \|QP_\tau^\perp y\|^2 - \left(2\lambda - \frac{\lambda^2}{2} - 1\right)\|P_\tau y\|^2 - \left(1-\frac{\lambda^2}{2}\right)\|P_\tau^\perp y\|^2 \geqslant 0.$$

Since

$$\|P_\tau y\|^2 - \|P_\tau^\perp y\|^2 + \|QP_\tau^\perp y\|^2 = \|QP_\tau y\|^2,$$

by (17), and $\|P_\tau y\|^4 \leqslant \|QP_\tau y\|^2$, it follows that

$$\begin{aligned}h(\lambda) &= \|QP_\tau y\|^2 + (\|QP_\tau^\perp y\|^2 + \|P_\tau y\|^2 - \|P_\tau^\perp y\|^2) \\ &\qquad - 2\lambda\|P_\tau y\|^2 + \frac{\lambda^2}{2}(\|P_\tau y\|^2 + \|P_\tau^\perp y\|^2) \\ &= \frac{\lambda^2}{2} - 2\lambda\|P_\tau y\|^2 + 2\|QP_\tau y\|^2 \\ &= 2\left(\frac{\lambda}{2} - \|P_\tau y\|^2\right)^2 + 2(\|QP_\tau y\|^2 - \|P_\tau y\|^4) \geqslant 0,\end{aligned}$$

and we arrive at the conclusion.

2.2. New proofs of Theorems 4, 5 and 6

The purpose of this section is to provide new proofs to Theorems 4, 5 and 6, and to show that the interval $[\frac{1}{2},+\infty)$ involved in Theorem 5 can be extended to $[0,+\infty)$.

2.2.1. Proof of Theorem 4

The inequalities stated in (11) can be rewritten as

$$\begin{aligned}&\|P_\tau U_{\mathcal{M}}x\|^2-\lambda\|P_\tau^\perp U_{\mathcal{M}}x\|^2\\&\qquad\leqslant\|QP_\tau U_{\mathcal{M}}x\|^2-\lambda\|QP_\tau^\perp U_{\mathcal{M}}x\|^2\\&\qquad\leqslant(\lambda^3-\lambda^2+1)\|P_\tau U_{\mathcal{M}}x\|^2+(\lambda^3-3\lambda^2+2\lambda-1)\|P_\tau^\perp U_{\mathcal{M}}x\|^2\end{aligned}\tag{20}$$

for each $x\in\mathcal{N}$. Denote by $y=U_{\mathcal{M}}x$ and normalize it to $\|y\|=1$. Then $Qy=y$. Now the left-hand inequality in (20) follows from the equality

$$\|P_\tau y\|^2-\|QP_\tau y\|^2=\|Q^\perp P_\tau y\|^2=\|Q^\perp P_\tau^\perp y\|^2=\|P_\tau^\perp y\|^2-\|QP_\tau^\perp y\|^2\tag{21}$$

for each $\lambda\geqslant1$.

For the inequality on the right in (20), we know, by combining (20) with (21), that it is equivalent to

$$(1-\lambda)\|QP_\tau^\perp y\|^2\leqslant(1-\lambda)[2\lambda\|P_\tau^\perp y\|^2-\lambda^2].\tag{22}$$

To show (22), it is equivalent to prove that, for $\lambda\geqslant1$, the quadratic function $f(\lambda)=\lambda^2-2\lambda\|P_\tau^\perp y\|^2+\|QP_\tau^\perp y\|^2\geqslant0$. Since the minimizer for $f(\lambda)$ is in $\lambda_0=\|P_\tau^\perp y\|^2$, and $\|QP_\tau^\perp y\|\geqslant\|P_\tau^\perp y\|^2$, we obtain

$$f(\lambda)\geqslant f(\lambda_0)=(\|QP_\tau^\perp y\|-\|P_\tau^\perp y\|^2)(\|QP_\tau^\perp y\|+\|P_\tau^\perp y\|^2)\geqslant0,$$

as desired.

