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A LOGARITHMIC WEIGHTED ADAMS–TYPE INEQUALITY IN THE WHOLE OF $\mathbb{R}^N$ WITH AN APPLICATION

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Abstract. In this paper, we will establish a logarithmic weighted Adams inequality in a logarithmic weighted second order Sobolev space in the whole set of $\mathbb{R}^N$. Using this result, we delve into the analysis of a weighted fourth-order equation in $\mathbb{R}^N$. We assume that the nonlinearity of the equation exhibits either critical or subcritical exponential growth, consistent with the Adams-type inequalities previously established. By applying the Mountain Pass Theorem, we demonstrate the existence of a weak solution to this problem. The primary challenge lies in the lack of compactness in the energy caused by the critical exponential growth of the non-linear term f.

1. Introduction

We start by providing an overview of Trudinger-Moser inequalities within classical first-order Sobolev spaces. Additionally, we'll explore Adams' inequalities in second-order Sobolev spaces. Subsequently, we'll extend these concepts to weighted Sobolev spaces. Moreover, we'll reference relevant works associated with these concepts.

In dimension $N \geqslant 2$ and for bounded domain $\Omega \subset \mathbb{R}^N$, the critical exponential growth is given by the well known Trudinger-Moser inequality [45, 47]

$$\sup_{\int_\Omega |\nabla u|^N \leqslant 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} dx < +\infty \text{ if and only if } \alpha \leqslant \alpha_N, \tag{1}$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ with ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. Later, the Trudinger-Moser inequality was improved to weighted inequalities [10, 12].

Equation (1) has been utilized to address elliptic problems that encompass nonlinearities exhibiting exponential growth. For instance, we refer to the following problems in dimensions where $N \geqslant 2$

$$-\Delta_N u = -\mathrm{div}(|\nabla u|^{N-2}\nabla u) = f(x,u) \text{ in } \Omega \subset \mathbb{R}^N,$$

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which have been studied considerably by Adimurthi [3, 4], Figueiredo et al. [31], Lam and Lu [36–39], Miyagaki and Souto [44] and Zhang and Chen [52].

Considerable focus has been directed toward weighted inequalities in weighted Sobolev spaces, notably known in mathematical literature as the weighted Trudinger-Moser inequality [10, 12]. The majority of studies have centered on radial functions owing to the radial nature of the weights involved. This quality enhances the maximum growth of integrability. When the weight is of logarithmic type, Calanchi and Ruf [11] extend the Trudinger-Moser inequality and proved the following results in the weighted Sobolev space, $W^{1,N}_{0,rad}(B,\rho) = \text{closure}\{u \in C^{\infty}_{0,rad}(B) \ \mid \ \int_B |\nabla u|^N \rho(x)dx < \infty\}$, where B denote the unit ball of $\mathbb{R}^N$.

THEOREM 1. [11]

(i) *Let* $\beta \in [0,1)$ *and let* ρ *given by* $\rho(x) = \left(\log \frac{1}{|x|}\right)^{\beta}$, *then*

$$\int_B e^{|u|^{\gamma}} dx < +\infty, \ \forall \ u \in W^{1,N}_{0,rad}(B,\rho),$$

if and only if

$$\gamma \leqslant \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and

$$\sup_{\substack{u\in W^{1,N}_{0,rad}(B,\rho) \\ \int_B |\nabla u|^N w(x)dx \leqslant 1}} \int_B e^{\alpha |u|^{\gamma_{N,\beta}}} dx < +\infty \quad \Leftrightarrow \quad \alpha \leqslant \alpha_{N,\beta} = N[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)]^{\frac{1}{1-\beta}}$$

where ω_{N-1} *is the area of the unit sphere* S^{N-1} *in* $\mathbb{R}^N$ *and* N' *is the Hölder conjugate of* N.

(ii) *Let* ρ *given by* $\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$, *then*

$$\int_B exp\{e^{|u|^{\frac{N}{N-1}}}\}dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,rad}(B,\rho)$$

and

$$\sup_{\substack{u\in W^{1,N}_{0,rad}(B,\rho) \\ \|u\|_{\rho}\leqslant 1}} \int_B exp\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{\frac{N}{N-1}}}\}dx < +\infty \quad \Leftrightarrow \quad \beta \leqslant N,$$

where ω_{N-1} *is the area of the unit sphere* S^{N-1} *in* $\mathbb{R}^N$ *and* N' *is the Hölder conjugate of* N.

