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A GROUP–THEORETICAL APPROACH TO EXTENDING THE SCHRÖDINGER–ROBERTSON UNCERTAINTY INEQUALITY

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Abstract. We provide a sufficient condition for expanding the domain in the Schrödinger-Robertson uncertainty inequality for infinitesimal operators derived from a unitary representation of a Lie group.

1. Introduction

The well-known *uncertainty principle inequality* (UPI) for Hilbert spaces states that if A and B are densely defined and are self-adjoint or skew-adjoint operators on a Hilbert space $\mathscr{H}$, with domains $\mathscr{D}(A)$ and $\mathscr{D}(B)$, and if $a,b \in \mathbb{C}$, then

$$\frac{1}{2}|\langle [A,B]x,x\rangle| \leqslant \|(A-aI)x\|\,\|(B-bI)x\|$$

for all $x \in \mathscr{D}([A,B])$. Here, as usual, the *commutator* $[A,B]$ is defined as $[A,B] := AB - BA$ on the domain $\mathscr{D}([A,B]) = \mathscr{D}(AB) \cap \mathscr{D}(BA)$ where the domain of the product AB is defined by $\mathscr{D}(AB) = \{x \in \mathscr{D}(B) \mid Bx \in \mathscr{D}(A)\}$, and similarly for $\mathscr{D}(BA)$. Observe that $\mathscr{D}([A,B]) \subseteq \mathscr{D}(A) \cap \mathscr{D}(B)$.

If $[A,B]$ is closable, then in general, it is *invalid* that the preceding inequality holds for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(\overline{[A,B]})$. However, in 1967, K. Kraus [8] proved that the inequality holds if A,B and $[A,B]$ are infinitesimal generators determined by a unitary representation of a Lie group of dimension less than four. Later, in 1997, G. B. Folland and A. Sitaram [5] improved Kraus's result to obtain the following statement: if $(\pi,\mathscr{H})$ is a unitary representation of a connected Lie group G with Lie algebra $\mathfrak{g}$, then for $X,Y \in \mathfrak{g}$ such that the linear span of X,Y, and $[X,Y]$ is an ideal in $\mathfrak{g}$, then the inequality

$$\frac{1}{2}\left|\langle \overline{[\pi(X),\pi(Y)]}x,x\rangle\right| \leqslant \|\pi(X)x\|\,\|\pi(Y)x\|$$

holds for all $x \in \mathscr{D}(\pi(X)) \cap \mathscr{D}(\pi(Y)) \cap \mathscr{D}(\overline{[\pi(X),\pi(Y)]})$. One notable application of this result arises when one consider the Schrödinger representation of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$ and this formulation leads to the n-dimensional

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UPI for functions in $L^2(\mathbb{R}^n)$, see [5]. In 2004, using a crucial tool given by I. E. Segal in [11], J. G. Christensen [2] made improvements to the above result by omitting the hypothesis that X,Y and $[X,Y]$ span an ideal in $\mathfrak{g}$. Additionally, J. G. Christensen and H. Schlichtkrull [3] employed the principal series representation of the Euclidean group $E(2)$ to establish the UPI for functions on the unit circle $\mathbb{S}^1$, which was initially introduced by E. Breiterberger in [1]. As a result, it is natural that one may apply this group theoretical technique to obtain the UPI for functions on the n-sphere $\mathbb{S}^n$. In this paper, we use this group theoretical method to obtain a *weaker* version of the UPI for functions on $\mathbb{S}^2$, as originally proposed by F. J. Narcowick and J. D. Ward in [9] and later extended to $\mathbb{S}^n$ by S. S. Goh and N. T. Goodman [6].

A well-known inequality stronger than the UPI is the following *Schrödinger-Robertson uncertainty inequality* (SRUI): if A and B are densely defined and are self-adjoint or skew-adjoint operators on a Hilbert space $\mathscr{H}$, then

$$\frac{1}{2}\left(|\langle [A,B]x,x\rangle|^2+|\langle \{A,B\}x,x\rangle|^2\right)^{1/2}\leqslant \|Ax\|\,\|Bx\|$$

for all $x\in\mathscr{D}([A,B])$. Here the *anticommutator* $\{A,B\}$ is defined as $\{A,B\}:=AB+BA$ on the domain $\mathscr{D}(\{A,B\})=\mathscr{D}(AB)\cap\mathscr{D}(BA)=\mathscr{D}([A,B])$. Detailed discussions and examples related to this inequality can be found in [4] and [10]. If $[A,B]$ and $\{A,B\}$ are closable, then in general, the SRUI does not hold for all $x\in\mathscr{D}(A)\cap\mathscr{D}(B)\cap\mathscr{D}(\overline{[A,B]})\cap\mathscr{D}(\overline{\{A,B\}})$. Therefore, an analogous question, as in the case of the UPI we discussed above, arises: can the SRUI hold on the extended domain when the operators involved come from a unitary representation of a Lie group? This leads us to investigate this question in a certain aspect. In this paper, we engage a group-theoretical framework in order to expand the domain of the infinitesimal operators, derived from a unitary representation of a Lie group, in the SRUI.

This paper is organized as follows. We review the notations and preliminary results, including the UPI from a group perspective, in Section 2. Additionally, by applying the result in [2] to a unitary representation of the Euclidean group $E(3)$, we obtain a weaker version of UPI for functions on the sphere $\mathbb{S}^2$. In Section 3, we employ a result of I. E. Segal in [11] to obtain a sufficient condition for the SRUI to hold on a larger domain of infinitesimal operators induced by a unitary representation of a Lie group. This is the main result of our paper.

2. Notation and auxiliary results

We recall basic results on unitary representations of Lie groups. These materials can be found in standard references such as in [7], [13] and [14]. Throughout this paper, we let G be a Lie group with its Lie algebra $\mathfrak{g}$, and let $(\pi,\mathscr{H})$ be a unitary representation of G. A vector x in $\mathscr{H}$ is *smooth* for G if the map $a\mapsto\pi(a)x$ is smooth from G to $\mathscr{H}$. Let $\mathscr{H}_\pi^\infty$ be the set of all smooth vectors for G. Then $\mathscr{H}_\pi^\infty$ is a subspace of $\mathscr{H}$. For each $X\in\mathfrak{g}$, we define *the infinitesimal generator* $\pi(X)$, with the domain $\mathscr{D}(\pi(X)) = \mathscr{H}_\pi^\infty$, as the limit