2.2.2. Proof of Theorem 5

Letting $y=U_{\mathcal{M}}x$ and normalize it to $\|y\|=1$ for any $x\in\mathcal{N}$. Then we can rewrite the inequalities in (12) as

$$\begin{aligned}(4\lambda-1)\|QP_\tau^\perp y\|^2+(1-\lambda^2)&\leqslant\|P_\tau y\|^2+(1+2\lambda)\|P_\tau^\perp y\|^2\\&\leqslant\|QP_\tau^\perp y\|^2+(1+\lambda^2).\end{aligned}\tag{23}$$

It is easy to see that the inequality on the left is equivalent to prove that

$$g(\lambda)=\lambda^2+2\lambda(\|P_\tau^\perp y\|^2-2\|QP_\tau^\perp y\|^2)+\|QP_\tau^\perp y\|^2\geqslant0,$$

which, actually, follows from the fact that

$$\begin{aligned}g(\lambda)&\geqslant\lambda^2-2\lambda\|P_\tau^\perp y\|^2+\|QP_\tau^\perp y\|^2\\&=(\lambda-\|P_\tau^\perp y\|^2)^2+(\|QP_\tau^\perp y\|^2-\|P_\tau^\perp y\|^4)\end{aligned}$$

for each $\lambda \in [0,+\infty)$, and that $\|P_\tau^\perp y\|^2 \leqslant \|QP_\tau^\perp y\|$.

As for the right-hand inequality in (23), it is equivalent to show that

$$\begin{aligned} 0 \leqslant h(\lambda) &= (1+\lambda^2) + \|QP_\tau^\perp y\|^2 - \|P_\tau y\|^2 - (1+2\lambda)\|P_\tau^\perp y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\tau^\perp y\|^2 + (1-\|P_\tau y\|^2 - \|P_\tau^\perp y\|^2) + \|QP_\tau^\perp y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\tau^\perp y\|^2 + \|QP_\tau^\perp y\|^2, \end{aligned}$$

which has already been presented in the proof of the left-hand inequality.

2.2.3. Proof of Theorem 6

For any $x \in \mathcal{N}$, taking $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$, then the inequalities in (13) can be rewritten as

$$\begin{aligned} (1+2\lambda)\|P_\tau^\perp y\|^2 - (1+\lambda^2) &\leqslant \|QP_\tau^\perp y\|^2 - \|P_\tau y\|^2 \\ &\leqslant (3-2\lambda)\|P_\tau^\perp y\|^2 + (\lambda^2 - 1). \end{aligned} \tag{24}$$

The left-hand inequality in (24) follows from the following calculation

$$\begin{aligned} &(1+\lambda^2) + \|QP_\tau^\perp y\|^2 - \|P_\tau y\|^2 - (1+2\lambda)\|P_\tau^\perp y\|^2 \\ &\quad = \lambda^2 + \|QP_\tau^\perp y\|^2 + \|P_\tau^\perp y\|^2 - (1+2\lambda)\|P_\tau^\perp y\|^2 \\ &\quad = \lambda^2 - 2\lambda\|P_\tau^\perp y\|^2 + \|QP_\tau^\perp y\|^2 \\ &\quad = (\lambda - \|P_\tau^\perp y\|^2)^2 + (\|QP_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^4) \geqslant 0 \end{aligned}$$

for any $\lambda \in \mathbb{R}$.

For the inequality on the right in (24), it is equivalent to show that

$$h(\lambda) = (3-2\lambda)\|P_\tau^\perp y\|^2 + (\lambda^2 - 1) + \|P_\tau y\|^2 - \|QP_\tau^\perp y\|^2 \geqslant 0,$$

which is indeed true, since

$$\begin{aligned} h(\lambda) &= \lambda^2 + (3-2\lambda)\|P_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2 \\ &= \lambda^2 + 2(1-\lambda)\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2 \\ &= (\lambda - \|P_\tau^\perp y\|^2)^2 + (\|P_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^4) + (\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2). \end{aligned}$$

3. Improved results

The following two results make an improvement to Theorems 7 and 8 respectively, which enables the inequalities on the left to be truly determined by the involved intervals and allows the parameter to take values from larger ranges.

