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ON SOME INEQUALITIES FOR THE h–FOURIER COSINE–LAPLACE DISCRETE GENERALIZED CONVOLUTION AND APPLICATIONS

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Abstract. In this article, we study some inequalities for the h-Fourier cosine-Laplace discrete generalized convolution on the time scale $\mathbb{T}_h^0$ and establish some norm estimations for this discrete generalized convolution on some function spaces. We present some sufficient conditions for the existence of the h-Fourier cosine-Laplace discrete generalized convolution. A Young-type inequality, a Saitoh-type inequality and a reverse Saitoh-type inequality for this discrete generalized convolution are obtained. As applications, we apply some of these inequalities to estimate the solutions of a class of equations of the h-Fourier cosine-Laplace discrete generalized convolution type.

1. Introduction

The subject of time scales was first introduced by Stefan Hilger in 1988. From time scales analysis, we can do away the discrepancy between continuous and discrete analysis. Let h be a fixed positive real number. In this article, we are interested in the discrete time scale $\mathbb{T}_h^0 = h\mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here,

$$\mathbb{N} = \{1, 2, 3, 4, \dots\} \text{ is the set of all positive integers.}$$

For the time scale $\mathbb{T}_h^0$, the concept of the h-Laplace transform was established by M. Bohner and G. Sh. Guseinov in [2, p. 78] as follows: For a function $x : \mathbb{T}_h^0 \to \mathbb{C}$, its h-Laplace transform is given by

$$\mathscr{L}\{x\}(u) = \frac{h}{1+hu} \sum_{n=0}^{\infty} \frac{x(nh)}{(1+hu)^n}, \qquad u \in \mathbb{C} \text{ and } u \neq \frac{-1}{h},$$

for those values of u such that this series converges.

For a function $x : \mathbb{T}_h^0 \to \mathbb{C}$ in which $\sum_{n=0}^{\infty} |x(nh)|$ is finite, its h-Fourier cosine transform is defined by (see [6, p. 914], [7, p. 267] and [13, p. 208])

$$\mathscr{F}_c\{x\}(u) = hx(0) + 2h \sum_{n=1}^{\infty} x(nh) \cos(unh), \qquad u \in \mathbb{R}. \tag{1}$$

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When we consider the Fourier integral transform on the time scale $\mathbb{R}$, we have some fundamental results about the Fourier convolution inequalities such as the Young's inequality (we refer to the books [11] and [1]), the Saitoh's inequality in [8] and the reverse Saitoh inequality in [9].

For the Fourier cosine integral transform and the Laplace integral transform, a Young's type theorem, a Saitoh's type inequality and a reverse Saitoh-type inequality for the Fourier cosine-Laplace generalized convolution with a weight function γ were investigated in [12], where the weight function γ can be rewritten in the following form: $\gamma(u) = e^{-\upsilon u}$ (here, $\upsilon > 0$).

For the time scale $\mathbb{T}_h^0$, some inequalities were established in [15] for the h-Fourier cosine-Laplace discrete generalized convolution with a weight function (we will denote this weight function in here by $\widehat{\gamma}_1$), which include a Young's type inequality, a Saitoh's type inequality and a reverse Saitoh's type inequality, where the weight function is given by $\widehat{\gamma}_1(u) = (1+hu)^{-\mu}$, $u \in \left[0, \pi h^{-1}\right]$ (here, $\mu \in \mathbb{N}$). However, as far as we know, prior to this article, for $p > 1$, there is no research published on the ℓ_p-norm estimate for the h-Fourier cosine-Laplace discrete generalized convolution. The estimate (3.25) for the h-Fourier cosine-Laplace generalized convolution of two functions $x : \mathbb{T}_h^0 \to \mathbb{R}$ and $y : \mathbb{T}_h^0 \to \mathbb{R}$ in which $\sum\limits_{n=0}^{\infty} |x(nh)| < \infty$ and $\sum\limits_{n=0}^{\infty} |y(nh)| < \infty$ was given in [14, p. 26]. The ℓ_1-norm estimate (3.8) for the h-Fourier sine-Laplace discrete generalized convolution was given in [17, p. 448].

In this article, we derive some inequalities for the h-Fourier cosine-Laplace discrete generalized convolution on some function spaces. The article is structured as follows. In Section 2, we give some sufficient conditions for the existence of the h-Fourier cosine-Laplace discrete generalized convolution. In Section 3, we investigate a Young-type inequality for the h-Fourier cosine-Laplace discrete generalized convolution. In Section 4, we establish a Saitoh-type inequality and a reverse Saitoh-type inequality for this discrete generalized convolution. In the final section, we apply some of these inequalities to estimate the solutions of a class of equations of the h-Fourier cosine-Laplace discrete generalized convolution type.

2. Some sufficient conditions for the existence of the h-Fourier cosine-Laplace discrete generalized convolution

For $m, n \in \mathbb{N}_0$, we put ([14, p. 22])

$$I(n,m) := \int_0^{\pi} \frac{\cos(nu)}{(1+u)^{m+1}} du. \tag{2}$$

We define the function $\theta : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ as follows ([14, p. 22]):

$$\theta(k,n,m) := I(n+k,m) + I\big(|n-k|, m\big), \quad k,n,m \in \mathbb{N}_0. \tag{3}$$

DEFINITION 1. [14] The h-Fourier cosine-Laplace discrete generalized convolution of two functions $\widetilde{x},\widetilde{y}:\mathbb{T}_h^0\to\mathbb{R}$ in which $\sum_{n=0}^{\infty}\big(|\widetilde{x}(nh)|+|\widetilde{y}(nh)|\big)<\infty$ on the time scale $\mathbb{T}_h^0$ is defined as:

$$(\widetilde{x}*\widetilde{y})(kh):=\widetilde{x}(0)\big[\widetilde{G}_1\{\widetilde{y}\}(k)\big]+\frac{h}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\widetilde{x}(nh)\widetilde{y}(mh)\theta(k,n,m),\quad k\in\mathbb{N}_0, \tag{4}$$

where the function θ is given by (3), and the function $\widetilde{G}_1\{\widetilde{y}\}:\mathbb{N}_0\to\mathbb{R}$ is given by

$$\widetilde{G}_1\{\widetilde{y}\}(j)=\frac{h}{2\pi}\sum_{m=0}^{\infty}\widetilde{y}(mh)\theta(j,0,m),\quad j\in\mathbb{N}_0.$$

Assume that $\widehat{x},\widehat{y}:\mathbb{T}_h^0\to\mathbb{C}$ are two functions in which at least one of the following three statements is false: $\widehat{x}:\mathbb{T}_h^0\to\mathbb{R}$, $\widehat{y}:\mathbb{T}_h^0\to\mathbb{R}$, and $\sum_{n=0}^{\infty}\big(|\widehat{x}(nh)|+|\widehat{y}(nh)|\big)<\infty$. In a way similar to Definition 3.1 in [14], the h-Fourier cosine-Laplace discrete generalized convolution of $\widehat{x}$ and $\widehat{y}$ on the time scale $\mathbb{T}_h^0$ is defined as follows: For $k\in\mathbb{N}_0$,

$$(\widehat{x}*\widehat{y})(kh):=\frac{h}{2\pi}\widehat{x}(0)\Big(\sum_{m=0}^{\infty}\widehat{y}(mh)\theta(k,0,m)\Big)+\frac{h}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\widehat{x}(nh)\widehat{y}(mh)\theta(k,n,m), \tag{5}$$

where the function θ is given by (3), provided that the right hand side of (5) converges for all $k\in\mathbb{N}_0$.

For the time scale $\mathbb{T}_h^0$, the h-Laplace discrete convolution was defined in [2, p. 80], the h-Fourier cosine discrete convolution of two functions $x,y:\mathbb{T}_h^0\to\mathbb{R}$ in which $\sum_{n=0}^{\infty}\big(|x(nh)|+|y(nh)|\big)<\infty$ was studied in [13] and the h-Fourier cosine-Laplace discrete generalized convolution with the weight function $\widehat{\gamma}_1$ was introduced in [15], where the weight function $\widehat{\gamma}_1$ is given by $\widehat{\gamma}_1(u)=(1+hu)^{-\mu}$, $u\in\big[0,\pi h^{-1}\big]$ (here, $\mu\in\mathbb{N}$).