$$\pi(X)x := \lim_{t\to 0} \frac{\pi(\exp tX)x - x}{t}$$

for $x \in \mathscr{H}_\pi^\infty$. Then $\pi(X)(\mathscr{H}_\pi^\infty) \subseteq \mathscr{H}_\pi^\infty$ for all $X \in \mathfrak{g}$ and π is a representation of $\mathfrak{g}$ on $\mathscr{H}_\pi^\infty$. Let da be a left Haar measure on G and $C_c^\infty(G)$ be the set of all infinitely differentiable functions with compact supports in G. The *Gårding subspace* of $\mathscr{H}$, denoted by $\mathscr{G}(\pi)$, is the vector space of elements $\pi(f)x$ which are of the form

$$\pi(f)x := \int_G f(a)\pi(a)x\,da$$

where $f \in C_c^\infty(G)$, $x \in \mathscr{H}$. Then the Gårding subspace $\mathscr{G}(\pi)$ is dense in $\mathscr{H}$. Since $\mathscr{G}(\pi) \subseteq \mathscr{D}(\pi(X))$, it follows that $\pi(X)$ is densely defined for all $X \in \mathfrak{g}$. Moreover, it can be shown that $\pi(X)$ is skew-symmetric on $\mathscr{H}_\pi^\infty$. Now $\pi(X)$ is closable and we denote the closure of $\pi(X)$ as $\overline{\pi}(X)$. Moreover, the closed operator $\overline{\pi}(X)$ is skew-adjoint, see [11]. For $X \in \mathfrak{g}$, one can define a *left- and a right- invariant differential operators*, respectively, on $C^\infty(G)$ by

$$(\mathfrak{X}_L^X f)(a) := \frac{d}{dt}\Big|_{t=0} f(a \cdot \exp tX), \quad (\mathfrak{X}_R^X f)(a) := \frac{d}{dt}\Big|_{t=0} f(\exp tX \cdot a)$$

for $a \in G$. Next, we state the following proposition which will be used later in this paper.

PROPOSITION 1. (Taylor, [14]) *Let $X \in \mathfrak{g}$, $f \in C_c^\infty(G)$ and let Δ be the modular function on G.*

(i) *For $x \in \mathscr{H}$, $\overline{\pi}(X)\pi(f)x = -\pi(\mathfrak{X}_R^X f)x$.*

(ii) *For $x \in \mathscr{D}(\overline{\pi}(X))$,*

$$\pi(f)\overline{\pi}(X)x = -\pi(\mathfrak{X}_L^X f)x + \Delta(X)\pi(f)x$$

where $\Delta(X) := \frac{d}{dt}\Big|_{t=0} \Delta(\exp tX)$.

After employing the above proposition and the results of I. E. Segal in [11], J. G. Christensen [2] eliminated the condition that "X, Y and $[X,Y]$ span an ideal of $\mathfrak{g}$" in the hypothesis of the result in [5], and proved the following theorem.

THEOREM 2. (Christensen, [2]) *Let G be a Lie group with its Lie algebra $\mathfrak{g}$, and let $(\pi, \mathscr{H})$ be a unitary representation of G. Suppose $X, Y \in \mathfrak{g}$. Then the inequality*

$$\frac{1}{2}\left|\langle \overline{\pi}[X,Y]x, x\rangle\right| \leqslant \|(\overline{\pi}(X) - aI)x\|\,\|(\overline{\pi}(Y) - bI)x\|$$

holds for all $x \in \mathscr{D}(\overline{\pi}(X)) \cap \mathscr{D}(\overline{\pi}(Y)) \cap \mathscr{D}(\overline{\pi}([X,Y]))$, and $a, b \in \mathbb{C}$.

REMARK 1. Since π is a representation of $\mathfrak{g}$ on $\mathscr{H}_\pi^\infty$, $[\pi(X),\pi(Y)] = \pi([X,Y])$ on $\mathscr{H}_\pi^\infty$ and hence $\overline{[\pi(X),\pi(Y)]} = \overline{\pi}([X,Y])$. Since $\overline{\pi}(X) = \pi(X)$ and $\overline{\pi}(Y) = \pi(Y)$ on $\mathscr{H}_\pi^\infty$, $[\overline{\pi}(X),\overline{\pi}(Y)]|_{\mathscr{H}_\pi^\infty} = [\pi(X),\pi(Y)]$. It follows that $\overline{[\overline{\pi}(X),\overline{\pi}(Y)]} = \overline{[\pi(X),\pi(Y)]} = \overline{\pi}([X,Y])$. Therefore,

$$\mathscr{D}(\overline{\pi}(X)) \cap \mathscr{D}(\overline{\pi}(Y)) \cap \mathscr{D}(\overline{[\overline{\pi}(X),\overline{\pi}(Y)]}) = \mathscr{D}(\overline{\pi}(X)) \cap \mathscr{D}(\overline{\pi}(Y)) \cap \mathscr{D}(\overline{\pi}([X,Y])).$$

In [3], J. G. Christensen and H. Schlichtkrull applied Theorem 2 to the principal series representation of the Euclidean motion group $E(2)$ in order to formulate the UPI for functions on the unit circle. Then it is natural that one can employ this method to the motion group $E(n)$ to get the UPI for functions on the sphere $\mathbb{S}^{n-1}$. In the following example, we apply Theorem 2 to the Euclidean motion group $E(3)$.

EXAMPLE 1. Let $E(3)$ be the Euclidean motion group on $\mathbb{R}^3$. One can realize $E(3)$ as

$$E(3) = \left\{ (A,x) := \begin{pmatrix} A & x \\ 0 & 1 \end{pmatrix} \middle| A \in SO(3), x \in \mathbb{R}^3 \right\},$$

and its Lie algebra $\mathfrak{e}(3)$ is given by

$$\mathfrak{e}(3) = \left\{ \begin{pmatrix} Z & x \\ 0 & 0 \end{pmatrix} \middle| Z \in \mathfrak{so}(3), x \in \mathbb{R}^3 \right\}.$$