THEOREM 9. *Suppose $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{N}$, and $\mathcal{P} = \{(\mathcal{P}_i, \upsilon_i)\}_{i\in\mathbb{I}}$ is an alternate dual fusion frame of $\mathcal{M}$. Then for each $\lambda \in [0,2]$, for any*

$\tau \subset \mathbb{I}$ *and any* $x \in \mathcal{N}$, *we have*

$$\begin{aligned}
&(1-\lambda-(2-\lambda)\|W^{\tau}\|^{2})\|x\|^{2} \\
&\quad \leqslant \Big\|\sum_{i\in\tau}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^{2} - \lambda \operatorname{Re}\sum_{i\in\tau}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle \\
&\quad \leqslant \frac{(2-\lambda)\|W^{\tau}\|^{2}+\lambda(\|W^{\tau^{c}}\|^{2}-1)}{2}\|x\|^{2}.
\end{aligned}$$

Proof. Since $W^{\tau}+W^{\tau^{c}}=\mathrm{Id}_{\mathcal{N}}$, the identity operator on $\mathcal{N}$, we have, for each $\lambda\in[0,2]$, for any $x\in\mathcal{N}$ and any $\tau\subset\mathbb{I}$, that

$$\begin{aligned}
&\Big\|\sum_{i\in\tau}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^{2} - \lambda \operatorname{Re}\sum_{i\in\tau}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle \\
&\quad = \|W^{\tau}x\|^{2}-\lambda\operatorname{Re}\langle W^{\tau}x,x\rangle \\
&\quad = \|W^{\tau}x\|^{2}+\lambda\frac{\|x-W^{\tau}x\|^{2}-\|W^{\tau}x\|^{2}-\|x\|^{2}}{2} \\
&\quad = \frac{2\|W^{\tau}x\|^{2}+\lambda\|W^{\tau^{c}}x\|^{2}-\lambda\|W^{\tau}x\|^{2}-\lambda\|x\|^{2}}{2} \\
&\quad \leqslant \frac{(2-\lambda)\|W^{\tau}\|^{2}+\lambda(\|W^{\tau^{c}}\|^{2}-1)}{2}\|x\|^{2}.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\Big\|\sum_{i\in\tau}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^{2} - \lambda \operatorname{Re}\sum_{i\in\tau}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle \\
&\quad = \|W^{\tau}x\|^{2}-\lambda\operatorname{Re}\langle W^{\tau}x,x\rangle \\
&\quad = \|x\|^{2}+\|W^{\tau^{c}}x\|^{2}-2\operatorname{Re}\langle W^{\tau^{c}}x,x\rangle-\lambda(\|x\|^{2}-\operatorname{Re}\langle W^{\tau^{c}}x,x\rangle) \\
&\quad = (1-\lambda)\|x\|^{2}+\|W^{\tau^{c}}x\|^{2}-(2-\lambda)\operatorname{Re}\langle W^{\tau^{c}}x,x\rangle \\
&\quad \geqslant (1-\lambda)\|x\|^{2}-(2-\lambda)\|W^{\tau^{c}}\|\|x\|^{2} \\
&\quad = (1-\lambda-(2-\lambda)\|W^{\tau^{c}}\|)\|x\|^{2}
\end{aligned}$$

for each $\lambda\in[0,2]$, for any $x\in\mathcal{N}$ and any $\tau\subset\mathbb{I}$, and the proof is finished. □

THEOREM 10. *Suppose* $\mathcal{M}=\{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ *is a fusion frame for* $\mathcal{N}$, *and* $\mathcal{P}=\{(\mathcal{P}_i,\upsilon_i)\}_{i\in\mathbb{I}}$ *is an alternate dual fusion frame of* $\mathcal{M}$. *Then for each* $\lambda\in[0,1]$, *for any* $\tau\subset\mathbb{I}$ *and any* $x\in\mathcal{N}$, *we have*

$$\begin{aligned}
&(1-2(1-\lambda)\|W^{\tau^{c}}\|)\|x\|^{2} \\
&\quad \leqslant \Big\|\sum_{i\in\tau}\beta_i\upsilon_i\pi_{\mathcal{P}_i}S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^{2} + 2\lambda \operatorname{Re}\sum_{i\in\tau^{c}}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle \qquad (25) \\
&\quad \leqslant \big((1-\lambda)\|W^{\tau}\|^{2}+\lambda(1+\|W^{\tau^{c}}\|^{2})\big)\|x\|^{2}.
\end{aligned}$$