Suppose that p is a real number satisfying $1\leqslant p<\infty$. The following function spaces and norms are used in this article:

$$\ell_p(\mathbb{T}_h^0):=\Big\{x:\mathbb{T}_h^0\to\mathbb{C}\;\Big|\;\sum_{n=0}^{\infty}|x(nh)|^p<\infty\Big\},$$

$$\ell_\infty(\mathbb{T}_h^0):=\Big\{y:\mathbb{T}_h^0\to\mathbb{C}\;\Big|\;\sup_{n\in\mathbb{N}_0}|y(nh)|<\infty\Big\},$$

$$\|x\|_p:=h\Big(\sum_{n=0}^{\infty}|x(nh)|^p\Big)^{\frac{1}{p}},\quad x\in\ell_p(\mathbb{T}_h^0), \tag{6}$$

$$\|x\|_p^{(1)}:=h\Big(|x(0)|^p+2^p\sum_{n=1}^{\infty}|x(nh)|^p\Big)^{\frac{1}{p}},\quad x\in\ell_p(\mathbb{T}_h^0),\ \text{and} \tag{7}$$

$$\|y\|_\infty:=h\sup_{n\in\mathbb{N}_0}|y(nh)|,\quad y\in\ell_\infty(\mathbb{T}_h^0). \tag{8}$$

Assume that ϑ is a function from $\mathbb{T}_h^0$ to $(0,\infty)$, where $(0,\infty)=\{\upsilon\in\mathbb{R}\mid \upsilon>0\}$ is the set of all positive real numbers.

We set the following weighted function space and norms:

$$\ell_p(\mathbb{T}_h^0,\vartheta):=\Big\{x:\mathbb{T}_h^0\to\mathbb{C}\ \Big|\ \sum_{n=0}^{\infty}|x(nh)|^p\vartheta(nh)<\infty\Big\},$$

$$\|x\|_{\ell_p(\mathbb{T}_h^0,\vartheta)}:=h\Big(\sum_{n=0}^{\infty}|x(nh)|^p\vartheta(nh)\Big)^{\frac{1}{p}},\quad x\in\ell_p(\mathbb{T}_h^0,\vartheta),\ \text{ and} \tag{9}$$

$$\|x\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\vartheta)}:=h\left(\frac{|x(0)|^p\vartheta(0)}{2}+\sum_{n=1}^{\infty}|x(nh)|^p\vartheta(nh)\right)^{\frac{1}{p}},\quad x\in\ell_p(\mathbb{T}_h^0,\vartheta). \tag{10}$$

For each function x from $\mathbb{T}_h^0$ to $\mathbb{C}$, we determine a function $H_1\{x\}$ from $\mathbb{T}_h^0$ to $\mathbb{C}$ by ([15, p. 323])

$$\big(H_1\{x\}\big)(0):=\frac{x(0)}{2}\quad\text{and}\quad\big(H_1\{x\}\big)(nh):=x(nh)\ \text{ for }\ n\in\mathbb{N}. \tag{11}$$

From [15, p. 323], (6) and (7), if $x\in\ell_p(\mathbb{T}_h^0)$, then $H_1\{x\}\in\ell_p(\mathbb{T}_h^0)$,

$$\big\|H_1\{x\}\big\|_p=\frac{\|x\|_p^{(1)}}{2}\quad\text{and}\quad\sum_{n=0}^{\infty}\big|H_1\{x\}(nh)\big|^p=\left(\frac{\|x\|_p^{(1)}}{2h}\right)^p. \tag{12}$$

For two functions $\tau_1,\tau_2:\mathbb{T}_h^0\to\mathbb{C}$, the product function $\tau_1\tau_2:\mathbb{T}_h^0\to\mathbb{C}$ is taken to be the pointwise product function as $(\tau_1\tau_2)(nh)=\tau_1(nh)\tau_2(nh)$, $n\in\mathbb{N}_0$. For a function τ from $\mathbb{T}_h^0$ to $\mathbb{C}$, we write $\tau\equiv 0$ if and only if $\tau(nh)=0$ for all $n\in\mathbb{N}_0$. For a function $\kappa:\mathbb{T}_h^0\to\mathbb{C}$, we write $\kappa\not\equiv 0$ if and only if there exists $\nu\in\mathbb{N}_0$ such that $\kappa(\nu h)\neq 0$.

In this article, let α be a fixed real number satisfying $\alpha>0$ and let ω be a function from $\mathbb{T}_h^0$ to $(0,\infty)$ given by

$$\omega(nh)=(1+n)^{\alpha},\quad n\in\mathbb{N}_0. \tag{13}$$

We define two functions C_0 and C from $(1,\infty)$ to $(0,\infty)$ as follows:

$$C_0(q):=\ln(1+\pi)+\sum_{m=1}^{\infty}\frac{1}{m(m+1)^{\frac{\alpha}{q-1}}}\left(1-\frac{1}{(1+\pi)^m}\right),\quad q\in(1,\infty),\ \text{ and} \tag{14}$$

$$C(q):=\frac{1}{\pi}\big[C_0(q)\big]^{1-\frac{1}{q}}\big[\pi+\ln(1+\pi)\big]^{\frac{1}{q}},\quad q\in(1,\infty), \tag{15}$$

where $(1,\infty)=\{\upsilon\in\mathbb{R}\mid\upsilon>1\}$.

For $j\in\mathbb{N}$ and $m\in\mathbb{N}_0$, using (2), we recall the following identity ([14, p. 22])

$$I(j,m)=\frac{1}{m!}\int_0^{\infty}\frac{t^{m+1}e^{-t}}{j^2+t^2}\Big[1-(-1)^je^{-\pi t}\Big]dt. \tag{16}$$

From the results in [14, pp. 22, 26], for $j,m \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we have

$$0 < I(j,m) \leqslant I(0,0) = \ln(1+\pi), \quad I(j,k) \leqslant I(0,k) = \frac{1}{k}\left(1 - \frac{1}{(1+\pi)^k}\right), \tag{17}$$

and

$$\theta(j,m,0) \leqslant 2\ln(1+\pi). \tag{18}$$

By doing the same arguments or similar arguments to the proof of Theorem 3.1 in [14], we can prove the following lemma.

LEMMA 1. *Suppose that $x,y \in \ell_1(\mathbb{T}_h^0)$ and at least one of the following two statements is false: (a) $x : \mathbb{T}_h^0 \to \mathbb{R}$ and (b) $y : \mathbb{T}_h^0 \to \mathbb{R}$. Then the generalized convolution $x{*}y$ is well defined and belongs to the space $\ell_1(\mathbb{T}_h^0)$. Additionally,*

$$\|x{*}y\|_1^{(1)} \leqslant \left(2 + \frac{\ln(1+\pi)}{\pi}\right)\|x\|_1^{(1)}\|y\|_1^{(1)}.$$

Furthermore, we have the following factorization identity:

$$\mathscr{F}_c\{x{*}y\}(u) = \mathscr{F}_c\{x\}(u)\mathscr{L}\{y\}(u), \quad \forall u \in \left[0, \frac{\pi}{h}\right].$$

LEMMA 2. *If two functions x and y belong to the space $\ell_1(\mathbb{T}_h^0)$, then*

$$\|x{*}y\|_1 \leqslant \left(1 + \frac{\ln(1+\pi)}{\pi}\right)\|x\|_1^{(1)}\|y\|_1. \tag{19}$$

The equality in (19) is attained if and only if $x \equiv 0$ or $y \equiv 0$.

Proof. From [15, p. 325], [17, p. 448], (3) and (17), we have

$$\theta(k,n,m) > 0, \quad \forall k,n,m \in \mathbb{N}_0.$$

We set $x_1 := H_1\{x\}$, where $H_1\{x\}$ is given by (11).

Using (4), (5), (11) and $x_1 = H_1\{x\}$, we then get

$$\sum_{k=0}^{\infty} |(x{*}y)(kh)| \leqslant \frac{h}{\pi}\sum_{n=0}^{\infty}|x_1(nh)|\sum_{m=0}^{\infty}|y(mh)|\sum_{k=0}^{\infty}\theta(k,n,m). \tag{20}$$

For $m \in \mathbb{N}_0$, the following inequality holds ([14, p. 26])

$$\sum_{j=1}^{\infty} I(j,m) < \pi. \tag{21}$$

From [15, p. 325] and (3), we find that

$$\sum_{k=0}^{\infty}\theta(k,n,m) = I(n,m) + I(0,m) + 2\sum_{j=1}^{\infty} I(j,m), \quad \forall m,n \in \mathbb{N}_0. \tag{22}$$

According to [15, p. 325], (22), (21) and (17), for $n,m\in\mathbb{N}_0,$ we obtain

$$\sum_{k=0}^{\infty}\theta(k,n,m)<2\big[\pi+\ln(1+\pi)\big]. \tag{23}$$

By virtue of (20), (23), $x_1=H_1\{x\},$ (12) and (6), we see that

$$\begin{aligned}\sum_{k=0}^{\infty}|(x*y)(kh)|&\leqslant\frac{2h}{\pi}\big[\pi+\ln(1+\pi)\big]\sum_{n=0}^{\infty}|x_1(nh)|\sum_{m=0}^{\infty}|y(mh)|\\&=2h\left(1+\frac{\ln(1+\pi)}{\pi}\right)\frac{\|x\|_1^{(1)}}{2h}\frac{\|y\|_1}{h}.\end{aligned}\tag{24}$$

Multiplying both sides of (24) by h and using (6), we derive the inequality (19). The equality holds if and only if $x\equiv 0$ or $y\equiv 0$. This completes the proof. $\square$

THEOREM 1. *Let $p,q\in(0,\infty)$ be such that $p\geqslant 1, q>1$ and $p^{-1}+q^{-1}\geqslant 1$. Let ω be the weight function defined by (13). Then for $x\in\ell_p(\mathbb{T}_h^0)$ and $y\in\ell_q(\mathbb{T}_h^0,\omega),$ the generalized convolution $x*y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$. Moreover, we get the following estimate:*

$$\|x*y\|_\infty\leqslant C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)},$$

where the constant $C(q)$ is given by (15).

