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BOUNDS AND MONOTONICITY OF THE NIELSEN BETA FUNCTION

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Abstract. In this paper, we establish bounds and prove the complete monotonicity of functions involving the Nielsen beta function. We derive a novel integral representation for the Dirichlet eta function. As an application, we construct several bounded Bernstein functions associated with the Nielsen beta function.

1. Introduction

The Nielsen β-function $\beta(x)$, first introduced in [10], can be represented through several equivalent froms

$$\beta(x)=\int_0^1 \frac{t^{x-1}}{1+t}\,dt=\int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt,\quad x>0, \tag{1}$$

$$\beta(x)=\sum_{n=0}^\infty \frac{(-1)^n}{n+x}=\frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right\},\quad x>0, \tag{2}$$

where $\psi(x)=\frac{d}{dx}\ln\Gamma(x)$ is the psi or digamma function and $\Gamma(x)$ is the Euler gamma function. It is well-known that the $\beta(x)$ function satisfies the following functional relations (see [7], [10]):

$$\beta(x+1)=\frac{1}{x}-\beta(x), \tag{3}$$

$$\beta(x)+\beta(1-x)=\frac{\pi}{\sin\pi x},\quad 0<x<1. \tag{4}$$

For additional properties and inequalities on the function refer to [2], [3], [6], [7], [8], [9], [14], and [15] and, in general, by successive differentiation of (1) and (2), we have

$$\begin{aligned}\beta^{(n)}(x)&=\int_0^1 \frac{(\ln t)^n t^{x-1}}{1+t}\,dt=(-1)^n\int_0^\infty \frac{t^n e^{-xt}}{1+e^{-t}}\,dt, \\ &=(-1)^n n!\sum_{k=0}^\infty \frac{(-1)^k}{(k+x)^{n+1}}=\frac{1}{2^{n+1}}\left\{\psi^{(n)}\left(\frac{x+1}{2}\right)-\psi^{(n)}\left(\frac{x}{2}\right)\right\},\quad x>0.\end{aligned} \tag{5}$$

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Recall from [11, equation 25.5.3 and 25.5.4] that, the integral representations of zeta (ζ) function are given by

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x+1}\,dx \quad \Re s > 1 \tag{6}$$

and

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s+1)} \int_0^\infty \frac{e^x x^s}{(e^x+1)^2}\,dx \quad \Re s > 1, \tag{7}$$

where, $\Gamma(s)$ is the classical Euler's gamma function. The Dirichlet eta (η) function is defined by

$$\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} \tag{8}$$

and the relation between ζ and η functions is given in [1, Equation 23.2.19] by

$$\eta(x) = (1-2^{1-x})\zeta(x), \quad \Re x > 0. \tag{9}$$

A function f is said to be completely monotonic on the interval (a,b), where $-\infty \leqslant a < b \leqslant \infty$, if f has derivatives of all orders on (a,b) and satisfies the inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0, \quad \text{for all } x \in (a,b) \text{ and } n \in \{0\} \cup \mathbb{N}, \tag{10}$$

see [13]. The famous Bernstein's theorem [13, p.161, Theorem 12b]: A function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\,d\sigma(t), x \in (0,\infty), \tag{11}$$

where $\sigma(\tau)$ is non decreasing and the integral in (11) converges for $x \in (0,\infty)$, and if a function $f(x)$ is completely monotonic, then the function

$$(-1)^m f^{(m)}(x) \tag{12}$$

is also completely monotonic for all $m \in \mathbb{N}$.

A function $f:(0,\infty) \to (0,\infty)$ is called a Bernstein function if f is infinitely differentiable and whose derivative is completely monotonic on $(0,\infty)$, that is

$$(-1)^n f^{(n+1)}(x) \geqslant 0 \quad \text{for all } x > 0 \text{ and } n = 0,1,2,\ldots$$

These functions play a vital role in the theory of convolution semigroups of measures supported on $(0,\infty)$ and related functional calculus, see [12].

2. Main results

In this section, we establish the complete monotonicity of functions involving Nielsen's beta function and derive certain bounds.