Define a unitary representation $\pi : E(3) \to U(L^2(\mathbb{S}^2))$ as follows:

$$(\pi(g)f)(s) := e^{ix\cdot s} f(A^{-1}s)$$

where $g = (A,x) \in E(3)$, $f \in L^2(\mathbb{S}^2)$, $s \in \mathbb{S}^2$ and $x \cdot s$ denotes the dot product in $\mathbb{R}^3$. For each $s \in \mathbb{S}^2$, we can express its spherical coordinates as $s = (s_1, s_2, s_3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$. Then we can identify functions in $L^2(\mathbb{S}^2)$ with functions in $L^2([0,\pi]\times[0,2\pi])$. The standard basis of $\mathfrak{e}(3)$ is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations are

$$[X_i, X_j] = \varepsilon_{ijk} X_k, \quad [X_i, Y_j] = \varepsilon_{ijk} Y_k, \quad [Y_i, Y_j] = 0,$$

where ε_{ijk} is the Levi-Civita symbol, and their corresponding differential operators are

$$\begin{aligned}
\pi(X_1) &= s_3\frac{\partial}{\partial s_2} - s_2\frac{\partial}{\partial s_3} = \sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\\
\pi(X_2) &= s_1\frac{\partial}{\partial s_3} - s_3\frac{\partial}{\partial s_1} = -\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\\
\pi(X_3) &= s_2\frac{\partial}{\partial s_1} - s_1\frac{\partial}{\partial s_2} = -\frac{\partial}{\partial\varphi}
\end{aligned}$$

and

$$\begin{aligned}
(\pi(Y_1)f)(s) &= is_1 f(s) = i\sin\theta\cos\varphi f(s),\\
(\pi(Y_2)f)(s) &= is_2 f(s) = i\sin\theta\sin\varphi f(s),\\
(\pi(Y_3)f)(s) &= is_3 f(s) = i\cos\theta f(s)
\end{aligned}$$

where for $j=1,2,3$, $\mathscr{D}(\pi(Y_j)) = L^2(\mathbb{S}^2)$ and

$$\mathscr{D}(\pi(X_j)) = \Big\{ f\in L^2(\mathbb{S}^2) : f \text{ is separately absolutely continuous and } \frac{\partial f}{\partial\theta}, \frac{\partial f}{\partial\varphi} \in L^2(\mathbb{S}^2) \Big\}.$$

When we apply Theorem 2 to the commutation relations $[X_1,Y_2]=Y_3$, $[X_2,Y_3]=Y_1$ and $[X_3,Y_1]=Y_2$, we obtain

$$\begin{aligned}
\frac{1}{2}|\langle\overline{\pi}(Y_3)f,f\rangle| &\leqslant \|\overline{\pi}(X_1)f - \langle\overline{\pi}(X_1)f,f\rangle f\|\,\|\overline{\pi}(Y_2)f - \langle\overline{\pi}(Y_2)f,f\rangle f\|,\\
\frac{1}{2}|\langle\overline{\pi}(Y_1)f,f\rangle| &\leqslant \|\overline{\pi}(X_2)f - \langle\overline{\pi}(X_2)f,f\rangle f\|\,\|\overline{\pi}(Y_3)f - \langle\overline{\pi}(Y_3)f,f\rangle f\|,\\
\frac{1}{2}|\langle\overline{\pi}(Y_2)f,f\rangle| &\leqslant \|\overline{\pi}(X_3)f - \langle\overline{\pi}(X_3)f,f\rangle f\|\,\|\overline{\pi}(Y_1)f - \langle\overline{\pi}(Y_1)f,f\rangle f\|
\end{aligned}$$

for all $f\in\cap_{j=1}^3\mathscr{D}(\overline{\pi}(X_j))$ with $\|f\|_2=1$. Let $d\sigma$ denote the surface measure on $\mathbb{S}^2$. Then for $f\in\cap_{j=1}^3\mathscr{D}(\overline{\pi}(X_j))$ with $\|f\|_2=1$,

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2} s|f(s)|^2 d\sigma\right\|_{\mathbb{R}^3}^2\\
&= \sum_{j=1}^3\left|\int_{\mathbb{S}^2} s_j|f(s)|^2 d\sigma\right|^2\\
&= |\langle\overline{\pi}(Y_1)f,f\rangle|^2 + |\langle\overline{\pi}(Y_2)f,f\rangle|^2 + |\langle\overline{\pi}(Y_3)f,f\rangle|^2\\
&= |\langle\overline{\pi}([X_2,Y_3])f,f\rangle|^2 + |\langle\overline{\pi}([X_3,Y_1])f,f\rangle|^2 + |\langle\overline{\pi}([X_1,Y_2])f,f\rangle|^2\\
&\overset{\text{UPI}}{\leqslant} 4\Big(\|\overline{\pi}(X_2)f - \langle\overline{\pi}(X_2)f,f\rangle f\|^2\,\|\overline{\pi}(Y_3)f - \langle\overline{\pi}(Y_3)f,f\rangle f\|^2\\
&\quad + \|\overline{\pi}(X_3)f - \langle\overline{\pi}(X_3)f,f\rangle f\|^2\,\|\overline{\pi}(Y_1)f - \langle\overline{\pi}(Y_1)f,f\rangle f\|^2\\
&\quad + \|\overline{\pi}(X_1)f - \langle\overline{\pi}(X_1)f,f\rangle f\|^2\,\|\overline{\pi}(Y_2)f - \langle\overline{\pi}(Y_2)f,f\rangle f\|^2 \Big)
\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_{k=1}^{3}\|\overline{\pi}(X_k)f-\langle\overline{\pi}(X_k)f,f\rangle f\|^2\sum_{k=1}^{3}\|\overline{\pi}(Y_k)f-\langle\overline{\pi}(Y_k)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_1)f-\langle\overline{\pi}(X_1)f,f\rangle f\|^2\|\overline{\pi}(Y_1)f-\langle\overline{\pi}(Y_1)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_1)f-\langle\overline{\pi}(X_1)f,f\rangle f\|^2\|\overline{\pi}(Y_3)f-\langle\overline{\pi}(Y_3)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_2)f-\langle\overline{\pi}(X_2)f,f\rangle f\|^2\|\overline{\pi}(Y_1)f-\langle\overline{\pi}(Y_1)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_2)f-\langle\overline{\pi}(X_2)f,f\rangle f\|^2\|\overline{\pi}(Y_2)f-\langle\overline{\pi}(Y_2)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_3)f-\langle\overline{\pi}(X_3)f,f\rangle f\|^2\|\overline{\pi}(Y_2)f-\langle\overline{\pi}(Y_2)f,f\rangle f\|^2\\
&\quad -4\|\overline{\pi}(X_3)f-\langle\overline{\pi}(X_3)f,f\rangle f\|^2\|\overline{\pi}(Y_3)f-\langle\overline{\pi}(Y_3)f,f\rangle f\|^2\\
&\overset{\text{UPI}}{\leqslant} 4\Big(\sum_{k=1}^{3}\|\overline{\pi}(X_k)f-\langle\overline{\pi}(X_k)f,f\rangle f\|^2\Big)\Big(\sum_{k=1}^{3}\|\overline{\pi}(Y_k)f-\langle\overline{\pi}(Y_k)f,f\rangle f\|^2\Big)\\
&\quad -|\langle\overline{\pi}([X_1,Y_3])f,f\rangle|^2-|\langle\overline{\pi}([X_2,Y_1])f,f\rangle|^2-|\langle\overline{\pi}([X_3,Y_2])f,f\rangle|^2\\
&= 4\sum_{k=1}^{3}\|\overline{\pi}(X_k)f-\langle\overline{\pi}(X_k)f,f\rangle f\|^2\sum_{k=1}^{3}\|\overline{\pi}(Y_k)f-\langle\overline{\pi}(Y_k)f,f\rangle f\|^2-\sum_{k=1}^{3}|\langle\overline{\pi}(Y_k)f,f\rangle|^2\\
&= 4\sum_{k=1}^{3}\|\overline{\pi}(X_k)f-\langle\overline{\pi}(X_k)f,f\rangle f\|^2\sum_{k=1}^{3}\|\overline{\pi}(Y_k)f-\langle\overline{\pi}(Y_k)f,f\rangle f\|^2-\left\|\int_{\mathbb{S}^2}s|f(s)|^2d\sigma\right\|_{\mathbb{R}^3}^2.
\end{aligned}
$$