Proof. We put $r:=q(q-1)^{-1}$ and $\beta:=\alpha(q-1)^{-1}$. We have $r>1,\ \beta>0$ and $q^{-1}+r^{-1}=1\leqslant q^{-1}+p^{-1}$. Hence, $r\geqslant p$.

Since $x\in\ell_p(\mathbb{T}_h^0)$ and $r\geqslant p,$ it follows that $x\in\ell_r(\mathbb{T}_h^0)$ and $\|x\|_r^{(1)}\leqslant\|x\|_p^{(1)}$.

We denote $x_1:=H_1\{x\},$ where $H_1\{x\}$ is determined by (11). We define

$$A_k:=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_1(nh)|^r\theta(k,n,m)(m+1)^{-\beta},\quad k\in\mathbb{N}_0,\quad\text{and}\tag{25}$$

$$B_k:=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|y(mh)|^q\theta(k,n,m)(m+1)^{\beta(q-1)},\quad k\in\mathbb{N}_0.\tag{26}$$

Using (4), (5), (11), $x_1=H_1\{x\},$ (25), (26), the equality $r^{-1}+q^{-1}=1$ and Hölder's inequality, we deduce that

$$\begin{aligned}&|(x*y)(kh)|\leqslant\frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_1(nh)||y(mh)|\theta(k,n,m)\\&=\frac{h}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[|x_1(nh)|\theta(k,n,m)^{\frac1r}(m+1)^{\beta\left(\frac1q-1\right)}\right]\left[|y(mh)|\theta(k,n,m)^{\frac1q}(m+1)^{\frac{\beta(q-1)}{q}}\right]\\&\leqslant\frac{h}{\pi}\big(A_k\big)^{\frac1r}\big(B_k\big)^{\frac1q},\quad\forall k\in\mathbb{N}_0.\end{aligned}\tag{27}$$

It follows from (3) and (17) that

$$\theta(k,n,m) \leqslant \frac{2}{m}\left(1-\frac{1}{(1+\pi)^m}\right), \quad \forall k,n \in \mathbb{N}_0 \text{ and } \forall m \in \mathbb{N}. \tag{28}$$

Because $\beta = \dfrac{\alpha}{q-1}$, for $k,n \in \mathbb{N}_0$, combining (18), (28) and (14), we recognize that

$$\sum_{m=0}^{\infty} \frac{\theta(k,n,m)}{(m+1)^{\beta}} \leqslant 2\ln(1+\pi) + \sum_{m=1}^{\infty} \frac{2}{m(m+1)^{\beta}}\left(1-\frac{1}{(1+\pi)^m}\right) = 2C_0(q). \tag{29}$$

From (25), (29), $x_1 = H_1\{x\}$ and (12), we obtain

$$0 \leqslant A_k \leqslant 2C_0(q)\sum_{n=0}^{\infty} |x_1(nh)|^r = 2C_0(q)\left(\frac{\|x\|_r^{(1)}}{2h}\right)^r, \quad \forall k \in \mathbb{N}_0. \tag{30}$$

By virtue of [15, p. 329] and (3), we get

$$\theta(k,n,m) = \theta(n,k,m), \quad \forall k,n,m \in \mathbb{N}_0.$$

Therefore, according to [15, p. 329] and (23), we attain

$$\sum_{n=0}^{\infty} \theta(k,n,m) = \sum_{n=0}^{\infty} \theta(n,k,m) < 2\big[\pi + \ln(1+\pi)\big], \quad \forall k,m \in \mathbb{N}_0. \tag{31}$$

Since $\beta = \alpha(q-1)^{-1}$, from (26), (31), the definition of the function ω in (13) and the definition of the weighted norm (9), we have

$$\begin{aligned} 0 \leqslant \frac{B_k}{2\big[\pi+\ln(1+\pi)\big]} &\leqslant \sum_{m=0}^{\infty} |y(mh)|^q (m+1)^{\alpha} \\ &= \left(\frac{\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}}{h}\right)^q, \quad \forall k \in \mathbb{N}_0. \end{aligned} \tag{32}$$

For $k \in \mathbb{N}_0$, due to (27), (30), (32) and the equality $r^{-1}+q^{-1}=1$, we find that

$$\begin{aligned} |(x*y)(kh)| &\leqslant \frac{1}{2h\pi}\big[2C_0(q)\big]^{\frac{1}{r}}\Big\{2\big[\pi+\ln(1+\pi)\big]\Big\}^{\frac{1}{q}}\|x\|_r^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)} \\ &= \frac{1}{2h\pi}\big[2C_0(q)\big]^{1-\frac{1}{q}}\Big\{2\big[\pi+\ln(1+\pi)\big]\Big\}^{\frac{1}{q}}\|x\|_r^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)} < \infty. \end{aligned} \tag{33}$$

From (33), we then get that $x*y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$. Moreover, thanks to (8), (33), (15) and the inequality $\|x\|_r^{(1)} \leqslant \|x\|_p^{(1)}$, we observe that

$$\begin{aligned} \|x*y\|_\infty &\leqslant \frac{1}{2\pi}\big[2C_0(q)\big]^{1-\frac{1}{q}}\Big\{2\big[\pi+\ln(1+\pi)\big]\Big\}^{\frac{1}{q}}\|x\|_r^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)} \\ &= C(q)\|x\|_r^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)} \leqslant C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}, \end{aligned}$$

where the constant $C(q)$ is given by (15). The theorem is proved. $\square$

3. A Young-type inequality

THEOREM 2. (A Young-type theorem). *Let p, q and r be three positive real numbers such that*

$$p>1,\ q>1,\ r>1 \ \text{ and } \ p^{-1}+q^{-1}+r^{-1}=2. \tag{34}$$

Let ω be the weight function defined by the formula (13). If $x\in\ell_p(\mathbb{T}_h^0)$, $y\in\ell_q(\mathbb{T}_h^0,\omega)$ and $z\in\ell_r(\mathbb{T}_h^0)$, then $x{*}y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$. Furthermore,

$$\Big|\sum_{k=0}^{\infty}(x{*}y)(kh)z(kh)\Big| \leqslant \frac{C(q)}{h^2}\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}\|z\|_r, \tag{35}$$

where the constant $C(q)$ is given by (15). The equality holds if and only if $x\equiv 0$ or $y\equiv 0$ or $z\equiv 0$.

Proof. From $r>1$ and $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}=2$, we obtain $p^{-1}+q^{-1}>1$.

Since $x\in\ell_p(\mathbb{T}_h^0)$, $y\in\ell_q(\mathbb{T}_h^0,\omega)$, $p>1$, $q>1$ and $p^{-1}+q^{-1}>1$, applying Theorem 1, we get that $x{*}y$ is well defined and belongs to $\ell_\infty(\mathbb{T}_h^0)$.

If $x\equiv 0$ or $y\equiv 0$ or $z\equiv 0$, then we can easily see that (35) becomes an equality.

Assume that $x\not\equiv 0$, $y\not\equiv 0$ and $z\not\equiv 0$, we will prove (35) with strict inequality.

Let p_1, q_1 and r_1 respectively be the conjugate exponents of p, q and r, i.e.

$$\frac{1}{p}+\frac{1}{p_1}=1, \quad \frac{1}{q}+\frac{1}{q_1}=1 \ \text{ and } \ \frac{1}{r}+\frac{1}{r_1}=1. \tag{36}$$

From (34) and (36), we deduce that

$$\frac{p}{q_1}+\frac{p}{r_1}=1, \quad \frac{q}{p_1}+\frac{q}{r_1}=1, \quad \frac{r}{p_1}+\frac{r}{q_1}=1 \ \text{ and } \ \frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=1. \tag{37}$$

We set $\beta:=\dfrac{\alpha}{q-1}$ and $x_1:=H_1\{x\}$, where $H_1\{x\}$ is given by (11).