Proof. For each $\lambda \in [0,1]$, for any $x \in \mathcal{N}$ and any $\tau \subset \mathbb{I}$, we can obtain the left-hand inequality in (25) by the following computation

$$\begin{aligned}
&\Big\|\sum_{i\in\tau}\beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^2 + 2\lambda \operatorname{Re}\sum_{i\in\tau^c}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle\\
&\quad= \|W^{\tau}x\|^2 + 2\lambda \operatorname{Re}\langle W^{\tau^c}x,x\rangle\\
&\quad= \|x\|^2 + \|W^{\tau^c}x\|^2 - 2\operatorname{Re}\langle W^{\tau^c}x,x\rangle + 2\lambda\operatorname{Re}\langle W^{\tau^c}x,x\rangle\\
&\quad= \|x\|^2 + \|W^{\tau^c}x\|^2 - 2(1-\lambda)\operatorname{Re}\langle W^{\tau^c}x,x\rangle\\
&\quad\geqslant \|x\|^2 - 2(1-\lambda)\|W^{\tau^c}\|\|x\|^2\\
&\quad= (1-2(1-\lambda)\|W^{\tau^c}\|)\|x\|^2.
\end{aligned}$$

For the inequality on the right hand side, we compute that

$$\begin{aligned}
&\Big\|\sum_{i\in\tau}\beta_i \upsilon_i \pi_{\mathcal{P}_i} S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x)\Big\|^2 + 2\lambda \operatorname{Re}\sum_{i\in\tau^c}\beta_i\upsilon_i\langle S_{\mathcal{M}}^{-1}\pi_{\mathcal{M}_i}(x),\pi_{\mathcal{P}_i}(x)\rangle\\
&\quad= \|W^{\tau}x\|^2 + 2\lambda \operatorname{Re}\langle W^{\tau^c}x,x\rangle\\
&\quad= \|W^{\tau}x\|^2 + 2\lambda\|x\|^2 - 2\lambda\operatorname{Re}\langle W^{\tau}x,x\rangle\\
&\quad= \|W^{\tau}x\|^2 + 2\lambda\|x\|^2 - \lambda(\|x\|^2 + \|W^{\tau}x\|^2 - \|x-W^{\tau}x\|^2)\\
&\quad= (1-\lambda)\|W^{\tau}x\|^2 + \lambda\|x\|^2 + \lambda\|W^{\tau^c}x\|^2\\
&\quad\leqslant ((1-\lambda)\|W^{\tau}\|^2 + \lambda(1+\|W^{\tau^c}\|^2))\|x\|^2
\end{aligned}$$

for each $\lambda \in [0,1]$, for any $x \in \mathcal{N}$ and any $\tau \subset \mathbb{I}$, and we have the result. □

4. New inequalities

In this section, we give several new bilateral inequalities for fusion frames and, following the approaches of the results, one can easily establish corresponding bilateral inequalities for some other generalized frames such as K-frames, continuous fusion frames, Hilbert-Schmidt frames, K-g-frames and continuous g-frames etc, which admit new types of structures compared with the ones given in [1, 18, 20, 23, 24].

THEOREM 11. *Suppose $\mathcal{M} = \{(\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{N}$ with the fusion frame operator $S_{\mathcal{M}}$, and $\mathcal{M}' = \{(S_{\mathcal{M}}^{-1}\mathcal{M}_i,\beta_i)\}_{i\in\mathbb{I}}$ is the dual fusion frame of $\mathcal{M}$. Then for each $\lambda \geqslant 0$, for any $\tau \subset \mathbb{I}$ and any $x \in \mathcal{N}$, we have*

$$\begin{aligned}
&(1-\lambda^3)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 + (2\lambda^2-2\lambda-1)\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2\\
&\quad\leqslant \sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 - \lambda\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau^c}(x)\|^2\\
&\quad\leqslant (1+\lambda^3)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 - (1+2\lambda^2)\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2.
\end{aligned}\tag{26}$$

Proof. For any $x \in \mathcal{N}$, taking $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$. Then the inequalities in (26) can be rewritten as

$$\begin{aligned}(1-\lambda^3)+(2\lambda^2-2\lambda-1)\|P_\tau^\perp y\|^2 \leqslant \|P_\tau y\|^2 - \lambda\|QP_\tau^\perp y\|^2 \\ \leqslant (1+\lambda^3)-(1+2\lambda^2)\|P_\tau^\perp y\|^2.\end{aligned} \tag{27}$$