Hence, for $f\in\cap_{j=1}^{3}\mathscr{D}(\overline{\pi}(X_j))$ with $\|f\|_2=1$,

$$
\begin{aligned}
2\left\|\int_{\mathbb{S}^2}s|f(s)|^2d\sigma\right\|_{\mathbb{R}^3}^2 &\leqslant 4\sum_{k=1}^{3}\|\overline{\pi}(X_k)f-\langle\overline{\pi}(X_k)f,f\rangle f\|^2\sum_{k=1}^{3}\|\overline{\pi}(Y_k)f-\langle\overline{\pi}(Y_k)f,f\rangle f\|^2\\
&= 4\left(\int_{\mathbb{S}^2}|\Omega f-a(f)f|^2\,d\sigma\right)\left(1-\left\|\int_{\mathbb{S}^2}s|f(s)|^2d\sigma\right\|_{\mathbb{R}^3}^2\right)
\end{aligned}
$$

where $\Omega f(x):=-ix\times\nabla f(x)$ and $a(f):=\int_{\mathbb{S}^2}(\Omega f)\overline{f}\,d\sigma$. The last equation is obtained due to the fact that

$$
-x\times\nabla=-\begin{pmatrix} i & j & k\\ s_1 & s_2 & s_3\\ \partial_1 & \partial_2 & \partial_3\end{pmatrix}=\begin{pmatrix} s_3\partial_2-s_2\partial_3\\ s_1\partial_3-s_3\partial_1\\ s_2\partial_1-s_1\partial_2\end{pmatrix}=\begin{pmatrix}\pi(X_1)\\ \pi(X_2)\\ \pi(X_3)\end{pmatrix},\quad a(f)=\begin{pmatrix}\langle i\pi(X_1)f,f\rangle\\ \langle i\pi(X_2)f,f\rangle\\ \langle i\pi(X_3)f,f\rangle\end{pmatrix}.
$$

Consequently, we derive an uncertainty inequality for functions on the sphere S^2:

$$
\frac{1}{2}\left\|\int_{\mathbb{S}^2}s|f(s)|^2d\sigma\right\|_{\mathbb{R}^3}^2\leqslant\left(\int_{\mathbb{S}^2}|\Omega f-a(f)f|^2\,d\sigma\right)\left(1-\left\|\int_{\mathbb{S}^2}s|f(s)|^2d\sigma\right\|_{\mathbb{R}^3}^2\right)
$$

for $f\in\cap_{j=1}^{3}\mathscr{D}(\overline{\pi}(X_j))$ with $\|f\|_2=1$.

Although using the above group theoretical approach yields a UPI for functions on $\mathbb{S}^2$ as demonstrated in the previous example, it turns out that the lower bound on the left-hand side of our inequality is only half of the one appeared in the results in [6] and [9]. So our inequality is a weaker version of the one in [6] and [9]. Nevertheless, our inequality is consistent with the one derived in [12]. We emphasize that in [12], the inequality holds for $f \in \mathscr{D}(\cap_{i=1}^{3} \pi(X_i))$, while in our result, the inequality holds for $f \in \mathscr{D}(\cap_{i=1}^{3} \overline{\pi}(X_i))$, which is a bigger set. As our inequality above is not the sharpest one, it suggests that the Euclidean group $E(3)$ is not the right group to gain the inequality in [6] and [9]. Therefore, this generates the following question: how can one determine a suitable pair of a Lie group and a corresponding unitary representation that would establish the UPI for functions on a sphere as given in [6] and [9]? Finding such pair would be a future work.

3. Schrödinger-Robertson uncertainty inequality

Let A,B be densely defined operators in a Hilbert space $\mathscr{H}$ with domains $\mathscr{D}(A)$ and $\mathscr{D}(B)$, respectively. Recall that the *anticommutator* $\{A,B\}$ is defined by

$$\{A,B\} := AB + BA$$

on the domain $\mathscr{D}(\{A,B\}) = \mathscr{D}(AB) \cap \mathscr{D}(BA)$, where $\mathscr{D}(AB) = \{x \in \mathscr{D}(B) : Bx \in \mathscr{D}(A)\}$ and vice versa for $\mathscr{D}(BA)$. The Schrödinger-Robertson uncertainty inequality (SRUI) states that if A,B are (skew) self-adjoint operators in a Hilbert space $\mathscr{H}$, then

$$\frac{1}{2}\left(|\langle [A,B]x,x\rangle|^2 + |\langle \{A,B\}x,x\rangle|^2\right)^{1/2} \leqslant \|Ax\|\,\|Bx\|$$

for $x \in \mathscr{D}([A,B])$. It can be seen that, in general, the domains $\mathscr{D}([A,B])$ and $\mathscr{D}(\{A,B\})$ may not be even densely defined. However, if $[A,B]$ and $\{A,B\}$ are closable along with some additional convergence conditions, we attain the following straightforward extension of the SRUI.

PROPOSITION 3. *Let A,B be densely defined operators in a Hilbert space $\mathscr{H}$ with the following properties:*

(i) *A and B are (skew) self-adjoint operators,*

(ii) *$[A,B]$ and $\{A,B\}$ are closable, and*

(iii) *for each $x \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(\overline{[A,B]}) \cap \mathscr{D}(\overline{\{A,B\}})$, there is a sequence $x_n \in \mathscr{D}([A,B])$ such that $x_n \to x$, $Ax_n \to Ax$, $Bx_n \to Bx$, and $[A,B]x_n$ and $\{A,B\}x_n$ converge in $\mathscr{H}$.*

Then

$$\frac{1}{2}\left(\left|\langle \overline{[A,B]}x,x\rangle\right|^2 + \left|\langle \overline{\{A,B\}}x,x\rangle\right|^2\right)^{1/2} \leqslant \|Ax\|\,\|Bx\|$$

for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(\overline{[A,B]}) \cap \mathscr{D}(\overline{\{A,B\}})$.