THEOREM 1. *(1) The function* $f_1(x) = x\beta(x) - \frac{1}{2}$ *is completely monotonic on* $(0,\infty)$.

(2) The function $f_2(x) = x\beta^{(2n)}(x) + 2n\beta^{(2n-1)}(x)$ *is completely monotonic on* $(0,\infty)$ *and maps* $(0,\infty)$ *onto* $(0,\Gamma(2n+1)\eta(2n))$ *for all* $n \in \mathbb{N}$.

(3) The negative of the function $f_3(x) = x\beta^{(2n-1)}(x) + (2n-1)\beta^{(2n-2)}(x)$ *is completely monotonic on* $(0,\infty)$ *and maps* $(0,\infty)$ *onto* $(-\Gamma(2n)\eta(2n-1),0)$ *for all* $n \in \mathbb{N}$.

Proof. Using the integral representation (1), we derive:

$$\begin{aligned} x\beta(x) &= \int_0^\infty \frac{xe^{-xt}}{1+e^{-t}}\,dt = -\int_0^\infty \frac{1}{1+e^{-t}}\frac{d}{dt}\left(e^{-xt}\right)\,dt \\ &= \left(-\frac{e^{-xt}}{1+e^{-t}}\right)\Big|_{t\to 0}^{t\to+\infty} + \int_0^\infty \left(\frac{1}{1+e^{-t}}\right)' e^{-xt}\,dt \\ &= \frac{1}{2} + \int_0^\infty \frac{e^t}{(1+e^t)^2}e^{-xt}\,dt. \end{aligned}$$

Thus, we obtain:

$$f_1(x) = x\beta(x) - \frac{1}{2} = \int_0^\infty \frac{e^t}{(1+e^t)^2}e^{-xt}\,dt. \tag{13}$$

This expression shows that the function $x\beta(x) - \frac{1}{2}$ is completely monotonic on $(0,\infty)$, and by (12), the function

$$f_2(x) = \left(x\beta(x) - \frac{1}{2}\right)^{(2n)} = x\beta^{(2n)}(x) + 2n\beta^{(2n-1)}(x) = \int_0^\infty \frac{t^{2n}e^t}{(1+e^t)^2}e^{-xt}\,dt$$

is completely monotonic, positive and decreasing on $(0,\infty)$. Therefore for $0 < x < \infty$, we have

$$\lim_{x\to\infty} f_2(x) < f_2(x) < \lim_{x\to 0^+} f_2(x).$$

Now,

$$\lim_{x\to 0^+} f_2(x) = \lim_{x\to 0^+} \int_0^\infty \frac{t^{2n}e^t}{(1+e^t)^2}e^{-xt}\,dt$$

by equation (9), we derive

$$\int_0^\infty \frac{t^{2n}e^t}{(1+e^t)^2}\,dt = \Gamma(2n+1)\eta(2n).$$

Further, $\lim_{x\to\infty} f_2(x) = 0$. Thus, we conclude that

$$0 < x\beta^{(2n)}(x) + 2n\beta^{(2n-1)}(x) < \Gamma(2n+1)\eta(2n).$$

This completes the proof of the second statement. Since $f_1(x) = x\beta(x) - \frac{1}{2}$ is completely monotonic on $(0,\infty)$, then by (12), we have

$$\begin{aligned} f_3(x) &= \left(x\beta(x) - \frac{1}{2}\right)^{(2n-1)} = (x\beta^{(2n-1)}(x) + (2n-1)\beta^{(2n-2)}(x)) \\ &= -\int_0^\infty \frac{t^{2n-1}e^t}{(1+e^t)^2} e^{-xt}\, dt. \end{aligned}$$

Thus, the function $-f_3(x)$ is completely monotonic, positive and decreasing on $(0,\infty)$. Therefore, for $0 < x < \infty$, we have:

$$\lim_{x\to 0^+} f_3(x) < f_3(x) < \lim_{x\to\infty} f_3(x).$$

Now,

$$\lim_{x\to 0^+} f_3(x) = -\lim_{x\to 0^+} \int_0^\infty \frac{t^{2n-1}e^t}{(1+e^t)^2} e^{-xt}\, dt$$

by equation (9), we find

$$-\int_0^\infty \frac{t^{2n-1}e^t}{(1+e^t)^2}\, dt = -\Gamma(2n)\eta(2n-1).$$

Further, $\lim_{x\to\infty} f_3(x) = 0$. Thus, we conclude:

$$-\Gamma(2n)\eta(2n-1) < x\beta^{(2n-1)}(x) + (2n-1)\beta^{(2n-2)}(x) < 0.$$

This completes the proof of the third statement. $\square$

As a direct consequence of the above theorem we have the following corollary.