Using (4), (5), (11) and $x_1=H_1\{x\}$, we have

$$\begin{aligned}\Big|\sum_{k=0}^{\infty}(x{*}y)(kh)z(kh)\Big| &= \Big|\frac{h}{\pi}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x_1(nh)y(mh)\theta(k,n,m)z(kh)\Big| \\ &\leqslant \frac{h}{\pi}\Big|\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_1(nh)||y(mh)||z(kh)|\theta(k,n,m)\Big|.\end{aligned} \tag{38}$$

We define the following three functions U, V and W from $\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{N}_0$ to $[0,\infty)$:

$$U(k,n,m):=|y(mh)|^{\frac{q}{p_1}}(1+m)^{\frac{\beta(q-1)}{p_1}}|z(kh)|^{\frac{r}{p_1}}\left[\theta(k,n,m)\right]^{\frac{1}{p_1}}, \qquad k,n,m\in\mathbb{N}_0, \tag{39}$$

$$V(k,n,m):=|x_1(nh)|^{\frac{p}{q_1}}|z(kh)|^{\frac{r}{q_1}}\left(\frac{\theta(k,n,m)}{(1+m)^{\beta}}\right)^{\frac{1}{q_1}}, \qquad k,n,m\in\mathbb{N}_0, \tag{40}$$

and

$$W(k,n,m) := |x_1(nh)|^{\frac{p}{r_1}} |y(mh)|^{\frac{q}{r_1}} (1+m)^{\frac{\beta(q-1)}{r_1}} \big[\theta(k,n,m)\big]^{\frac{1}{r_1}}, \ \ k,n,m \in \mathbb{N}_0, \tag{41}$$

where $[0,\infty) = (0,\infty) \cup \{0\}$.

For $k,n,m \in \mathbb{N}_0$, by virtue of (39), (40), (41) and (37), it follows that

$$U(k,n,m)V(k,n,m)W(k,n,m) = |x_1(nh)||y(mh)||z(kh)|\theta(k,n,m). \tag{42}$$

Since $\beta = \alpha(q-1)^{-1}$, according to (39), $\sum\limits_{n=0}^{\infty} \theta(k,n,m) < 2\big[\pi + \ln(1+\pi)\big]$ for all $k,m \in \mathbb{N}_0$ in (31), $y \not\equiv 0$, $z \not\equiv 0$, (13), (9) and (6), we get

$$\begin{aligned} S_1 &:= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \big|U(k,n,m)\big|^{p_1} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |y(mh)|^q (1+m)^{\beta(q-1)} |z(kh)|^r \theta(k,n,m) \\ &< 2\big[\pi+\ln(1+\pi)\big] \sum_{m=0}^{\infty} |y(mh)|^q (1+m)^{\alpha} \sum_{k=0}^{\infty} |z(kh)|^r \\ &= 2\big[\pi+\ln(1+\pi)\big] \left(\frac{\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}}{h}\right)^q \left(\frac{\|z\|_r}{h}\right)^r \quad \text{and} \quad S_1 > 0. \end{aligned} \tag{43}$$

Combining (40), (29), $x_1 = H_1\{x\}$, (12), (6), $x \not\equiv 0$ and $z \not\equiv 0$, we thus deduce that

$$\begin{aligned} S_2 &:= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \big|V(k,n,m)\big|^{q_1} = \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{k=0}^{\infty} |z(kh)|^r \sum_{m=0}^{\infty} \frac{\theta(k,n,m)}{(1+m)^{\beta}} \\ &\leqslant 2C_0(q) \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{k=0}^{\infty} |z(kh)|^r = 2C_0(q) \left(\frac{\|x\|_p^{(1)}}{2h}\right)^p \left(\frac{\|z\|_r}{h}\right)^r \quad \text{and} \quad S_2 > 0. \end{aligned} \tag{44}$$

Because $\beta = \alpha(q-1)^{-1}$, from (41), (23), $x_1 = H_1\{x\}$, $x \not\equiv 0$, $y \not\equiv 0$, (12), (13) and (9), we arrive at

$$\begin{aligned} S_3 &:= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \big|W(k,n,m)\big|^{r_1} = \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{m=0}^{\infty} |y(mh)|^q (1+m)^{\beta(q-1)} \sum_{k=0}^{\infty} \theta(k,n,m) \\ &< 2\big[\pi+\ln(1+\pi)\big] \sum_{n=0}^{\infty} |x_1(nh)|^p \sum_{m=0}^{\infty} |y(mh)|^q (1+m)^{\alpha} \\ &= 2\big[\pi+\ln(1+\pi)\big] \left(\frac{\|x\|_p^{(1)}}{2h}\right)^p \left(\frac{\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}}{h}\right)^q \quad \text{and} \quad S_3 > 0. \end{aligned} \tag{45}$$

Using (38), (42), (37), (43), (44), (45) and Hölder's inequality, we have

$$\begin{aligned} \Big|\sum_{k=0}^{\infty} (x*y)(kh)z(kh)\Big| &\leqslant \frac{h}{\pi}\Big|\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} U(k,n,m)V(k,n,m)W(k,n,m)\Big| \\ &\leqslant \frac{h}{\pi}(S_1)^{\frac{1}{p_1}}(S_2)^{\frac{1}{q_1}}(S_3)^{\frac{1}{r_1}}. \end{aligned} \tag{46}$$

Due to (46), (43), (44), (45), (37) and (36), we obtain the inequality

$$\Big|\sum_{k=0}^{\infty}(x*y)(kh)z(kh)\Big| < \frac{C(q)}{h^2}\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}\|z\|_r,$$

where the constant $C(q)$ is given by (15). So we complete the proof. $\square$

COROLLARY 1. (A Young-type inequality). *Let p,q and r be three positive real numbers satisfying $p>1,\ q>1,\ r>1$ and $p^{-1}+q^{-1}=1+r^{-1}$. Let ω be the weight function defined by (13). If $x\in\ell_p(\mathbb{T}_h^0)$ and $y\in\ell_q(\mathbb{T}_h^0,\omega)$, then $x*y\in\ell_r(\mathbb{T}_h^0)$ and*

$$\|x*y\|_r \leqslant C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}, \tag{47}$$

where the constant $C(q)$ is given by (15). The equality in (47) is attained if and only if $x\equiv 0$ or $y\equiv 0$.

Proof. According to $x\in\ell_p(\mathbb{T}_h^0),\ y\in\ell_q(\mathbb{T}_h^0,\omega),\ p>1,\ q>1$ and $p^{-1}+q^{-1}=1+r^{-1}>1$, applying Theorem 1, we deduce that $x*y$ is well defined and belongs to the space $\ell_\infty(\mathbb{T}_h^0)$.

If $x\equiv 0$ or $y\equiv 0$, then we can easily see that (47) becomes an equality.

Suppose that $x\not\equiv 0$ and $y\not\equiv 0$, we will prove (47) with strict inequality.

Let p_1, q_1 and r_1 be the conjugate exponents of p, q and r, respectively, i.e.

$$\frac{1}{p}+\frac{1}{p_1}=1,\qquad \frac{1}{q}+\frac{1}{q_1}=1 \ \text{ and } \ \frac{1}{r}+\frac{1}{r_1}=1.$$

From $p^{-1}+q^{-1}=1+r^{-1}$ and $r^{-1}+{r_1}^{-1}=1$, it follows that $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r_1}=2.$

We set $A_k:=\big[(x*y)(kh)\big]^r,\ k\in\mathbb{N}_0$. Let N be an arbitrary positive integer.