The theorem 1 has enabled the exploration of second-order weighted elliptic problems in dimensions where $N \geqslant 2$. As a result, Calanchi et al. [13] established the existence of a non-trivial radial solution for an elliptic problem defined on the unit ball in $\mathbb{R}^2$, where the nonlinearities exhibit double exponential growth at infinity. Following this, Deng et al. investigated the subsequent problem

$$\begin{cases} -\mathrm{div}(\sigma(x)|\nabla u(x)|^{N-2}\nabla u(x)) = f(x,u) & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{2}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geqslant 2$ and the nonlinearity $f(x,u)$ is continuous in $B \times \mathbb{R}$ and has critical growth in the sense of Theorem 1. The authors have proved that there is a non-trivial solution to this problem, using the mountain pass Theorem. Similar results are proven by Chetouane and Jaidane [24], Dridi [28] and Zhang [51]. Furthermore, problem (2), involving a potential, has been studied by Baraket and Jaidane [8]. Moreover, we mention that Abid et al. [1] have proved the existence of a positive ground state solution for a weighted second-order elliptic problem of Kirchhoff type, with nonlinearities having a double exponential growth at infinity, using minimax techniques combined with Trudinger-Moser inequality.

We should notice the work of J. Li et al. [40] in the n-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ where they extend the well-known concentration-compactness principle on finite domains in the Euclidean spaces of Lions [43] to the setting of the Heisenberg group. Their results improve the sharp Trudinger-Moser inequality on domains of finite measure in $\mathbb{H}^n$ by Cohn and Lu. As an application of the concentration-compactness principle, the authors establish the existence of ground state solutions for a class of Q- Laplacian subelliptic equations on H^n:

$$-div\left(\left|\nabla_H u\right|^{Q-2}\nabla_H u\right) + V(\zeta)|u|^{Q-2}u = \frac{f(u)}{\rho(\zeta)^\beta}, \quad Q = 2n+2$$

with nonlinear terms f of maximal exponential growth $\exp\left(t^{\frac{Q}{Q-1}}\right)$ as $t \to +\infty$. Also, the same authors in [41], established the concentration-compactness principle of Trudinger-Moser type on any compact Riemannian manifolds as well as on the entire complete noncompact Riemannian manifolds with Ricci curvature lower bound.

Let's also note that several recent works [34, 49, 52, 54] have studied the existence of solutions for elliptic operators involving nonlinearities with exponential growth with respect to Trudinguer or Adams-type inequalities.

In recent years, Aouaoui and Jlel [7] have extended the work of Calanchi and Ruf to the whole $\mathbb{R}^2$ space, by considering the following weight

$$\rho_\beta(x) = \begin{cases} \left(\log\left(\frac{e}{|x|}\right)\right)^\beta & \text{if } |x| < 1, \\ \chi(|x|) & \text{if } |x| \geqslant 1, \end{cases} \tag{3}$$

where, $0 < \beta \leqslant 1$ and $\chi : [1, +\infty[\to]0, +\infty[$ is a continuous function such that $\chi(1) = 1$ and $\inf_{t \in [1,+\infty[} \chi(t) > 0$. The authors consider the space E_β as the space of all radial functions of the completion of $C_0^\infty\left(\mathbb{R}^2\right)$ with respect to the norm

$$\|u\|_{E_\beta}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 \rho_\beta(x)dx + \int_{\mathbb{R}^2} u^2 dx = |\nabla u|^2_{L^2\left(\mathbb{R}^2, \rho_\beta\right)} + |u|^2_{L^2\left(\mathbb{R}^2\right)}.$$

The authors proved the following result:

THEOREM 2. *Let $\beta \in (0,1)$ and ρ_β be defined by (3). For all $u \in E_\beta$, we have*

$$\int_{\mathbb{R}^2} \left(e^{|u|^{\frac{2}{1-\beta}}} - 1 \right) dx < +\infty.$$

Moreover, if $\alpha < \tau_\beta$, then

$$\sup_{u \in E_\beta, \|u\|_{E_\beta} \leqslant 1} \int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{2}{1-\beta}}} - 1 \right) dx < +\infty \tag{4}$$

where $\tau_\beta = 2\left[2\pi(1-\beta)\right]^{\frac{1}{1-\beta}}$.