Since

$$\begin{aligned}f(\lambda) &= \|P_\tau y\|^2 - \lambda\|QP_\tau^\perp y\|^2 - (1-\lambda^3) - (2\lambda^2-2\lambda-1)\|P_\tau^\perp y\|^2 \\ &= \lambda^3 - (2\lambda^2-2\lambda)\|P_\tau^\perp y\|^2 + (\|P_\tau y\|^2 + \|P_\tau^\perp y\|^2 - 1) - \lambda\|QP_\tau^\perp y\|^2 \\ &= \lambda^3 - 2\lambda^2\|P_\tau^\perp y\|^2 + 2\lambda\|P_\tau^\perp y\|^2 - \lambda\|QP_\tau^\perp y\|^2 \\ &= \lambda(\lambda^2 - 2\lambda\|P_\tau^\perp y\|^2 + 2\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2) \\ &= \lambda\big((\lambda - \|P_\tau^\perp y\|^2)^2 + (\|P_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^4) + (\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2)\big) \geqslant 0\end{aligned}$$

for each $\lambda \geqslant 0$, meaning that the left-hand inequality in (27) holds. As for the inequality on the right, it is equivalent to show that

$$g(\lambda) = (1+\lambda^3) - (1+2\lambda^2)\|P_\tau^\perp y\|^2 - \|P_\tau y\|^2 + \lambda\|QP_\tau^\perp y\|^2 \geqslant 0$$

for each $\lambda \geqslant 0$, which is obvious since

$$\begin{aligned}g(\lambda) &= \lambda^3 - 2\lambda^2\|P_\tau^\perp y\|^2 - (\|P_\tau^\perp y\|^2 + \|P_\tau y\|^2 - 1) + \lambda\|QP_\tau^\perp y\|^2 \\ &= \lambda^3 - 2\lambda^2\|P_\tau^\perp y\|^2 + \lambda\|QP_\tau^\perp y\|^2 \\ &= \lambda\big((\lambda - \|P_\tau^\perp y\|^2)^2 + (\|QP_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^4)\big). \quad \square\end{aligned}$$

THEOREM 12. *Suppose $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{N}$ with the fusion frame operator $S_{\mathcal{M}}$, and $\mathcal{M}' = \{(S_{\mathcal{M}}^{-1}\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is the dual fusion frame of $\mathcal{M}$. Then for each $\lambda \in \mathbb{R}$, for any $\tau \subset \mathbb{I}$ and any $x \in \mathcal{N}$, we have*

$$\begin{aligned}\lambda\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 + (\lambda-\lambda^2)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 \\ \leqslant \sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_{\mathcal{M}}^{-1}S_{\mathcal{M}}^{\tau}(x)\|^2 + \lambda\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 \\ \leqslant (2-3\lambda)\sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 + (\lambda^2+\lambda)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2.\end{aligned} \tag{28}$$

Proof. For all $x \in \mathcal{N}$, we let $y = U_{\mathcal{M}}x$ and normalize it to $\|y\| = 1$. Then, we can rewrite the inequalities in (28) as follows:

$$\lambda\|P_\tau y\|^2 + \lambda - \lambda^2 \leqslant \|QP_\tau y\|^2 + \lambda\|P_\tau^\perp y\|^2 \leqslant (2-3\lambda)\|P_\tau y\|^2 + \lambda^2 + \lambda, \tag{29}$$

which, actually, follow from the computations

$$\begin{aligned}f(\lambda) &= \lambda^2 + \lambda + (2-3\lambda)\|P_\tau y\|^2 - \lambda\|P_\tau^\perp y\|^2 - \|QP_\tau y\|^2 \\ &= \lambda^2 + (2-3\lambda)\|P_\tau y\|^2 + \lambda\|P_\tau y\|^2 - \|QP_\tau y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\tau y\|^2 + 2\|P_\tau y\|^2 - \|QP_\tau y\|^2 \\ &= (\lambda - \|P_\tau y\|^2)^2 + (\|P_\tau y\|^2 - \|P_\tau y\|^4) + (\|P_\tau y\|^2 - \|QP_\tau y\|^2) \geqslant 0,\end{aligned}$$

and

$$\begin{aligned} g(\lambda) &= \|QP_\tau y\|^2 + \lambda\|P_\tau^\perp y\|^2 - \lambda\|P_\tau y\|^2 + \lambda^2 - \lambda \\ &= \lambda^2 + \lambda(\|P_\tau^\perp y\|^2 - 1) - \lambda\|P_\tau y\|^2 + \|QP_\tau y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\tau y\|^2 + \|QP_\tau y\|^2 \\ &= (\lambda - \|P_\tau y\|^2)^2 + (\|QP_\tau y\|^2 - \|P_\tau y\|^4) \geqslant 0. \quad \square \end{aligned}$$