For $v\in\mathbb{C}$, we denote its complex conjugate by $\overline{v}$. We define a function $z:\mathbb{T}_h^0\to\mathbb{C}$ by

$$z(kh):=\begin{cases} h^r\big[(x*y)(kh)\big]^{r-1}\dfrac{\overline{A_k}}{|A_k|}, & \text{if } 0\leqslant k\leqslant N \text{ and } (x*y)(kh)\neq 0,\\ 0, & \text{if } 0\leqslant k\leqslant N \text{ and } (x*y)(kh)=0,\\ 0, & \text{if } k>N,\end{cases}$$

for $k\in\mathbb{N}_0$. It is straightforward that $z\in\ell_{r_1}(\mathbb{T}_h^0)$. Using the inequality (35), where we use r_1 instead of r, we then get

$$S_N:=h^r\sum_{k=0}^{N}|(x*y)(kh)|^r\leqslant\frac{C(q)}{h^2}\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)}\|z\|_{r_1}. \tag{48}$$

Because $\dfrac{1}{r}+\dfrac{1}{r_1}=1$, we have $rr_1=r+r_1$. Hence, it is easy to see that $S_N\geqslant 0$ and

$$\left(\frac{\|z\|_{r_1}}{h}\right)^{r_1}=h^{rr_1}\sum_{k=0}^{N}|(x*y)(kh)|^{r_1(r-1)}=h^{r_1}h^r\sum_{k=0}^{N}|(x*y)(kh)|^r=h^{r_1}S_N.$$

This shows that $\|z\|_{r_1} = h^2(S_N)^{\frac{1}{r_1}}$. Plugging this equality into (48) and using the equality $\dfrac{1}{r} + \dfrac{1}{r_1} = 1$, we deduce that

$$(S_N)^{\frac{1}{r}} \leqslant C(q)\|x\|_p^{(1)}\|y\|_{\ell_q(\mathbb{T}_h^0,\omega)} < \infty. \tag{49}$$

Since N is an arbitrary positive integer, letting $N \to \infty$ in (49), we attain that the function $x{*}y$ belongs to the space $\ell_r(\mathbb{T}_h^0)$ and the inequality (47) holds.

Assume that there exists $x_0 \in \ell_p(\mathbb{T}_h^0)$ and $y_0 \in \ell_q(\mathbb{T}_h^0, \omega)$ such that

$$x_0 \not\equiv 0, \;\; y_0 \not\equiv 0 \;\; \text{and} \;\; \|x_0{*}y_0\|_r = C(q)\|x_0\|_p^{(1)}\|y_0\|_{\ell_q(\mathbb{T}_h^0,\omega)}. \tag{50}$$

We put $B_k := \big[(x_0{*}y_0)(kh)\big]^r$, $k \in \mathbb{N}_0$. We define a function $z_0 : \mathbb{T}_h^0 \to \mathbb{C}$ by

$$z_0(kh) := \begin{cases} h^r\big[(x_0{*}y_0)(kh)\big]^{r-1}\dfrac{\overline{B_k}}{|B_k|}, & \text{if } (x_0{*}y_0)(kh) \neq 0, \\ 0, & \text{if } (x_0{*}y_0)(kh) = 0, \end{cases}$$

for $k \in \mathbb{N}_0$. By virtue of $\|x_0{*}y_0\|_r = C(q)\|x_0\|_p^{(1)}\|y_0\|_{\ell_q(\mathbb{T}_h^0,\omega)} \neq 0$, we obtain $z_0 \not\equiv 0$. Using some similar arguments or the same arguments as in the first part of this proof with x_0, y_0 and z_0 instead of x, y and z, respectively, we get

$$\|x_0{*}y_0\|_r < C(q)\|x_0\|_p^{(1)}\|y_0\|_{\ell_q(\mathbb{T}_h^0,\omega)},$$

which is a contradiction with (50). The proof is completed. $\square$

4. A Saitoh-type inequality and a reverse Saitoh-type inequality

In this section, let p and q be two positive real numbers satisfying

$$p > 1, \;\; q > 1 \;\; \text{and} \;\; p^{-1} + q^{-1} = 1.$$

We suppose that $\rho_1, \rho_2 : \mathbb{T}_h^0 \to \mathbb{R}$ are two given functions in $\ell_1(\mathbb{T}_h^0)$ such that

$$\rho_1(nh) > 0 \;\; \text{and} \;\; \rho_2(nh) > 0, \quad \forall n \in \mathbb{N}_0.$$

THEOREM 3. (A Saitoh-type inequality). *Let F_1 be a function in the space $\ell_p(\mathbb{T}_h^0, \rho_1)$ and let F_2 be a function in the space $\ell_p(\mathbb{T}_h^0, \rho_2)$. Then we have*

$$F_1\rho_1, F_2\rho_2 \in \ell_1(\mathbb{T}_h^0) \;\; \text{and} \;\; \big((F_1\rho_1){*}(F_2\rho_2)\big)(\rho_1{*}\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0).$$

Moreover, the following inequality for the h-Fourier cosine-Laplace discrete generalized convolution holds

$$\Big\|\big((F_1\rho_1){*}(F_2\rho_2)\big)(\rho_1{*}\rho_2)^{\frac{1}{p}-1}\Big\|_p \leqslant C\|F_1\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho_1)}\|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)}, \tag{51}$$

where $C = h^{\frac{1-p}{p}}\left(2 + \dfrac{2\ln(1+\pi)}{\pi}\right)^{\frac{1}{p}}$. The inequality in (51) becomes an equality if and only if $F_1 \equiv 0$ or $F_2 \equiv 0$.

Proof. For $j=1,2,$ since $\rho_j:\mathbb{T}_h^0\to(0,\infty)$, $F_j\in\ell_p(\mathbb{T}_h^0,\rho_j)$ and $\rho_j\in\ell_1(\mathbb{T}_h^0)$, we derive that $F_j\big(\rho_j\big)^{\frac{1}{p}}\in\ell_p(\mathbb{T}_h^0)$ and $\big(\rho_j\big)^{\frac{1}{q}}\in\ell_q(\mathbb{T}_h^0)$. Hence, using $p^{-1}+q^{-1}=1,\ p>1$ and $q>1,$ we observe that

$$F_j\rho_j=F_j\big(\rho_j\big)^{\frac{1}{p}}\big(\rho_j\big)^{\frac{1}{q}}\in\ell_1(\mathbb{T}_h^0)\ \text{ for }\ j=1,2.$$

Because ρ_1, ρ_2, $F_1\rho_1$ and $F_2\rho_2$ belong to the space $\ell_1(\mathbb{T}_h^0)$, applying Theorem 3.1 in [14, p. 26] and Lemma 1, we deduce that $(F_1\rho_1)*(F_2\rho_2)$ and $\rho_1*\rho_2$ are well defined.

If $F_1\equiv 0$ or $F_2\equiv 0,$ then we can easily see that (51) becomes an equality.

Assume that $F_1\not\equiv 0$ and $F_2\not\equiv 0,$ we will prove (51) with strict inequality.

We denote $\widehat{\rho}_1:=H_1\{\rho_1\},$ where the function $H_1\{\rho_1\}:\mathbb{T}_h^0\to\mathbb{C}$ is given by ([15, p. 323])

$$\big(H_1\{\rho_1\}\big)(0)=\frac{\rho_1(0)}{2}\quad\text{ and }\quad\big(H_1\{\rho_1\}\big)(nh)=\rho_1(nh)\ \text{ for }\ n\in\mathbb{N}.$$

Let N be an arbitrary positive integer. We define

$$S_1(N):=h^p\sum_{k=0}^{N}\big|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\big|^p\big|(\rho_1*\rho_2)(kh)\big|^{1-p}\quad\text{and}\tag{52}$$

$$S_2(k):=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m)=\frac{\pi}{h}(\rho_1*\rho_2)(kh),\quad k\in\mathbb{N}_0.\tag{53}$$

By virtue of (52) and (53), we find that

$$S_1(N)=\frac{h}{\pi^{1-p}}\sum_{k=0}^{N}\big|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\big|^p[S_2(k)]^{1-p}.\tag{54}$$

Using (4), (5), $\widehat{\rho}_1=H_1\{\rho_1\}$, the equality $p^{-1}+q^{-1}=1,$ (53) and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{\pi}{h}\big|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\big|\leqslant\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|F_1(nh)|\widehat{\rho}_1(nh)|F_2(mh)|\rho_2(mh)\theta(k,n,m)\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|F_1(nh)||F_2(mh)|\big[\widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m)\big]^{\frac{1}{p}}\big[\widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m)\big]^{\frac{1}{q}}\\
&\leqslant\Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|F_1(nh)|^p\widehat{\rho}_1(nh)|F_2(mh)|^p\rho_2(mh)\theta(k,n,m)\Big]^{\frac{1}{p}}\big[S_2(k)\big]^{\frac{1}{q}},\quad\forall k\in\mathbb{N}_0.
\end{aligned}\tag{55}$$

Due to (55), for $k\in\mathbb{N}_0,$ we attain

$$\begin{aligned}
&\big|(F_1\rho_1)*(F_2\rho_2)(kh)\big|^p\\
&\leqslant\Big(\frac{h}{\pi}\Big)^p\big[S_2(k)\big]^{\frac{p}{q}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|F_1(nh)|^p\widehat{\rho}_1(nh)|F_2(mh)|^p\rho_2(mh)\theta(k,n,m).
\end{aligned}\tag{56}$$