If $\alpha > \tau_\beta$, then

$$\sup_{u \in E_\beta, \|u\|_{E_\beta} \leqslant 1} \int_{\mathbb{R}^2} \left(e^{\alpha |u|^{\frac{2}{1-\beta}}} - 1 \right) dx = +\infty.$$

The concept of critical exponential growth was further expanded to higher order Sobolev spaces by Adams [2]. Specifically, Adams demonstrated the following outcome: for $m \in \mathbb{N}$ and Ω as an open bounded set in $\mathbb{R}^N$ where $m < N$, there exists a positive constant $C_{m,N}$ such that

$$\sup_{u \in W_0^{m,\frac{N}{m}}(\Omega), |\nabla^m u|_{\frac{N}{m}} \leqslant 1} \int_\Omega e^{\beta_0 |u|^{\frac{N}{N-m}}} \, dx \leqslant C_{m,N} |\Omega|, \tag{5}$$

where $W_0^{m,\frac{N}{m}}(\Omega)$ denotes the m^{th}-order Sobolev space, $\nabla^m u$ denotes the m^{th}-order gradient of u, namely

$$\nabla^m u := \begin{cases} \Delta^{\frac{m}{2}} u, & \text{if } m \text{ is even} \\ \nabla \Delta^{\frac{m-1}{2}} u, & \text{if } m \text{ odd} \end{cases}$$

and

$$\beta_0 = \beta_0(m,N) := \frac{N}{\omega_{N-1}} \begin{cases} \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})} \right]^{\frac{N}{N-m}}, & \text{if } m \text{ is even} \\ \left[\dfrac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})} \right]^{\frac{N}{N-m}}, & \text{if } m \text{ odd.} \end{cases}$$

In the particular case where $N = 4$ and $m = 2$, the inequality (5) takes the form

$$\sup_{u \in W_0^{2,2}(\Omega), |\Delta u|_2 \leqslant 1} \int_\Omega e^{32\pi^2 u^2} \, dx \leqslant C|\Omega|. \tag{6}$$

Also, for bounded domains $\Omega \subset \mathbb{R}^4$, in [2] the authors proved the following inequality

$$\sup_{u \in S} \int_\Omega (e^{\alpha u^2} - 1) dx < +\infty \quad \Leftrightarrow \quad \alpha \leqslant 32\pi^2$$

where

$$S=\Big\{u\in W_0^{2,2}(\Omega)\ \mid\ \big(\int_\Omega|\Delta u|^2dx\big)^{\frac{1}{2}}\leqslant 1\Big\}.$$

When Ω is replaced by the whole space $\mathbb{R}^4$, Ruff and Sani [38] established the corresponding Adams type inequality as follows:

$$\sup_{\|u\|_{W^{2,2}}\leqslant 1}\int_\Omega (e^{\alpha u^2}-1)dx<+\infty \quad\Leftrightarrow\quad \alpha\leqslant 32\pi^2 \tag{7}$$

where $\|u\|^2_{W^{2,2}(\mathbb{R}^4)}=\int_{\mathbb{R}^4}|\triangle u|^2dx+\int_{\mathbb{R}^4}|\nabla u|^2dx+\int_{\mathbb{R}^4}u^2dx.$

In [15] Chen et al. provided a sharp critical and subcritical trace Trudinger-Moser and Adams inequalities on the half-spaces and prove the existence of their extremals through the method based on the Fourier rearrangement, harmonic extension and scaling invariance. Also, in [14], Chen et al. established a sharp concentration-compactness principle associated with a singular Adams inequality on the second-order Sobolev spaces in $\mathbb{R}^4$. As applications, they proved the existence of ground state solutions to the following bi-Laplacian equation with critical nonlinearity:

$$\Delta^2u+V(x)u=\frac{f(x,u)}{|x|^\beta},$$

where $V(x)$ has a positive lower bound and $f(x,t)$ behaves like $\exp(\alpha|t|^2)$ as $t\to\infty$. In [16–20, 22] Chen et al. studied the existence and nonexistence of extremals for critical Adams inequalities in $\mathbb{R}^4$. Also, they've worked on ground state of bi-harmonic equations with critical exponential growth.