THEOREM 13. *Suppose $\mathcal{M} = \{(\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is a fusion frame for $\mathcal{N}$ with the fusion frame operator $S_\mathcal{M}$, and $\mathcal{M}' = \{(S_\mathcal{M}^{-1}\mathcal{M}_i, \beta_i)\}_{i\in\mathbb{I}}$ is the dual fusion frame of $\mathcal{M}$. Then for each $\lambda \geqslant 2$, for any $\tau \subset \mathbb{I}$ and any $x \in \mathcal{N}$, we have*

$$\begin{aligned} &(1-\lambda^2)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 + (\lambda-2)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_\mathcal{M}^{-1}S_\mathcal{M}^{\tau^c}(x)\|^2 \\ &\quad\leqslant \sum_{i\in\tau}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 - \lambda\sum_{i\in\tau^c}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 \\ &\quad\leqslant (1+\lambda^3)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}(x)\|^2 - (1+2\lambda^2)\sum_{i\in\mathbb{I}}\beta_i^2\|\pi_{\mathcal{M}_i}S_\mathcal{M}^{-1}S_\mathcal{M}^{\tau^c}(x)\|^2. \end{aligned} \tag{30}$$

Proof. The inequalities in (30) can be rewritten as

$$1-\lambda^2+(\lambda-2)\|QP_\tau^\perp y\|^2 \leqslant \|P_\tau y\|^2 - \lambda\|P_\tau^\perp y\|^2 \leqslant 1+\lambda^3-(1+2\lambda^2)\|QP_\tau^\perp y\|^2, \tag{31}$$

if we take $y = U_\mathcal{M}x$ and normalize it to $\|y\| = 1$ for each $x \in \mathcal{N}$. To prove the inequality to the left, it is sufficient to show that

$$f(\lambda) = \|P_\tau y\|^2 - \lambda\|P_\tau^\perp y\|^2 - (1-\lambda^2) - (\lambda-2)\|QP_\tau^\perp y\|^2 \geqslant 0$$

for any $\lambda \geqslant 2$, which is obvious, since a simple calculation gives

$$\begin{aligned} f(\lambda) &= \lambda^2 - (1-\|P_\tau y\|^2) - \lambda\|P_\tau^\perp y\|^2 - (\lambda-2)\|QP_\tau^\perp y\|^2 \\ &= \lambda^2 - \|P_\tau^\perp y\|^2 - \lambda\|P_\tau^\perp y\|^2 - (\lambda-2)\|QP_\tau^\perp y\|^2 \\ &\geqslant \lambda^2 - \|P_\tau^\perp y\|^2 - \lambda\|P_\tau^\perp y\|^2 - (\lambda-2)\|P_\tau^\perp y\|^2 \\ &= \lambda^2 - 2\lambda\|P_\tau^\perp y\|^2 + \|P_\tau^\perp y\|^2 \\ &= (\lambda - \|P_\tau^\perp y\|^2)^2 + (\|P_\tau^\perp y\|^2 - \|P_\tau^\perp y\|^4). \end{aligned}$$

Additionally, observe that

$$\begin{aligned} g(\lambda) &= 1+\lambda^3-(1+2\lambda^2)\|QP_\tau^\perp y\|^2 - \|P_\tau y\|^2 + \lambda\|P_\tau^\perp y\|^2 \\ &= \lambda^3 - 2\lambda^2\|QP_\tau^\perp y\|^2 + \|P_\tau^\perp y\|^2 + \lambda\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2 \\ &= \lambda(\lambda - \|QP_\tau^\perp y\|^2)^2 + (\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^2) + \lambda(\|P_\tau^\perp y\|^2 - \|QP_\tau^\perp y\|^4) \geqslant 0, \end{aligned}$$

implying that the right-hand inequality in (31) is satisfied. $\square$

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