According to (54), (56) and the equality $\frac{p}{q}+1-p=0$, we have the following estimate

$$S_1(N) \leqslant \frac{h}{\pi^{1-p}}\left(\frac{h}{\pi}\right)^p \sum_{k=0}^{N}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |F_1(nh)|^p \widehat{\rho}_1(nh)|F_2(mh)|^p \rho_2(mh)\theta(k,n,m). \quad (57)$$

From (57), taking the limit of (57) as $N \to \infty$ and using (23), $\widehat{\rho}_1 = H_1\{\rho_1\}$, (10) and (9), it follows that

$$\begin{aligned}
\lim_{N\to\infty} S_1(N) &\leqslant \frac{h}{\pi^{1-p}}\left(\frac{h}{\pi}\right)^p \sum_{n=0}^{\infty} |F_1(nh)|^p \widehat{\rho}_1(nh)| \sum_{m=0}^{\infty} F_2(mh)|^p \rho_2(mh) \sum_{k=0}^{\infty} \theta(k,n,m) \\
&< \frac{2h^{1+p}\left[\pi+\ln(1+\pi)\right]}{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |F_1(nh)|^p \widehat{\rho}_1(nh)|F_2(mh)|^p \rho_2(mh) \\
&= \frac{2h^{1+p}\left[\pi+\ln(1+\pi)\right]}{\pi} \left(\frac{\|F_1\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho_1)}}{h}\right)^p \left(\frac{\|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)}}{h}\right)^p < \infty. \quad (58)
\end{aligned}$$

Combining (58), (52) and (6), we deduce that $\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0)$ and the following inequality holds

$$\left\| \big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \right\|_p < C\|F_1\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho_1)} \|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)},$$

where $C = h^{\frac{1-p}{p}}\left(2+\frac{2\ln(1+\pi)}{\pi}\right)^{\frac{1}{p}}$. The theorem is proved. $\square$

For the remaining part of this section, we will investigate a reverse Saitoh-type inequality for the h-Fourier cosine-Laplace discrete generalized convolution.

DEFINITION 2. [5, 10, 16] The Specht's ratio is determined by

$$S(t) = \frac{t^{\frac{1}{t-1}}}{e\ln\left(t^{\frac{1}{t-1}}\right)}, \qquad t\in\mathbb{R},\ \ t>0 \ \text{ and } \ t\neq 1. \quad (59)$$

Here, the function $\ln : (0,\infty) \to \mathbb{R}$ is the natural logarithm function and the number e is the constant $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

The value of the Specht's ratio $S(.)$ at the point 1 is ([16])

$$S(1) = 1. \quad (60)$$

LEMMA 3. [16] *The function $S(t)$ is strictly decreasing for $0<t<1$ and strictly increasing for $t>1$. Furthermore, the following equations hold*

$$\lim_{u\to 1} \ln\left[S(u)\right] = 0 \quad \textit{and} \quad S(t) = S\left(\frac{1}{t}\right) \ \textit{for all } t>0.$$

The following corollary is a consequence of [4, Corollary 3.17] and Lemma 3.

COROLLARY 2. *Let a and b be two functions from $\mathbb{N}_0 \times \mathbb{N}_0$ to $(0,\infty)$ satisfying*

$$B = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b(n,m) < \infty \quad \text{and} \quad 0 < M_1 \leqslant \frac{a(n,m)}{b(n,m)} \leqslant M_2 < \infty, \quad \forall n,m \in \mathbb{N}_0.$$

Then $A = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a(n,m) < \infty$ and $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left[a(n,m)\right]^{\frac{1}{p}} \left[b(n,m)\right]^{\frac{1}{q}} < \infty$. In addition,

$$S\left(\frac{M_1}{M_2}\right) \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left[a(n,m)\right]^{\frac{1}{p}} \left[b(n,m)\right]^{\frac{1}{q}} \geqslant A^{\frac{1}{p}} B^{\frac{1}{q}}.$$

The following inequality was given in [15, p. 332] without proof, we will give a proof of this inequality.

LEMMA 4. [15] *For $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, it holds an inequality*

$$\sum_{k=0}^{\infty} \theta(k,n,m) > \frac{\pi}{2} - \frac{1}{m(1+\pi)^m}. \tag{61}$$

Proof. We call A the left hand side of (61). Using the identity (16), it is easily proven that

$$\sum_{j=0}^{\infty} I(2j+1,m) > \frac{1}{m!} \int_0^{\infty} t^{m+1} e^{-t} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2 + t^2} dt. \tag{62}$$

For $t > 0$, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2+t^2} &> \frac{1}{2}\sum_{j=1}^{\infty} \frac{1}{j^2+t^2} > \frac{1}{2}\sum_{j=1}^{\infty} \int_j^{j+1} \frac{dx}{x^2+t^2} = \frac{1}{2}\int_1^{\infty} \frac{dx}{x^2+t^2} \\ &= \frac{1}{2}\left(\frac{1}{t}\arctan\frac{x}{t}\right)\Big|_1^{\infty} = \frac{1}{2t}\left(\frac{\pi}{2} - \arctan\frac{1}{t}\right) > \frac{1}{2t}\left(\frac{\pi}{2} - \frac{1}{t}\right). \end{aligned} \tag{63}$$

From (62) and (63), using the gamma function, we deduce that

$$\sum_{j=0}^{\infty} I(2j+1,m) > \frac{\pi}{4}\frac{1}{m!}\int_0^{\infty} t^m e^{-t} dt - \frac{1}{2}\frac{1}{m!}\int_0^{\infty} t^{m-1} e^{-t} dt = \frac{\pi}{4} - \frac{1}{2m}. \tag{64}$$

Due to (22), (64), $I(0,m) = \frac{1}{m}\left(1 - \frac{1}{(1+\pi)^m}\right)$ and (17), it follows that

$$A > \frac{1}{m}\left(1 - \frac{1}{(1+\pi)^m}\right) + 2\left(\frac{\pi}{4} - \frac{1}{2m}\right) = \frac{\pi}{2} - \frac{1}{m(1+\pi)^m}.$$

The proof is completed. $\square$

For $n \in \mathbb{N}_0$, it is easy to see that

$$\sum_{k=0}^{\infty} \theta(k,n,0) > \theta(n,n,0) > I(0,0) = \ln(1+\pi) > \frac{\pi}{2} - \frac{1}{1+\pi}. \tag{65}$$

Combining (4.14) in [15, p. 332] and (65), we arrive at

$$\sum_{k=0}^{\infty} \theta(k,n,m) > \frac{\pi}{2} - \frac{1}{1+\pi}, \quad \forall m,n \in \mathbb{N}_0. \tag{66}$$

THEOREM 4. (A reverse Saitoh-type inequality). *If F_1 and F_2 are two functions from $\mathbb{T}_h^0$ to $(0,\infty)$ satisfying*

$$0 < M_1^{\frac{1}{p}} \leqslant F_1(nh) \leqslant M_2^{\frac{1}{p}} < \infty \quad \textit{and} \quad 0 < M_3^{\frac{1}{p}} \leqslant F_2(nh) \leqslant M_4^{\frac{1}{p}} < \infty, \quad \forall n \in \mathbb{N}_0, \tag{67}$$

then for $j = 1, 2$, we have $F_j\rho_j \in \ell_1(\mathbb{T}_h^0)$ and $F_j \in \ell_p(\mathbb{T}_h^0, \rho_j)$. Additionally,

$$\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0).$$

Moreover, the following inequality holds

$$\left\| \big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \right\|_p > C\|F_1\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho_1)} \|F_2\|_{\ell_p(\mathbb{T}_h^0,\rho_2)}, \tag{68}$$

where

$$C = \left(\frac{\pi^2+\pi-2}{2\pi(1+\pi)}\right)^{\frac{1}{p}} \left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-1} h^{\frac{1-p}{p}}.$$

Here, the Specht's ratio $S(.)$ is determined by (59) and (60).