Recently, Adams' type inequalities on the logarithmic weighted Sobolev space $W^{2,2}_{0,rad}(B_1)$ of radial function in the unit ball B_1 of $\mathbb{R}^4$ has been established. More precisely, in [48] the authors proved the following result

THEOREM 3. [48] *Let $\beta\in(0,1)$ and let $w=(\log(\frac{e}{|x|}))^\beta$, then*

$$\sup_{\substack{u\in W^{2,2}_{0,rad}(B_1,w)\\ \int_{B_1}w(x)|\Delta u|^2dx\leqslant 1}}\int_{B_1}e^{\alpha|u|^{\frac{2}{1-\beta}}}dx<+\infty \quad\Leftrightarrow\quad \alpha\leqslant\alpha_\beta=4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}}.$$

This last result allowed the authors in [29] to study the following weighted problem

$$\begin{cases}\Delta(w(x)\Delta u)-\Delta u+V(x)u=f(x,u) & \text{in}\quad B_1\\ \qquad\qquad\qquad\qquad u=\frac{\partial u}{\partial n}=0 & \text{on}\quad \partial B_1.\end{cases}$$

The weight $w(x)$ is given by

$$w(x)=\left(\log\frac{e}{|x|}\right)^\beta,\ \ \beta\in(0,1),$$

B_1 is the unitary disk in $\mathbb{R}^4$, $f(x,t)$ is continuous in $B_1\times\mathbb{R}$ and behaves like $\exp\{\alpha t^{\frac{2}{1-\beta}}\}$ as $|t|\to+\infty$, for some $\alpha>0$ uniformly with respect to $x\in B_1$. The potential $V:\overline{B_1}\to\mathbb{R}$ is a positive continuous function and bounded away from zero in B_1. The authors establish the existence of radial solution by variational techniques and using Adams' inequality [48].

It should be noted that several works concerning weighted elliptic equations of Kirchhoff type with critical nonlinearities in the sense of Theorem 1 or Theorem 3 have been studied (see [1, 32, 33]).

The result of Theorem 3 has been generalised by H. Zhao and M. Zhu to the unit ball of $\mathbb{R}^N$. More precisely they proved the following result:

THEOREM 4. [55] *Let* $N\geqslant 4$, $\beta\in(0,1)$ *and* $w_\beta=\left(\log\left(\frac{e}{|x|}\right)\right)^{\beta(\frac{N}{2}-1)}$. *Then*

$$\sup_{\substack{u\in W_{0,rad}^{2,\frac{N}{2}}(B,w)\\ \int_B w(x)|\Delta u|^{\frac{N}{2}}dx\leqslant 1}}\int_B e^{\alpha|u|^{\frac{N}{(N-2)(1-\beta)}}}dx<+\infty$$

$$\Leftrightarrow\quad \alpha\leqslant\alpha_\beta=N[(N-2)NV_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}}, \tag{8}$$

where V_N *is the volume of the unit ball* B *in* $\mathbb{R}^N$ *and the subspace of radial functions* $W_{0,rad}^{2,\frac{N}{2}}(B,w_\beta)$ *is defined as*

$$W_{0,rad}^{2,\frac{N}{2}}(B,w_\beta)=closure\left\{u\in C_{0,rad}^{\infty}(B)\ \mid\ \int_B\left(\log\left(\frac{e}{|x|}\right)\right)^{\beta(\frac{N}{2}-1)}|\Delta u|^{\frac{N}{2}}dx<\infty\right\}, \tag{9}$$

endowed with the norm $\|u\|_{w_\beta}=\left(\int_B\left(\log\left(\frac{e}{|x|}\right)\right)^{\beta(\frac{N}{2}-1)}|\Delta u|^{\frac{N}{2}}dx\right)^{\frac{2}{N}}$.