Proof. Let $x \in \mathscr{D}(A) \cap \mathscr{D}(B) \cap \mathscr{D}(\overline{[A,B]}) \cap \mathscr{D}(\overline{\{A,B\}})$. By (iii), there is a sequence $x_n \in \mathscr{D}([A,B])$ such that $x_n \to x$, $Ax_n \to Ax$, $Bx_n \to Bx$, $[A,B]x_n$ and $\{A,B\}x_n$ converge in $\mathscr{H}$. Since $\overline{[A,B]}x := \lim_{n\to\infty}[A,B]x_n$ and $\overline{\{A,B\}}x := \lim_{n\to\infty}\{A,B\}x_n$, it follows that

$$|\langle [A,B]x_n, x_n\rangle|^2 + |\langle \{A,B\}x_n, x_n\rangle|^2 \leqslant 4\,\|Ax_n\|\,\|Bx_n\|\,.$$

Taking $n \to \infty$, one get

$$\left|\langle \overline{[A,B]}x, x\rangle\right|^2 + \left|\langle \overline{\{A,B\}}x, x\rangle\right|^2 \leqslant 4\,\|Ax\|\,\|Bx\|\,. \quad \square$$

We note that if the operators A and B are generated by a unitary representation of a Lie group G, namely $A = \overline{\pi}(X), B = \overline{\pi}(Y)$ for some $X, Y \in \mathfrak{g}$, then it is guaranteed that the domains of $[A,B]$ and $\{A,B\}$ are densely defined and closable. Therefore, to apply the above proposition in order to obtain an extended version of the SRUI in this case, we need the convergence condition (iii) to hold. To achieve this condition, we employ the following result of I. E. Segal in [11].

LEMMA 1. (Segal, [11]) *Let U_n be a sequence of neighborhoods of the identity $e \in G$ such that $\cap_n U_n = \{e\}$. Suppose that $(f_n) \subseteq L^1(G)$ is a sequence of real-valued functions such that* supp $f_n \subseteq U_n$, $\|f_n\|_1$ *is bounded and* $\int_G f_n(a)da \longrightarrow \lambda \in \mathbb{R}$. *If $\phi(a)$ is a bounded continuous function from G to a Banach space $\mathscr{B}$, then $\int_G f_n(a)\phi(a)da$ converges to $\lambda\phi(e)$ as $n \to \infty$.*

Next, we derive some useful equations. Let $f \in C_c^\infty(G)$, $a \in G$ and $t \in \mathbb{R}$. By Taylor's theorem, there exists $t'_a \in \mathbb{R}$ such that $|t'_a| < |t|$ and

$$\frac{f(\exp tX \cdot a) - f(a)}{t} = (\mathfrak{X}_R^X f)(a) + \frac{t}{2} g(a, t'_a)$$

where $g(b,u) := \left.\frac{d^2}{ds^2}\right|_{s=u} f(\exp sX \cdot b)$ for $b \in G, u \in \mathbb{R}$. This implies

$$\left|\int_K \frac{f(\exp tX \cdot a) - f(a)}{t} da - \int_K (\mathfrak{X}_R^X f)(a)da\right| = \left|\frac{t}{2}\int_K g(a,t'_a)da\right|$$

where K is a compact support of f. Since $g(\cdot,\cdot)$ is continuous on a compact set $K \times [-1,1]$, it follows that

$$\int_K (\mathfrak{X}_R^X f)(a)da = \lim_{t\to 0}\int_K \frac{f(\exp tX \cdot a) - f(a)}{t} da = 0 \tag{1}$$

where the last equality holds since da is left-invariant. Similarly, we have

$$\int_K (\mathfrak{X}_L^X f)(a)\,da \;=\; \lim_{t\to 0}\int_K \frac{f(a\cdot \exp tX) - f(a)}{t}\,da \;=\; -\Delta(X)\int_G f(a)\,da.$$

This observation allows us for interchanging the limit and the integral sign and will be used in the proof of the succeeding lemma.

LEMMA 2. *Let (U_n) be a sequence of neighborhoods of e such that $\cap_n U_n = \{e\}$. Suppose that for each $X,Y \in \mathfrak{g}$, there is a sequence of real-valued functions (f_n) in $C_c^\infty(G)$ satisfying the following properties:*

(P1) supp $f_n \subseteq U_n$ *for each $n \in \mathbb{N}$, the sequence $(\|f_n\|_1)$ is bounded and $\int_G f_n(a)da$ converges to $\lambda \in \mathbb{R}$, and*

(P2) *the sequence*

$$\left(\left\|\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X f_n + \Delta(X)\mathfrak{X}_L^Y f_n\right\|_1\right)$$

is bounded.

Then, for each $u \in \mathscr{H}$,

$$\pi\left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n\right)u - \pi\left(\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n\right)u + \Delta(Y)\pi(\mathfrak{X}_L^X f_n)u + \Delta(X)\pi(\mathfrak{X}_L^Y f_n)u - \Delta(X)\Delta(Y)\pi(f_n)u$$

converges to $-4\lambda\Delta(X)\Delta(Y)u$ as $n \to \infty$.

Proof. Let $X,Y \in \mathfrak{g}$. Then there is a sequence (f_n) having the properties (P1) and (P2). Let $u \in \mathscr{H}$. We have

$$\begin{aligned}
&\pi\left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n\right)u - \pi\left(\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n\right)u + \Delta(Y)\pi(\mathfrak{X}_L^X f_n)u + \Delta(X)\pi(\mathfrak{X}_L^Y f_n)u - \Delta(X)\Delta(Y)\pi(f_n)u \\
&= \int_G \left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X f_n + \Delta(X)\mathfrak{X}_L^Y f_n - \Delta(X)\Delta(Y)f_n\right)(a)\pi(a)u\,da.
\end{aligned}$$

Since $a \mapsto \pi(a)u$ is continuous and bounded for every $u \in \mathscr{H}$ and (f_n) serves the properties (P1) and (P2), by Lemma 1, it suffices to show that

$$\int_G (\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X f_n + \Delta(X)\mathfrak{X}_L^Y f_n - \Delta(X)\Delta(Y)f_n)(a)\,da$$

is convergent. The integral of the first term vanishes by the equation (1). We recall that for all $f \in C_c^\infty(G)$ and all $Z \in \mathfrak{g}$,

$$\int_G (\mathfrak{X}_L^Z f)(a)\,da \;=\; \Delta(-Z)\int_G f(a)\,da.$$

This implies that

$$\int_G \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n(a)\,da = \int_G \Delta(-Y)\Delta(-X)f_n(a)\,da.$$