Proof. From (67) and $\rho_1, \rho_2 \in \ell_1(\mathbb{T}_h^0)$, for $j = 1, 2$, we have

$$F_j\rho_j \in \ell_1(\mathbb{T}_h^0) \quad \text{and} \quad F_j \in \ell_p(\mathbb{T}_h^0, \rho_j).$$

Since $F_1 \in \ell_p(\mathbb{T}_h^0, \rho_1)$ and $F_2 \in \ell_p(\mathbb{T}_h^0, \rho_2)$, applying Theorem 3, we get

$$\big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \in \ell_p(\mathbb{T}_h^0).$$

We set $\widehat{\rho}_1 := H_1\{\rho_1\}$, where $H_1\{\rho_1\} : \mathbb{T}_h^0 \to \mathbb{C}$ is defined by ([15, p. 323])

$$\big(H_1\{\rho_1\}\big)(0) = \frac{\rho_1(0)}{2} \quad \text{and} \quad \big(H_1\{\rho_1\}\big)(nh) = \rho_1(nh) \ \text{ for } \ n \in \mathbb{N}.$$

We denote

$$S_1 := \left\| \big((F_1\rho_1)*(F_2\rho_2)\big)(\rho_1*\rho_2)^{\frac{1}{p}-1} \right\|_p \quad \text{and} \tag{69}$$

$$S_2(k) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m) = \frac{\pi}{h}(\rho_1*\rho_2)(kh), \quad k \in \mathbb{N}_0. \tag{70}$$

Combining (69), (6) and (70), we obtain

$$S_1^p = \frac{h}{\pi^{1-p}} \sum_{k=0}^{\infty} \left|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\right|^p [S_2(k)]^{1-p}. \tag{71}$$

For each $k \in \mathbb{N}_0,$ we define two functions a_k and b_k from $\mathbb{N}_0 \times \mathbb{N}_0$ to $(0,\infty)$ as follows:

$$a_k(n,m) := \big[F_1(nh)F_2(mh)\big]^p \widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m), \quad n,m \in \mathbb{N}_0, \ \text{ and} \tag{72}$$

$$b_k(n,m) := \widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m), \quad n,m \in \mathbb{N}_0. \tag{73}$$

From (67), (72) and (73), for $n,m,k \in \mathbb{N}_0,$ we deduce that

$$0 < M_1M_3 \leqslant \frac{a_k(n,m)}{b_k(n,m)} \leqslant M_2M_4 < \infty. \tag{74}$$

According to (4), (5), $\widehat{\rho}_1 = H_1\{\rho_1\}$, (72), (73) and the equality $p^{-1}+q^{-1}=1,$ we find that

$$\frac{\pi}{h}\left|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\right| = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \big[a_k(n,m)\big]^{\frac{1}{p}} \big[b_k(n,m)\big]^{\frac{1}{q}}, \quad \forall k \in \mathbb{N}_0. \tag{75}$$

For each $k \in \mathbb{N}_0,$ due to (75), (73) and (74), applying Corollary 2 for the two functions a_k and b_k from $\mathbb{N}_0 \times \mathbb{N}_0$ to $(0,\infty),$ we attain

$$\frac{\pi}{h}\left|\big((F_1\rho_1)*(F_2\rho_2)\big)(kh)\right| \geqslant \left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-1} \big[A(k)\big]^{\frac{1}{p}} \big[B(k)\big]^{\frac{1}{q}}, \tag{76}$$

where

$$A(k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_k(n,m) \quad \text{and} \tag{77}$$

$$B(k) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_k(n,m) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widehat{\rho}_1(nh)\rho_2(mh)\theta(k,n,m) = S_2(k). \tag{78}$$

Using (71), (76) and (78), it follows that

$$S_1^p \geqslant \frac{h}{\pi^{1-p}} \sum_{k=0}^{\infty} \left(\frac{h}{\pi}\right)^p \left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-p} A(k)[S_2(k)]^{\frac{p}{q}+1-p}. \tag{79}$$

Since $p^{-1}+q^{-1}=1,$ we have $\dfrac{p}{q}+1-p=0.$

By virtue of (79), (77), (72), $\dfrac{p}{q}+1-p=0$ and (66), we observe that

$$\begin{aligned} S_1^p &\geqslant \frac{h^{1+p}}{\pi}\left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-p} \sum_{n=0}^{\infty} \big[F_1(nh)\big]^p \widehat{\rho}_1(nh) \sum_{m=0}^{\infty} \big[F_2(mh)\big]^p \rho_2(mh) \sum_{k=0}^{\infty} \theta(k,n,m) \\ &> \frac{h^{1+p}}{\pi}\left(\frac{\pi}{2} - \frac{1}{1+\pi}\right)\left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-p} \sum_{n=0}^{\infty} \big[F_1(nh)\big]^p \widehat{\rho}_1(nh) \sum_{m=0}^{\infty} \big[F_2(mh)\big]^p \rho_2(mh) \\ &= \left(\frac{\pi^2+\pi-2}{2\pi(1+\pi)}\right) h^{1+p}\left[S\left(\frac{M_1M_3}{M_2M_4}\right)\right]^{-p} \sum_{n=0}^{\infty} \big[F_1(nh)\big]^p \widehat{\rho}_1(nh) \sum_{m=0}^{\infty} \big[F_2(mh)\big]^p \rho_2(mh). \end{aligned} \tag{80}$$

From (80), (69), $\widehat{\rho}_1 = H_1\{\rho_1\}$, (10) and (9), we obtain the inequality (68).

The proof of Theorem 4 is completed. $\square$

5. Applications

Let M be a given number in $\mathbb{N}$ and let p, λ_k $(k = 0, 1, \ldots, \text{M})$ be given complex numbers satisfying $p \in \mathbb{R}$, $p > 1$ and $\sum_{k=1}^{\text{M}} |\lambda_k|^2 > 0$. Assume that ρ and ψ are two given functions in $\ell_1(\mathbb{T}_h^0)$ such that $\rho : \mathbb{T}_h^0 \to \mathbb{R}$ and $\rho(nh) > 0$, $\forall n \in \mathbb{N}_0$.

We define a function $f : \left[0, \frac{\pi}{h}\right] \to \mathbb{C}$ by

$$f(u) := \mathscr{F}_c\{\psi\}(u), \quad u \in \left[0, \frac{\pi}{h}\right]. \tag{81}$$

DEFINITION 3. [13] The h-Fourier cosine convolution on the time scale $\mathbb{T}_h^0$ of two functions $\widetilde{x}, \widetilde{z} \in \ell_1(\mathbb{T}_h^0)$ in which $\widetilde{x}, \widetilde{z} : \mathbb{T}_h^0 \to \mathbb{R}$ is defined as

$$(\widetilde{x} \underset{1}{*} \widetilde{z})(t) := h\Big\{\widetilde{G}_2(\widetilde{x}, \widetilde{z}, t) + \widetilde{x}(0)\widetilde{z}(t)\Big\}, \quad t \in \mathbb{T}_h^0,$$

where

$$\widetilde{G}_2(\widetilde{x}, \widetilde{z}, t) = \sum_{n=1}^{\infty} \widetilde{x}(nh)\left[\widetilde{z}(|t - nh|) + \widetilde{z}(t + nh)\right], \quad t \in \mathbb{T}_h^0.$$

Suppose that $\widehat{x}$ and $\widehat{z}$ are two functions in $\ell_1(\mathbb{T}_h^0)$ such that at least one of the following two conditions is false: (a) $\widehat{x} : \mathbb{T}_h^0 \to \mathbb{R}$ and (b) $\widehat{z} : \mathbb{T}_h^0 \to \mathbb{R}$. In a manner analogous to Definition 1 in [13], the h-Fourier cosine convolution on the time scale $\mathbb{T}_h^0$ of $\widehat{x}$ and $\widehat{z}$ is given by

$$(\widehat{x} \underset{1}{*} \widehat{z})(t) := h\left\{\left(\sum_{n=1}^{\infty} \widehat{x}(nh)\left[\widehat{z}(|t - nh|) + \widehat{z}(t + nh)\right]\right) + \widehat{x}(0)\widehat{z}(t)\right\}, \quad t \in \mathbb{T}_h^0.$$

By performing the same arguments or some analogous arguments as in the proof of Theorem 4 in [13], we have $\widehat{x} \underset{1}{*} \widehat{z} \in \ell_1(\mathbb{T}_h^0)$ and

$$\mathscr{F}_c\{\widehat{x} \underset{1}{*} \widehat{z}\}(u) = \mathscr{F}_c\{\widehat{x}\}(u)\mathscr{F}_c\{\widehat{z}\}(u), \quad \forall u \in \left[0, \frac{\pi}{h}\right]. \tag{82}$$

We define an operator $\mathbf{P} : \ell_1(\mathbb{T}_h^0) \to \ell_1(\mathbb{T}_h^0)$ by $\mathbf{P}(z) := z \underset{1}{*} \psi$ for $z \in \ell_1(\mathbb{T}_h^0)$.