Denote by E as the space of all radial functions of the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^{\frac{N}{2}}=\int_{\mathbb{R}^N}|\triangle u|^{\frac{N}{2}}w_\beta(x)dx+\int_{\mathbb{R}^N}|\nabla u|^{\frac{N}{2}}dx+\int_{\mathbb{R}^N}|u|^{\frac{N}{2}}dx,$$

where the weight $w_\beta(x)$ is given by

$$w_\beta(x)=\begin{cases}\left(\log\left(\dfrac{e}{|x|}\right)\right)^{\beta(\frac{N}{2}-1)} & \text{if } |x|<1,\\ \chi(|x|) & \text{if } |x|\geqslant 1,\end{cases} \tag{10}$$

where, $\frac{N^2-4N+2}{N(N-2)}<\beta<1$, $N\geqslant 4$ and $\chi:[1,+\infty[\to[1,+\infty[$ is a continuous function such that $\chi(1)=1$ and $\inf_{t\in[1,+\infty[}\chi(t)\geqslant 1$. Also, we suppose that there exists a positive constant $M>0$ such that

$$\frac{1}{r^{\frac{N^2}{2}}}\left(\int_1^r t^{N-1}\chi(t)dt\right)\left(\int_1^r\frac{t^{N-1}}{\chi(t)}dt\right)^{\frac{N}{2}-1}\leqslant M,\ \ \forall r\geqslant 1, \tag{11}$$

$$\frac{1}{r^{\frac{N^2}{2}}}\left(\int_1^r t^{N-1}\chi(t)dt\right) \leqslant M, \quad \forall r \geqslant 1, \tag{12}$$

and

$$\frac{\max_{r\leqslant t\leqslant 4r}\chi(t)}{\min_{r\leqslant t\leqslant 4r}\chi(t)} \leqslant M, \quad \forall r \geqslant 1. \tag{13}$$

Inspired by the examples given in [7], we can take the following examples of functions $\chi : [1,+\infty[\to [1,+\infty[$ satisfying the conditions (11), (12) and (13):

- Any continuous function χ such that $\chi(1) = 1$ and

$$1 \leqslant \inf_{t\geqslant 1}\chi(t) \leqslant \sup_{t\geqslant 1}\chi(t) < +\infty.$$

- $\chi(t) = t^{\delta}$, $0 < \delta < \frac{N^2}{2} - N$.
- $\chi(t) = 1 + \log^{\sigma} t$, $\sigma > 1$.

Following S. Aouaoui's proof [7], we prove that w_β belongs to the Muckenhoupt's class $A_{\frac{N}{2}}$ and therefore $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in the space E (see Lemma 1). It follows that the space E can be seen as

$$E = \left\{u \in L_{rad}^{\frac{N}{2}}\left(\mathbb{R}^N\right), \int_{\mathbb{R}^N}\left(|\triangle u|^{\frac{N}{2}} w_\beta(x) + \ |\nabla u|^{\frac{N}{2}}\right)dx < +\infty\right\},$$

endowed with the norm

$$\|u\|^{\frac{N}{2}} = \int_{\mathbb{R}^N}|\triangle u|^{\frac{N}{2}} w_\beta(x)dx + \int_{\mathbb{R}^N}|\nabla u|^{\frac{N}{2}}dx + \int_{\mathbb{R}^N}|u|^{\frac{N}{2}}dx.$$

In this paper, we prove a weighted Adams' inequality analogous to (7) in the whole of $\mathbb{R}^N$ that is:

THEOREM 5. *Let* $\gamma := \dfrac{N}{(N-2)(1-\beta)}$, $\beta \in (\frac{N^2-4N+2}{N(N-2)}, 1)$ *and* w_β *given by* (10), *then*

(i)

$$\int_{\mathbb{R}^N}\left(e^{|u|^{\gamma}} - 1\right)dx < +\infty, \quad \forall u \in E \tag{14}$$

(ii)

$$\sup_{\substack{u\in E\\ \|u\|\leqslant 1}} \int_{\mathbb{R}^N}(e^{\alpha|u|^{\gamma}} - 1)dx < +\infty \quad \Leftrightarrow \quad \alpha \leqslant \alpha_\beta \tag{15}$$

with $\alpha_\beta = N[(N-2)NV_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}}$ *and* V_N *is the volume of the unit sphere.*