COROLLARY 1. *For $x \in (0,\infty)$, the following inequalities hold*
(1) $0 < x\beta(x) - \frac{1}{2} < \frac{1}{2}$
(2) $-\ln(2) < x\beta'(x) + \beta(x) < 0$ *and*
(3) $0 < x\beta''(x) + 2\beta'(x) < \frac{\pi^2}{6}$.

REMARK 1. Theorem 2.1 (1) improves the result mentioned in [5, p. 292]

THEOREM 2. *The function* $f_4(x) = \frac{1}{4} + \frac{x}{2} - x^2\beta(x)$ *is completely monotonic on* $(0,\infty)$, *and bounded by*

$$0 < \frac{1}{4} + \frac{x}{2} - x^2\beta(x) < \frac{1}{4}. \tag{14}$$

Proof. Using equation (13), we obtain

$$\begin{aligned}
x^2\beta(x) - \frac{x}{2} &= -\int_0^\infty \frac{e^t}{(1+e^t)^2}\frac{d}{dt}(e^{-xt})\,dt \\
&= -\left(\frac{e^t}{(1+e^t)^2}e^{-xt}\right)_0^\infty + \int_0^\infty \left(\frac{e^t}{(1+e^t)^2}\right)' e^{-xt}\,dt \\
&= \frac{1}{4} - \int_0^\infty \frac{e^t(e^t-1)}{(1+e^t)^3}e^{-xt}\,dt.
\end{aligned}$$

Thus, the function

$$f_4(x) = \frac{1}{4} + \frac{x}{2} - x^2\beta(x) = \int_0^\infty \frac{e^t(e^t-1)}{(1+e^t)^3}e^{-xt}\,dt \tag{15}$$

is completely monotonic on $(0,\infty)$. Since the function $f_4(x) = \frac{1}{4} + \frac{x}{2} - x^2\beta(x)$ is completely monotonic, it implies that $f_4(x)$ is decreasing on $(0,\infty)$. Therefore, for $0 < x < \infty$, we have $\lim\limits_{x\to\infty} f_4(x) < f_4(x) < \lim\limits_{x\to 0^+} f_4(x)$. Now,

$$\begin{aligned}
\lim_{x\to 0^+} f_4(x) &= \lim_{x\to 0^+}\int_0^\infty \frac{e^t(e^t-1)}{(1+e^t)^3}e^{-xt}\,dt \\
&= \int_0^\infty \frac{e^t(e^t-1)}{(1+e^t)^3}\,dt.
\end{aligned}$$

Using the substitution $u = 1+e^t$, the above integral reduces to

$$\int_2^\infty \frac{u-2}{u^3}\,du = \frac{1}{4}.$$

Furthermore, $\lim\limits_{x\to\infty} f_4(x) = 0$. Thus, we have $0 < \frac{1}{4} + \frac{x}{2} - x^2\beta(x) < \frac{1}{4}$. $\square$

THEOREM 3. *The following limits hold*

(1) $\lim\limits_{x\to 0^+}\left(\frac{x}{2} - x^2\beta(x)\right) = 0,$

(2) $\lim\limits_{x\to\infty}\left(x^2\beta(x) - \frac{x}{2}\right) = \frac{1}{4}$, *and*

(3) The function $(-1)^n f_4^{(n)}(x)$ *is completely monotonic on* $(0,\infty)$ *for all* $n \in \mathbb{N}$ *with the limit* $\lim\limits_{x\to 0^+}(-1)^n f_4^{(n)}(x) = \Gamma(n+1)\eta(n-1)$.