Now, we have

$$\begin{aligned}
&\int_G (\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X f_n + \Delta(X)\mathfrak{X}_L^Y f_n - \Delta(X)\Delta(Y)f_n)(a)\,da \\
&= -4\Delta(X)\Delta(Y)\int_G f_n(a)\,da
\end{aligned}$$

which converges to $-4\lambda\Delta(X)\Delta(Y)$ as $n\to\infty$. Therefore,

$$\pi\left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n\right)u-\pi\left(\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n\right)u+\Delta(Y)\pi(\mathfrak{X}_L^X f_n)u+\Delta(X)\pi(\mathfrak{X}_L^Y f_n)u-\Delta(X)\Delta(Y)\pi(f_n)u$$

converges to $-4\lambda\Delta(X)\Delta(Y)u$ as $n\to\infty$. This completes the proof. □

We recall that a family of real-valued functions $\{f_n\}$ in $C_c^\infty(G)$ is said to be an *approximate identity* for $L^1(G)$ if there exists a sequence of neighborhoods U_n of e such that $\cap_n U_n=\{e\}$ along with the following properties:

(i) $\operatorname{supp} f_n\subseteq U_n$ and there is $c>0$ such that $\|f_n\|_{L^1(G)}<c$ for all $n\in\mathbb{N}$,

(ii) $\displaystyle\int_G f_n(a)\,da=1$ for all $n\in\mathbb{N}$.

Let $x\in\mathscr{G}(\pi)$ and $f_n\in C_c^\infty(G)$ be an approximate identity. Following the idea presented in [2], in order to establish on our main theorem, we consider the convergence of $\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)x$. For $f\in C_c^\infty(G)$ and $X,Y\in\mathfrak{g}$, by Proposition 1, we have

$$\overline{\pi}(X)\overline{\pi}(Y)\pi(f)x=\overline{\pi}(X)\pi(-\mathfrak{X}_R^Y f)x=\pi(\mathfrak{X}_R^X\mathfrak{X}_R^Y f)x. \tag{2}$$

and

$$\begin{aligned}\pi(f)\overline{\pi}(X)\overline{\pi}(Y)x &= \left(-\pi(\mathfrak{X}_L^X f)+\Delta(X)\pi(f)\right)\overline{\pi}(Y)x\\ &= -\pi(\mathfrak{X}_L^X f)\overline{\pi}(Y)x+\Delta(X)\pi(f)\overline{\pi}(Y)x\\ &= \pi(\mathfrak{X}_L^Y\mathfrak{X}_L^X f)x-\Delta(Y)\pi(\mathfrak{X}_L^X f)x-\Delta(X)\pi(\mathfrak{X}_L^Y f)x\\ &\quad+\Delta(X)\Delta(Y)\pi(f)x.\end{aligned} \tag{3}$$

Now using the above lemma and the equations (2) and (3), we are able to give a sufficient condition for expanding the domain in the SRUI. This is our main result.

THEOREM 4. *Let π be a unitary representation of a Lie group G on a Hilbert space $\mathscr{H}$ and $\mathfrak{g}$ be the corresponding Lie algebra. Suppose that for each $X,Y\in\mathfrak{g}$, there exists an approximate identity $(f_n)\subseteq C_c^\infty(G)$ such that the sequence*

$$\left(\left\|\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n-\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n+\Delta(Y)\mathfrak{X}_L^X(f_n)+\Delta(X)\mathfrak{X}_L^Y f_n\right\|_1\right) \tag{4}$$

is bounded. Then, for each $X,Y\in\mathfrak{g}$,

$$\frac{1}{2}\left(|\langle\overline{\pi}([X,Y])x,x\rangle|^2+\left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}}x,x\rangle\right|^2\right)^{1/2}\leqslant\ \|\overline{\pi}(X)x\|\,\|\overline{\pi}(Y)x\|$$

where $x\in\mathscr{D}(\overline{\pi}(X))\cap\mathscr{D}(\overline{\pi}(Y))\cap\mathscr{D}(\overline{\pi}([X,Y]))\cap\mathscr{D}(\overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}})=:\mathscr{D}$.

Proof. Let $X,Y\in\mathfrak{g}$. Since

$$\mathscr{G}(\pi)\subseteq\mathscr{D}([\overline{\pi}(X),\overline{\pi}(Y)])=\mathscr{D}(\{\overline{\pi}(X),\overline{\pi}(Y)\}),$$

we have $\mathscr{D}([\overline{\pi}(X),\overline{\pi}(Y)])$ and $\mathscr{D}(\{\overline{\pi}(X),\overline{\pi}(Y)\})$ are dense in $\mathscr{H}$. We have seen that $[\overline{\pi}(X),\overline{\pi}(Y)]$ is closable and $\overline{[\overline{\pi}(X),\overline{\pi}(Y)]} = \overline{\pi}([X,Y])$. To show that $\{\overline{\pi}(X),\overline{\pi}(Y)\}$ is closable, it suffices to prove that $\mathscr{D}(\{\overline{\pi}(X),\overline{\pi}(Y)\}^*)$ is dense in $\mathscr{H}$. We observe that

$$\begin{aligned}(\overline{\pi}(X)\overline{\pi}(Y)+\overline{\pi}(Y)\overline{\pi}(X))^* &\supseteq (\overline{\pi}(X)\overline{\pi}(Y))^* + (\overline{\pi}(Y)\overline{\pi}(X))^*\\ &\supseteq \overline{\pi}(Y)^*\overline{\pi}(X)^* + \overline{\pi}(X)^*\overline{\pi}(Y)^*\\ &= \pi(Y)\pi(X)+\pi(X)\pi(Y)\\ &= \{\pi(X),\pi(Y)\}.\end{aligned}$$

The first two superset relationships above hold since the domains $\mathscr{D}(\pi(X)\pi(Y))$ and $\mathscr{D}(\pi(Y)\pi(X))$ contain the Gårding subspace, which is dense in $\mathscr{H}$. Now, we have that $\mathscr{D}(\{\overline{\pi}(X),\overline{\pi}(Y)\}^*)$ is dense in $\mathscr{H}$. Therefore, $\{\overline{\pi}(X),\overline{\pi}(Y)\}$ is closable.

Let $x \in \mathscr{D}$. We will find a sequence (x_n) in $\mathscr{D}([\overline{\pi}(X),\overline{\pi}(Y)])$ such that $x_n \to x$ and

(I) $\overline{\pi}(X)x_n \to \overline{\pi}(X)x$, $\overline{\pi}(Y)x_n \to \overline{\pi}(Y)x$,

(II) $[\overline{\pi}(X),\overline{\pi}(Y)]x_n \to \overline{[\overline{\pi}(X),\overline{\pi}(Y)]}x$ and

(III) $\{\overline{\pi}(X),\overline{\pi}(Y)\}x_n \to \overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}}x$.