As an application of this last result, we study the following weighted problem

$$L(u) := \Delta(w(x)|\Delta u|^{\frac{N}{2}-2}\Delta u) - \operatorname{div}(|\nabla u|^{\frac{N}{2}-2}\nabla u) + |u|^{\frac{N}{2}-2}u = f(u) \text{ in } \mathbb{R}^N, \tag{16}$$

where the weight is given by (10), $N \geqslant 4$. The non linearity $f(t)$ is continuous in $\mathbb{R}$ and behaves like $\exp\{\alpha t^{\frac{N}{(N-2)(1-\beta)}}\}$ as $|t| \to +\infty$, for some $\alpha > 0$.

In view of inequality (15), we say that f has critical growth at $+\infty$ if there exists some $\alpha_0 > 0$,

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall\, \alpha \quad \text{such that} \quad \alpha > \alpha_0 \quad \text{and} \quad \lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = +\infty, \quad \forall\, \alpha < \alpha_0. \tag{17}$$

In view of inequality (14), we say that f has subcritical growth at $+\infty$ if

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall\, \alpha > 0.$$

Let us now state our results. In this paper, we always assume that the nonlinearities $f(t)$ has critical growth with $\alpha_0 > 0$ or $f(t)$ has subcritical growth and satisfies these conditions:

(H_1) The non-linearity $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exists $\theta > N$, such that $0 < \theta F(t) = \theta \int_0^t f(s)ds \leqslant t f(t), \ \forall t \in \mathbb{R} \setminus \{0\}$.

(H_3) $\lim_{t\to 0} \frac{f(t)}{t^{\frac{N}{2}-1}} = 0.$

(H_4) There exist $t_0,\ M_0 > 0$ such that

$$0 < F(t) \leqslant M_0 |f(t)| \ \text{ for all } \ |t| \geqslant t_0.$$

(H_5) The asymptotic condition

$$\lim_{t\to\infty} \frac{f(t)t}{e^{\alpha_0 t^\gamma}} \geqslant \gamma_0 \ \text{ with } \ \gamma_0 > \frac{\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}}{V_N e^{N(1-\log(2e))}}.$$

We say that u is a solution to the problem (16), if u is a weak solution in the following sense.

DEFINITION 1. A function u is called a solution to (16) if $u \in E$ and

$$\int_{\mathbb{R}^N} \left(w_\beta(x)\, |\Delta u|^{\frac{N}{2}-2} \Delta u\ \Delta\varphi + |\nabla u|^{\frac{N}{2}-2} \nabla u . \nabla\varphi + |u|^{\frac{N}{2}-2} u\varphi\right) dx = \int_{\mathbb{R}^N} f(u)\, \varphi\, dx, \tag{18}$$

for all $\varphi \in E$.

It is easy to see that seeking weak solutions of the problem (16) is equivalent to find nonzero critical points of the following functional on E:

$$\mathcal{J}(u) = \frac{2}{N}\left(\int_{\mathbb{R}^N} w_\beta(x)|\Delta u|^{\frac{N}{2}} + |\nabla u|^{\frac{N}{2}} + |u|^{\frac{N}{2}} dx\right) - \int_{\mathbb{R}^N} F(x,u)dx, \tag{19}$$

where $F(u)=\int_0^u f(t)dt$.

In the critical case, we prove the following Theorem.

THEOREM 6. *Assume that the function f has a critical growth at $+\infty$ and satisfies the conditions (H_1), (H_2), (H_3), (H_4) and (H_5). Then the problem (16) has a nontrivial solution.*

In the subcritical case, we prove the following Theorem.

THEOREM 7. *Assume that the function f has subcritical growth at $+\infty$ and satisfies the conditions (H_1), (H_2), (H_3), and (H_4). Then the problem (16) has a nontrivial solution.*

In general the study of fourth order partial differential equations is considered an interesting topic. The interest in studying such equations was stimulated by their applications in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see [25,30]. However many applications are generated by elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example [6,37]).