Proof. The first two statements follow directly from Theorem (2). We now proceed to prove the third statement. Since the function

$$\frac{1}{4} + \frac{x}{2} - x^2\beta(x)$$

is completely monotonic on $(0,\infty)$, it follows that

$$(-1)^n\left(\frac{1}{4} + \frac{x}{2} - x^2\beta(x)\right)^{(n)} = \int_0^\infty \frac{e^t(e^t-1)t^n}{(1+e^t)^3}e^{-xt}\,dt.$$

Thus, the function $(-1)^n f_4^{(n)}(x)$ is completely monotonic on $(0,\infty)$. Now, taking the limit as $x\to 0^+$, we have:

$$\begin{aligned}\lim_{x\to 0^+}(-1)^n\left(\frac{1}{4}+\frac{x}{2}-x^2\beta(x)\right)^{(n)} &= \lim_{x\to 0^+}\int_0^\infty \frac{e^t(e^t-1)t^n}{(1+e^t)^3}e^{-xt}\,dt\\ &= \int_0^\infty \frac{e^t(e^t-1)t^n}{(1+e^t)^3}\,dt.\end{aligned}$$

Next, we consider:

$$\int_0^\infty \frac{e^t(e^t-1)t^n}{(1+e^t)^3}\,dt = \int_0^\infty \frac{e^{2t}t^n}{(1+e^t)^3}\,dt - \int_0^\infty \frac{e^t t^n}{(1+e^t)^3}\,dt.$$

Using integration by parts, we obtain:

$$\begin{aligned}&= \left[t^n\left(-\frac{1}{1+e^t}+\frac{1}{2(1+e^t)^2}\right)\right]_0^\infty - \int_0^\infty nt^{n-1}\left(-\frac{1}{1+e^t}+\frac{1}{2(1+e^t)^2}\right)dt\\ &\quad+\left[-\frac{t^n}{2(1+e^t)^2}\right]_0^\infty - \int_0^\infty \frac{nt^{n-1}}{2(1+e^t)^2}dt\\ &= n\int_0^\infty \frac{t^{n-1}}{1+e^t}dt - n\int_0^\infty \frac{t^{n-1}}{(1+e^t)^2}dt. \end{aligned} \tag{16}$$

From equation (6) and [4, Lemma 3.1], the expression in (16) simplifies to:

$$\int_0^\infty \frac{e^t(e^t-1)t^n}{(1+e^t)^3}\,dt = \Gamma(n+1)\eta(n) - n\Gamma(n)\left(\eta(n)-\eta(n-1)\right) = \Gamma(n+1)\eta(n-1).$$

Thus, the proof of the theorem is complete. □

REMARK 2. Let $n = x+1 > 0$ in the above theorem. The novel integral representation for the Dirichlet eta function $\eta(x)$ is given by:

$$\eta(x) = \frac{1}{\Gamma(x+2)}\int_0^\infty \frac{e^t(e^t-1)t^{x+1}}{(1+e^t)^3}\,dt. \tag{17}$$

THEOREM 4. *The function* $f_5(x) = \frac{\beta(x+\frac{1}{2})}{\beta(x)}$ *is monotonically increasing on* $(0,\infty)$ *and maps* $(0,\infty)$ *onto* $(0,1)$.

Proof. Differentiating $f_5(x)$ with respect to x, we obtain

$$f_5'(x) = \frac{\beta(x)\beta'(x+\frac{1}{2})-\beta(x+\frac{1}{2})\beta'(x)}{\beta^2(x)}. \tag{18}$$

It is sufficient to show that the numerator in (18) is positive. Let

$$N_1(x) = \beta(x)\beta'(x+\frac{1}{2})-\beta(x+\frac{1}{2})\beta'(x).$$

By equation 2 of Corollary 2.2 we have

$$-\beta'(x) > \frac{\beta(x)}{x} \tag{19}$$

Multiplying equation (19) by $\beta(x+\frac{1}{2})$ and adding $\beta(x)\beta'(x+\frac{1}{2})$, we obtain

$$N_1(x) > \beta(x)\left(\frac{1}{x}\beta(x+\frac{1}{2}) + \beta'(x+\frac{1}{2})\right) = N_2(x)\beta(x),$$