If we can find such a sequence (x_n) in $\mathscr{D}([\overline{\pi}(X),\overline{\pi}(Y)])$, then by Lemma 3, we obtain our desired inequality and we are done. So it remains to construct the sequence (x_n) in $\mathscr{D}([\overline{\pi}(X),\overline{\pi}(Y)])$ satisfying property (I), (II) and (III) above.

By the assumption, there exists a sequence (f_n) such that the sequence in (4) is bounded. For each n, let $x_n := \pi(f_n)x$. We must verify (I)–(III). Recall that $\{\pi(f_n)x\} \subseteq \mathscr{G}(\pi) \subseteq \mathscr{D}$. To demonstrate (III), we observe that

$$\begin{aligned}\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)x &= \overline{\pi}(X)\overline{\pi}(Y)\pi(f_n)x + \overline{\pi}(Y)\overline{\pi}(X)\pi(f_n)x\\ &= -\overline{\pi}(X)\pi(\mathfrak{X}_R^Y f_n)x - \overline{\pi}(Y)\pi(\mathfrak{X}_R^X f_n)x\\ &= \pi(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n)x + \pi(\mathfrak{X}_R^Y\mathfrak{X}_R^X f_n)x\\ &= \int_G \mathfrak{X}_R^X\mathfrak{X}_R^Y f_n(a)\pi(a)x\,da + \int_G \mathfrak{X}_R^Y\mathfrak{X}_R^X f_n(a)\pi(a)x\,da\\ &= \int_G \left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n(a)\pi(a)x + \mathfrak{X}_R^Y\mathfrak{X}_R^X f_n(a)\right)\pi(a)x\,da\\ &= \int_G \left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n(a) + \mathfrak{X}_R^Y\mathfrak{X}_R^X f_n(a)\right)\pi(a)x\,da,\end{aligned}$$

the second and third equalities are due to the equation (2). We next deal with the sequence

$$\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)u - \pi(f_n)\{\overline{\pi}(X),\overline{\pi}(Y)\}u$$

for $u \in \mathscr{G}(\pi)$. From the equations (2) and (3), we have

$$\begin{aligned}
&\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)u - \pi(f_n)\{\overline{\pi}(X),\overline{\pi}(Y)\}u \\
&= \pi\left(\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n\right)u - \pi\left(\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n\right)u + \Delta(Y)\pi(\mathfrak{X}_L^X f_n)u + \Delta(X)\pi(\mathfrak{X}_L^Y f_n)u \\
&\quad + \pi\left(\mathfrak{X}_R^Y\mathfrak{X}_R^X f_n\right)u - \pi\left(\mathfrak{X}_L^X\mathfrak{X}_L^Y f_n\right)u + \Delta(X)\pi(\mathfrak{X}_L^Y f_n)u + \Delta(Y)\pi(\mathfrak{X}_L^X f_n)u \\
&\quad - \Delta(X)\Delta(Y)\pi(f_n)u - \Delta(Y)\Delta(X)\pi(f_n)u \\
&= \int_G (\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X f_n + \Delta(X)\mathfrak{X}_R^Y f_n - \Delta(X)\Delta(Y)f_n)(a)\pi(a)u\,da \\
&\quad + \int_G (\mathfrak{X}_R^Y\mathfrak{X}_R^X f_n - \mathfrak{X}_L^X\mathfrak{X}_L^Y f_n + \Delta(X)\mathfrak{X}_L^Y f_n + \Delta(Y)\mathfrak{X}_R^X f_n - \Delta(Y)\Delta(X)f_n)(a)\pi(a)u\,da.
\end{aligned}$$

The assumption implies that the sequence

$$\begin{aligned}
\Big\| &\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n - \mathfrak{X}_L^Y\mathfrak{X}_L^X f_n + \Delta(Y)\mathfrak{X}_L^X(f_n) + \Delta(X)\mathfrak{X}_R^Y f_n \\
&+ \mathfrak{X}_R^Y\mathfrak{X}_R^X f_n - \mathfrak{X}_L^X\mathfrak{X}_L^Y f_n + \Delta(X)\mathfrak{X}_L^Y(f_n) + \Delta(Y)\mathfrak{X}_R^X f_n \Big\|_1
\end{aligned}$$

is bounded. Therefore, by Lemma 3,

$$\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)u - \pi(f_n)\{\overline{\pi}(X),\overline{\pi}(Y)\}u$$

converges to $-4\Delta(X)\Delta(Y)u$. As (f_n) is an approximate identity,

$$\pi(f_n)\overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}}u \to \overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}}u.$$

This means that $\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)u$ also converges in $\mathscr{H}$. This result holds not only for all $u \in \mathscr{G}(\pi)$, but is also applicable to all elements in $\mathscr{H}$, since the Gårding subspace $\mathscr{G}(\pi)$ is dense in $\mathscr{H}$. Therefore,

$$\{\overline{\pi}(X),\overline{\pi}(Y)\}\pi(f_n)x \to \overline{\{\overline{\pi}(X),\overline{\pi}(Y)\}}x,$$

obtaining (III). Now, since (f_n) is an approximate identity in $C_c^\infty(G)$, we have that (I) and (II) are accomplished immediately because of the proof of the main theorem in [2]. This completes the proof. □

Our theorem relies significantly on the presence of required approximate identities. It is still an open problem: for each $X,Y \in \mathfrak{g}$, can one find a suitable approximate identity with the property (P2)? Finding such an approximate identity would not be an easy task. On the other hand, showing that there is no approximate identity satisfying the property (P2) seems to be nontrivial. Therefore, these two issues should be investigated in future work.

Analogous to [3], we apply the principal series representation of the Euclidean motion group $E(2)$ to Theorem 4 in order to formulate an SRUI for functions on the unit circle. The following example demonstrate this formulation.