This paper is structured as follows:

Section 2 presents essential background information on functional spaces. Section 3 establishes preliminary results crucial for our proofs. Section 4 focuses on proving Theorem 5. Section 5 demonstrates a concentration compactness result akin to Lions' Theorem. Section 6 verifies that the energy $\mathcal{J}$ adheres to two specific geometric properties and a compactness condition, albeit under a specified level. Section 7 offers the proof of Theorem 5. Finally, in Section 8, we conclude by proving Theorem 6 and Theorem 7.

Through this paper, the constants C or c may change from line to another and we sometimes index the constants in order to show how they change.

2. Weighted Lebesgue and Sobolev spaces setting

Let $\Omega\subset\mathbb{R}^N$, $N\geqslant 2$, bounded or unbounded, possibly even equal to the whole $\mathbb{R}^N$ and let $w\in L^1(\Omega)$ be a nonnegative function. In order to work with a weighted operator, it becomes necessary to introduce specific functional spaces denoted as $L^p(\Omega,w)$, $W^{m,p}(\Omega,w)$, and $W_0^{m,p}(\Omega,w)$. Later on, these spaces and some of their properties will be utilized. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on Ω and two measurable functions are considered as the same element if they are equal almost everywhere. Following Drabek et al. [27] and Kufner in [35], the weighted Lebesgue space $L^p(\Omega,w)$ is defined as follows:

$$L^p(\Omega,w)=\left\{u:\Omega\to\mathbb{R}\text{ measurable};\quad \int_\Omega w(x)|u|^p\,dx<\infty\right\}$$

for any real number $1\leqslant p<\infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \Big(\int_\Omega w(x)|u|^p \, dx \Big)^{\frac{1}{p}}.$$

For $m \geqslant 2$, let w be a given family of weight functions $w_\tau, \ |\tau| \leqslant m, \ w = \{w_\tau(x), \ x \in \Omega, \ |\tau| \leqslant m\}$.

In [27], the corresponding weighted Sobolev space was defined as

$$W^{m,p}(\Omega, w) = \{u \in L^p(\Omega), D^\tau u \in L^p(\Omega, w) \ \ \forall \ \ 1 \leqslant |\tau| \leqslant m-1, \\ D^\tau u \in L^p(\Omega, w) \ \ \forall \ \ |\tau| = m\}$$

endowed with the following norm:

$$\|u\|_{W^{m,p}(\Omega,w)} = \Bigg(\sum_{|\tau| \leqslant m-1} \int_\Omega |D^\tau u|^p dx + \sum_{|\tau|=m} \int_\Omega w(x) |D^\tau u|^p dx \Bigg)^{\frac{1}{p}},$$

where $w_\tau = 1$ for all $|\tau| < k, \ w_\tau = w$ for all $|\tau| = k$.

If we suppose also that $w(x) \in L^1_{loc}(\Omega)$, then $C_0^\infty(\Omega)$ is a subset of $W^{m,p}(\Omega, w)$ and we can introduce the space

$$W_0^{m,p}(\Omega, w)$$

as the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega, w)$. Moroever, the injection

$$W^{m,p}(\Omega, w) \hookrightarrow W^{m-1,p}(\Omega) \ \text{ is compact.}$$

Also, $(L^p(\Omega, w), \|\cdot\|_{p,w})$ and $(W^{m,p}(\Omega, w), \|\cdot\|_{W^{m,p}(\Omega,w)})$ are separable, reflexive Banach spaces provided that $w(x)^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$. Then the space

$$E = \Big\{ u \in L^{\frac{N}{2}}_{rad}(\mathbb{R}^N) \ \ | \ \ \int_{\mathbb{R}^N} \big(w_\beta(x) |\Delta u|^{\frac{N}{2}} + |\nabla u|^{\frac{N}{2}} \big) dx < +\infty \Big\}$$

is a Banach and reflexive space.

We have the following result.