where

$$N_2(x) = \frac{1}{x}\beta(x+\frac{1}{2}) + \beta'(x+\frac{1}{2}).$$

Using the integral representation and the convolution property of the Laplace transform, we have

$$\begin{aligned} N_2(x) &= \int_0^\infty e^{-xt}\left(\int_0^\infty \frac{e^{-u/2}e^{-xt}}{1+e^{-u}}\,du - \frac{te^{-t/2}e^{-xt}}{1+e^{-t}}\right) dt \\ &= \int_0^\infty \left(\int_0^t \frac{e^{-u/2}}{1+e^{-u}}\,du - \frac{te^{-t/2}}{1+e^{-t}}\right) e^{-xt}\,dt. \end{aligned}$$

Define

$$\vartheta(t) = \int_0^t \frac{e^{-u/2}}{1+e^{-u}}\,du - \frac{te^{-t/2}}{1+e^{-t}}. \tag{20}$$

We have $\vartheta(0) = 0$, and

$$\vartheta'(t) = \frac{te^{-t/2}}{1+e^{-t}}\left(\frac{1}{2} - \frac{1}{e^t+1}\right) > 0 \quad \text{for all } t > 0,$$

which implies that $\vartheta(t)$ is increasing on $(0,\infty)$. Therefore, for $t > 0$, we have $\vartheta(t) > 0$. Thus, $N_2(x) > 0$ for all $x \in (0,\infty)$, which implies that $f_5'(x) = N_2(x)\beta(x) > 0$ for all $x \in (0,\infty)$. Consequently, $f_5'(x)$ is increasing on $(0,\infty)$. For $0 < x < \infty$, we have

$$f_5(0) < f_5(x) < f_5(\infty).$$

By the asymptotic formula [1]

$$\psi^{(n)}(x) \sim \frac{(-1)^{n+1}n!}{x^{n+1}}, \quad x \to 0^+ \tag{21}$$

we derive

$$\beta^{(n)} \sim \frac{(-1)^{n+1}n!}{2}\left(\frac{1}{(x+1)^n} - \frac{1}{x^n}\right), x \to 0^+. \tag{22}$$

Using equation (22), we obtain

$$\lim_{x\to 0^+} f_5(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_5(x) = 1.$$

Therefore, we have $0 < \frac{\beta(x+\frac{1}{2})}{\beta(x)} < 1$. $\square$

THEOREM 5. *The function* $f_6(x) = \frac{\beta'(x+\frac{1}{2})}{\beta'(x)}$ *is monotonically increasing on* $(0,\infty)$ *and maps* $(0,\infty)$ *onto* $(0,1)$.

Proof. Differentiating $f_6(x)$ with respect to x, we obtain

$$f_6'(x) = \frac{\beta'(x)\beta''(x+\frac{1}{2}) - \beta'(x+\frac{1}{2})\beta''(x)}{(\beta'(x))^2}. \tag{23}$$

It is sufficient to show that the numerator in (23) is positive. Let

$$N_3(x) = \beta'(x)\beta''(x+\frac{1}{2}) - \beta'(x+\frac{1}{2})\beta''(x).$$

By equation 3 of Corollary 2.2 we have $2\beta'(x) + x\beta''(x) > 0$ for al $x \in (0,\infty)$, and multiplying this with $\beta'(x+\frac{1}{2})$ and adding $\beta'(x)\beta''(x+\frac{1}{2})$, we obtain

$$N_3(x) > \beta'(x)\left(\frac{2}{x}\beta'(x+\frac{1}{2}) + \beta''(x+\frac{1}{2})\right) = N_4(x)\beta'(x),$$

where

$$N_4(x) = \frac{2}{x}\beta'(x+\frac{1}{2}) + \beta''(x+\frac{1}{2}).$$

Using the integral representation of $\beta'(x+\frac{1}{2})$, $\beta''(x+\frac{1}{2})$ and the convolution property of the Laplace transform, we obtain

$$\begin{aligned} N_4(x) &= 2\int_0^\infty e^{-xt}\left(\int_0^\infty \frac{-te^{-t/2}e^{-xt}}{1+e^{-t}}\,dt\right) + \int_0^\infty \frac{t^2e^{-t/2}e^{-xt}}{1+e^{-t}}\,dt \\ &= \int_0^\infty \left(-2\int_0^t \frac{ue^{-u/2}}{1+e^{-u}}\,du + \frac{t^2e^{-t/2}}{1+e^{-t}}\right)e^{-xt}\,dt. \end{aligned}$$