EXAMPLE 2. Let $E(2)$ be the Euclidean motion group on $\mathbb{R}^2$ with its Lie algebra $\mathfrak{g}$. One can realize G as

$$E(2)=\left\{(r,z):=\begin{pmatrix} e^{ir} & z \\ 0 & 1\end{pmatrix}\middle|\, r\in\mathbb{R},\ z\in\mathbb{C}\right\},$$

and then the Lie algebra of $E(2)$, $\mathfrak{e}(2)$, is given by

$$\mathfrak{e}(2)=\left\{\begin{pmatrix} ir & z \\ 0 & 0\end{pmatrix}\middle|\, r\in\mathbb{R},\ z\in\mathbb{C}\right\}.$$

The *principal series representation* is a unitary representation $\pi : E(2)\to U\left(L^2(\mathbb{T})\right)$ given by

$$(\pi(r,z)f)(s):=e^{i\mathrm{Re}(z\overline{s})}f(se^{-ir})$$

for $f\in L^2(\mathbb{T})$ and $s\in\mathbb{T}:=\{s\in\mathbb{C}:\ |s|=1\}$. Let X, Y_1, and Y_2 be

$$X=\begin{pmatrix} i & 0 \\ 0 & 0\end{pmatrix},\quad Y_1=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\quad\text{and}\quad Y_2=\begin{pmatrix} 0 & i \\ 0 & 0\end{pmatrix}.$$

Then, writing $s=e^{i\theta}$, we have

$$(\overline{\pi}(X)f)(s)=-f'(s),\ (\overline{\pi}(Y_1)g)(s)=(i\cos\theta)g(s),\ (\overline{\pi}(Y_2)h)(s)=(i\sin\theta)h(s)$$

for $f\in\mathscr{D}(\overline{\pi}(X))=\{f\in L^2(\mathbb{T}):\ f\in AC(\mathbb{T})\text{ and } f'\in L^2(\mathbb{T})\}$ and $g,h\in L^2(\mathbb{T})=\mathscr{D}(\overline{\pi}(Y_1))=\mathscr{D}(\overline{\pi}(Y_2))$. Notice that we have the commutative relations $[X,Y_1]=Y_2$, $[X,Y_2]=-Y_1$.

Now, suppose that we can find an approximate identity $\{f_n\}$ on the unit circle $\mathbb{T}$ such that $\left\|f_n'\right\|_1$ is uniformly bounded. Since $\mathbb{S}^1$ is abelian,

$$\begin{aligned}
&\mathfrak{X}_R^X\mathfrak{X}_R^Y f_n-\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n+\Delta(Y)\mathfrak{X}_L^X f_n+\Delta(X)\mathfrak{X}_L^Y f_n\\
&=\mathfrak{X}_L^X\mathfrak{X}_L^Y f_n-\mathfrak{X}_L^Y\mathfrak{X}_L^X f_n+\mathfrak{X}_L^X f_n+\mathfrak{X}_L^Y f_n\\
&=[\mathfrak{X}_L^X,\mathfrak{X}_L^Y]f_n+\mathfrak{X}_L^X f_n+\mathfrak{X}_L^Y f_n\\
&=\mathfrak{X}_L^{[X,Y]}f_n+\mathfrak{X}_L^X f_n+\mathfrak{X}_L^Y f_n
\end{aligned}$$

Substituting $Y=Y_1$ and $Y=Y_2$ in the above equation, we get that

$$\begin{aligned}
\left\|\mathfrak{X}_L^{[X,Y_1]}f_n+\mathfrak{X}_L^X f_n+\mathfrak{X}_L^{Y_1}f_n\right\|_1&=\left\|(i\sin\theta)f_n-f_n'+(i\cos\theta)f_n\right\|_1\\
\left\|\mathfrak{X}_L^{[X,Y_2]}f_n+\mathfrak{X}_L^X f_n+\mathfrak{X}_L^{Y_2}f_n\right\|_1&=\left\|(-i\cos\theta)f_n-f_n'+(i\sin\theta)f_n\right\|_1.
\end{aligned}$$

Since $\left\|f_n'\right\|_1$ is uniformly bounded,

$$\left(\left\|(i\sin\theta)f_n-f_n'+(i\cos\theta)f_n\right\|_1\right)\quad\text{and}\quad\left(\left\|(-i\cos\theta)f_n-f_n'+(i\sin\theta)f_n\right\|_1\right)$$

are uniformly bounded. Applying Theorem 4, we have

$$4\|\overline{\pi}(X)f\|^2\|\overline{\pi}(Y_2)f\|^2 \geqslant \left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y_2)\}}f,f\rangle\right|^2 + |\langle\overline{\pi}([X,Y_2])f,f\rangle|^2$$
$$= \left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y_2)\}}f,f\rangle\right|^2 + |\langle\overline{\pi}(Y_1)f,f\rangle|^2$$

and

$$4\|\overline{\pi}(X)f\|^2\|\overline{\pi}(Y_1)f\|^2 \geqslant \left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y_1)\}}f,f\rangle\right|^2 + |\langle\overline{\pi}(Y_2)f,f\rangle|^2$$

for all $f \in \mathscr{D}(\overline{\pi}(X))$. Let σ denote the push-forward measure on the unit circle $\mathbb{T}$. By combining the above two inequalities, we obtain

$$\begin{aligned}
4\left\|f'\right\|^2\|f\|^2 &= 4\|\overline{\pi}(X)f\|^2\left(\|\overline{\pi}(Y_2)f\|^2 + \|\overline{\pi}(Y_1)f\|^2\right)\\
&= \|\overline{\pi}(X)f\|^2\|\overline{\pi}(Y_2)f\|^2 + 4\|\overline{\pi}(X)f\|^2\|\overline{\pi}(Y_1)f\|^2\\
&\geqslant |\langle\overline{\pi}(Y_1)f,f\rangle|^2 + |\langle\overline{\pi}(Y_2)f,f\rangle|^2\\
&\quad + \left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y_1)\}}f,f\rangle\right|^2 + \left|\langle\overline{\{\overline{\pi}(X),\overline{\pi}(Y_2)\}}f,f\rangle\right|^2\\
&= \left\|\int_{\mathbb{T}} s|f(s)|^2 d\sigma\right\|_{\mathbb{R}^2}^2 + \left|\int_{\mathbb{T}}(-2(i\cos\theta)f'(s) + (i\sin\theta)f(s))\,d\sigma\right|^2\\
&\quad + \left|\int_{\mathbb{T}}(-2(i\sin\theta)f'(s) - (i\cos\theta)f(s))\,d\sigma\right|^2
\end{aligned}$$

Therefore, assuming the existence of the required approximate identity (f_n), we then attain the above uncertainty inequality for all $f \in \mathscr{D}(\overline{\pi}(X))$, which is sharper than the uncertainty inequality in [3].

The result in the previous example is valid if there exists an approximate identity f_n on the unit circle such that

$$\sup_n \|f_n'\|_{L^1} < \infty.$$

However, verifying that a certain classical approximate identities on the unit circle satisfies the above condition turns out to be a difficult task. Alternatively, one could try to construct a new approximate identity on the unit circle so that it would satisfy the above condition. This could be a possible future work.

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