We consider the following equation

$$\lambda_0 x + \left(\sum_{k=1}^{\text{M}} \lambda_k \mathbf{P}^k\right)(x) = y\rho, \tag{83}$$

where y is a given function in $\ell_p(\mathbb{T}_h^0,\rho)$ and $x\in\ell_1(\mathbb{T}_h^0)$ is an unknown function. Here, for $z\in\ell_1(\mathbb{T}_h^0)$, $\mathbf{P}^1(z):=\mathbf{P}(z)$ and

$$\mathbf{P}^j(z):=\mathbf{P}^{j-1}\big(\mathbf{P}(z)\big),\quad j\in\mathbb{N}\ \text{and}\ j\geqslant 2.$$

For $u\in\left[0,\frac{\pi}{h}\right]$ and $z\in\ell_1(\mathbb{T}_h^0)$, from [13, p. 210], (82) and (81), we get

$$\mathscr{F}_c\big\{\mathbf{P}(z)\big\}(u)=\mathscr{F}_c\big\{z\underset{1}{*}\psi\big\}(u)=\mathscr{F}_c\{z\}(u)\mathscr{F}_c\{\psi\}(u)=f(u)\mathscr{F}_c\{z\}(u)\quad\text{and}$$

$$\mathscr{F}_c\big\{\mathbf{P}^j(z)\big\}(u)=\big[f(u)\big]^j\mathscr{F}_c\{z\}(u),\quad\forall j\in\mathbb{N}.\tag{84}$$

THEOREM 5. *Let $\eta:\mathbb{T}_h^0\to\mathbb{R}$ be a given function in $\ell_1(\mathbb{T}_h^0)$ such that $\eta(nh)>0$, $\forall n\in\mathbb{N}_0$. Assume that there exists a function Q in the space $\ell_p(\mathbb{T}_h^0,\eta)$ satisfying*

$$\mathscr{L}\{Q\eta\}(u)=\frac{1}{\lambda_0+\sum\limits_{k=1}^{\mathrm{M}}\lambda_k\big[f(u)\big]^k},\quad\forall u\in\left[0,\frac{\pi}{h}\right].\tag{85}$$

Then the equation (83) has a unique solution in $\ell_1(\mathbb{T}_h^0)$ and the solution is given by $x=(y\rho)*(Q\eta)$. Furthermore, the following estimate holds

$$\left\|x(\rho*\eta)^{\frac{1}{p}-1}\right\|_p\leqslant h^{\frac{1-p}{p}}\left(2+\frac{2\ln(1+\pi)}{\pi}\right)^{\frac{1}{p}}\|y\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho)}\|Q\|_{\ell_p(\mathbb{T}_h^0,\eta)}.\tag{86}$$

The equality in (86) is attained if and only if $y\equiv 0$.

Proof. Since $\rho,\eta:\mathbb{T}_h^0\to(0,\infty)$, $\rho,\eta\in\ell_1(\mathbb{T}_h^0)$, $y\in\ell_p(\mathbb{T}_h^0,\rho)$ and $Q\in\ell_p(\mathbb{T}_h^0,\eta)$, using Theorem 3 with ρ,η,y and Q instead of ρ_1,ρ_2,F_1 and F_2, respectively, we have $y\rho, Q\eta\in\ell_1(\mathbb{T}_h^0)$. Applying the h-Fourier cosine transform to both sides of (83) and using (84), we see that

$$\Big(\lambda_0+\sum_{k=1}^{\mathrm{M}}\lambda_k\big[f(u)\big]^k\Big)\mathscr{F}_c\{x\}(u)=\mathscr{F}_c\{y\rho\}(u),\quad\forall u\in\left[0,\frac{\pi}{h}\right].\tag{87}$$

By virtue of (87) and (85), we arrive at

$$\mathscr{F}_c\{x\}(u)=\mathscr{F}_c\{y\rho\}(u)\mathscr{L}\{Q\eta\}(u),\quad\forall u\in\left[0,\frac{\pi}{h}\right].$$

Therefore, $x=(y\rho)*(Q\eta)$. According to Theorem 3, we obtain the estimate (86). The equality is attained if and only if $y\equiv 0$. This completes the proof. □

REMARK 1. In Theorem 5, if the function y satisfies the condition

$$0<M_1^{\frac{1}{p}}\leqslant y(nh)\leqslant M_2^{\frac{1}{p}}<\infty,\quad\forall n\in\mathbb{N}_0,$$

and the function Q satisfies the condition

$$0 < M_3^{\frac{1}{p}} \leqslant Q(nh) \leqslant M_4^{\frac{1}{p}} < \infty, \quad \forall n \in \mathbb{N}_0,$$

then by using Theorem 4, we have the following estimate

$$\left\| x(\rho * \eta)^{\frac{1}{p}-1} \right\|_p > \left(\frac{\pi^2+\pi-2}{2\pi(1+\pi)} \right)^{\frac{1}{p}} \left[S\left(\frac{M_1 M_3}{M_2 M_4} \right) \right]^{-1} h^{\frac{1-p}{p}} \|y\|^{(2)}_{\ell_p(\mathbb{T}_h^0,\rho)} \|Q\|_{\ell_p(\mathbb{T}_h^0,\eta)}.$$

REMARK 2. Let ψ_0 be the function defined on the time scale $\mathbb{T}_h^0$ by

$$\psi_0(0) := \frac{-\pi^2}{3h^3} \quad \text{and} \quad \psi_0(nh) := \frac{2(-1)^{n+1}}{h^3 n^2} \quad \text{for } n \in \mathbb{N}.$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we derive that the function ψ_0 belongs to the space $\ell_1(\mathbb{T}_h^0)$.

From the definition of the h-Fourier cosine transform (1), we have

$$\mathscr{F}_c\{\psi_0\}(u) = \frac{-\pi^2}{3h^2} + \frac{4}{h^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(unh), \quad \forall u \in \mathbb{R}. \tag{88}$$

Using the following Fourier cosine series of the function $g(t) = t^2$ on the interval $[-\pi,\pi]$ ([3, p. 99])

$$t^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nt)}{n^2}, \quad \forall t \in [-\pi,\pi],$$

we then get $(uh)^2 = \frac{\pi^2}{3} + 4 \sum\limits_{n=1}^{\infty} (-1)^n \frac{\cos(unh)}{n^2}, \quad \forall u \in \left[0, \frac{\pi}{h}\right]$.

Combining the above identity with (88) yields

$$\mathscr{F}_c\{\psi_0\}(u) = -u^2, \quad \forall u \in \left[0, \frac{\pi}{h}\right]. \tag{89}$$

Let us define the operator $\mathbf{P}_0 : \ell_1(\mathbb{T}_h^0) \to \ell_1(\mathbb{T}_h^0)$ by $\mathbf{P}_0(z) := z \underset{1}{*} \psi_0$ for $z \in \ell_1(\mathbb{T}_h^0)$.

It is easily proven that the operator $\mathbf{P}_0$ in here is the same as the operator K in [15, p. 338]. The formula (5.18) was given in [15, p. 338] without proof, we will prove this formula. From [13, p. 210], (82) and (89), for $u \in \left[0, \frac{\pi}{h}\right]$ and $z \in \ell_1(\mathbb{T}_h^0)$, we deduce that

$$\mathscr{F}_c\{\mathbf{P}_0(z)\}(u) = \mathscr{F}_c\{z \underset{1}{*} \psi_0\}(u) = \mathscr{F}_c\{z\}(u)\mathscr{F}_c\{\psi_0\}(u) = -u^2 \mathscr{F}_c\{z\}(u).$$

Hence, the formula (5.18) in [15, p. 338] is proved.

Let M_0 be a given number in $\mathbb{N}$ and let μ, p_0, w_k ($k = 0, 1, \ldots, \mathrm{M}_0$) be given complex numbers satisfying $\mu \in \mathbb{N}$, $p_0 \in \mathbb{R}$, $p_0 > 1$ and $w_0 = 1$. Suppose that $\rho_0, \eta_0 : \mathbb{T}_h^0 \to \mathbb{R}$ are two given functions in $\ell_1(\mathbb{T}_h^0)$ such that $\rho_0(nh) > 0$ and $\eta_0(nh) > 0$ for all $n \in \mathbb{N}_0$. In Lemma 5.1 in [15, p. 338], the authors studied the equation that can be rewritten in the form

$$w_0\varphi + \Big(\sum_{k=1}^{M_0}(-1)^k w_k \mathbf{P}_0^k\Big)(\varphi) = y_0\rho_0,$$

where $y_0 \in \ell_{p_0}(\mathbb{T}_h^0, \rho_0)$ is a given function and φ is an unknown function in $\ell_1(\mathbb{T}_h^0)$, assuming that the following condition is satisfied: There exists a function Q_0 in the weighted function space $\ell_{p_0}(\mathbb{T}_h^0, \eta_0)$ such that

$$\mathscr{L}\{Q_0\eta_0\}(u) = \frac{(1+hu)^\mu}{w_0 + \sum\limits_{k=1}^{M_0} w_k u^{2k}}, \quad \forall u \in \Big[0, \frac{\pi}{h}\Big].$$

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