Define

$$\vartheta_1(t) = -2\int_0^t \frac{ue^{-u/2}}{1+e^{-u}}\,du + \frac{t^2e^{-t/2}}{1+e^{-t}}. \tag{24}$$

We have $\vartheta_1(0) = 0$, and

$$\vartheta_1'(t) = -\frac{t^2e^{-t/2}}{1+e^{-t}}\left(\frac{1}{2} - \frac{1}{e^t+1}\right) < 0 \quad \text{for all } t > 0,$$

which implies that $\vartheta_1(t)$ is decreasing on $(0,\infty)$. Therefore, for $t > 0$, we have $\vartheta_1(t) < 0$. Thus, $N_4(x) < 0$ for all $x \in (0,\infty)$, which implies that $f_6'(x) = N_4(x)\beta'(x) > 0$ for all $x \in (0,\infty)$. Consequently, $f_6'(x)$ is increasing on $(0,\infty)$. For $0 < x < \infty$, we have

$$f_6(0) < f_6(x) < f_6(\infty).$$

Using equation (22), we obtain

$$\lim_{x\to 0} f_6(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_6(x) = 1.$$

Therefore, we have $0 < f_6(x) < 1$ which is equivalent to $0 < \frac{\beta'(x+\frac{1}{2})}{\beta'(x)} < 1$. $\square$

3. Bernstein functions involving Nielsen beta function

In this section, we establish several bounded Bernstein functions involving the Nielsen beta function. To achieve this, we make use of the following key proposition, which provides a standard method for constructing bounded Bernstein functions from completely monotonic functions.

Proposition. [5, Proposition 4, p. 291] If g is a completely monotonic function on $(0,\infty)$ with $g(0+) < \infty$, then the function $f(x) := g(0+) - g(x)$ is a bounded Bernstein function on $(0,\infty)$.

COROLLARY 2. *(1) The function* $f_1(x) = x\beta(x) - \frac{1}{2}$ *is completely monotonic on* $(0,\infty)$*, and* $\lim_{x\to 0^+} f_1(x) = \frac{1}{2}$*. Then, by the above proposition, the function* $1 - x\beta(x)$ *is a bounded Bernstein function on* $(0,\infty)$*.*

(2) The function $f_2(x) = \left(x\beta(x) - \frac{1}{2}\right)^{(2n)}$ *is completely monotonic on* $(0,\infty)$*, and* $\lim_{x\to 0^+} f_2(x) = \Gamma(2n+1)\eta(2n)$*. Then, by the above proposition, the function* $\Gamma(2n+1)\eta(2n) - \left(x\beta(x) - \frac{1}{2}\right)^{(2n)}$ *is a bounded Bernstein function on* $(0,\infty)$ *for all* $n \in \mathbb{N}$*.*

(3) The function $f_3(x) = -\left(x\beta(x) - \frac{1}{2}\right)^{(2n-1)}$ *is completely monotonic on* $(0,\infty)$*, and* $\lim_{x\to 0^+} f_3(x) = -\Gamma(2n)\eta(2n-1)$*. Then, by the above proposition, the function* $\Gamma(2n)\eta(2n-1) + \left(x\beta(x) - \frac{1}{2}\right)^{(2n-1)}$ *is a bounded Bernstein function on* $(0,\infty)$ *for all* $n \in \mathbb{N}$*.*

(4) The function $f_4(x) = \frac{1}{4} + \frac{x}{2} - x^2\beta(x)$ *is completely monotonic on* $(0,\infty)$*, and* $\lim_{x\to 0^+} f_4(x) = \frac{1}{4}$*. Then, by the above proposition, the function* $x^2\beta(x) - \frac{x}{2}$ *is a bounded Bernstein function on* $(0,\infty)$*.*

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