LEMMA 1. *$C_0^\infty \left(\mathbb{R}^N\right)$ is dense in the space*

$$\Big\{ u \in L^{\frac{N}{2}}\left(\mathbb{R}^N\right), \int_{\mathbb{R}^N} \big(|\triangle u|^{\frac{N}{2}} w_\beta(x) + \ |\nabla u|^{\frac{N}{2}} \big) dx < +\infty \Big\}.$$

Proof. It suffice to see that ω_β belongs to the Muckenhoupt's class $A_{\frac{N}{2}}$ (we also say that ω_β is an $A_{\frac{N}{2}}$-weight), that is

$$\sup \left(\frac{1}{|B|} \int_B w_\beta(x) dx \right) \left(\frac{1}{|B|} \int_B \big(w_\beta(x)\big)^{-1} dx \right)^{\frac{N}{2}-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^N$.

Let $r>0$ and $x_0\in\mathbb{R}^N$. Denote by $B(x_0,r)$ (resp. $B(0,r)$) the open ball of $\mathbb{R}^N$ of center x_0 and radius r (resp. of center 0 and radius r).

First case. Suppose that $B(x_0,r)\cap B(0,r)\neq\emptyset$. Thus, $B(x_0,r)\subset B(0,3r)$ which implies that

$$\frac{1}{|B(x_0,r)|^{\frac{N}{2}}}\left(\int_{B(x_0,r)} w_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{w_\beta(x)}\right)^{\frac{N}{2}-1}$$
$$\leqslant \frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{3r} w_\beta(t)t^{N-1}dt\right)\left(\int_0^{3r}\frac{t^{N-1}}{w_\beta(t)}dt\right)^{\frac{N}{2}-1}. \tag{20}$$

If $3r<1$, then

$$\frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{3r} w_\beta(t)t^{N-1}dt\right)\left(\int_0^{3r}\frac{t^{N-1}}{w_\beta(t)}dt\right)^{\frac{N}{2}-1}$$
$$=\frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{3r} t^{N-1}(1-\log t)^{\beta(\frac{N}{2}-1)}dt\right)\left(\int_0^{3r}\frac{t^{N-1}}{(1-\log t)^{\beta(\frac{N}{2}-1)}}dt\right)^{\frac{N}{2}-1}.$$

But, a simple computation gives

$$\limsup_{r\to 0^+}\frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{3r} t^{N-1}(1-\log t)^{\beta(\frac{N}{2}-1)}dt\right)\left(\int_0^{3r}\frac{t^{N-1}}{(1-\log t)^{\beta(\frac{N}{2}-1)}}dt\right)^{\frac{N}{2}-1}<+\infty. \tag{21}$$

If $3r\geqslant 1$, then

$$\frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{3r}\omega_\beta(t)t^{N-1}dt\right)\left(\int_0^{3r}\frac{t^{N-1}}{w_\beta(t)}dt\right)^{\frac{N}{2}-1}$$
$$=\frac{c}{r^{\frac{N^2}{2}}}\left(\int_0^{1} t^{N-1}(1-\log t)^{\beta(\frac{N}{2}-1)}dt+\int_1^{3r} t\chi(t)dt\right)$$
$$\times\left(\int_0^{1}\frac{t^{N-1}}{(1-\log t)^{\beta(\frac{N}{2}-1)}}dt+\int_1^{3r}\frac{t^{N-1}}{\chi(t)}dt\right)^{\frac{N}{2}-1}. \tag{22}$$

Since $\inf_{t\geqslant 1}\chi(t)\geqslant 1$, then

$$\limsup_{r\to+\infty}\frac{1}{r^{\frac{N^2}{2}}}\int_1^{3r}\frac{t^{N-1}}{\chi(t)}dt=0<+\infty. \tag{23}$$

On the other hand, by (12), we infer

$$\limsup_{r\to+\infty}\frac{1}{r^{\frac{N^2}{2}}}\int_1^{3r} t^{N-1}\chi(t)dt<+\infty. \tag{24}$$

Hence, in view of (23), (24) and (22), it remains to show that

$$\limsup_{r\to+\infty}\frac{1}{r^{\frac{N^2}{2}}}\left(\int_1^{3r}t^{N-1}\chi(t)dt\right)\left(\int_1^{3r}\frac{t^{N-1}}{\chi(t)}dt\right)^{\frac{N}{2}-1}<+\infty.$$

But this fact can immediately be deduced from (12). Combining (21) and (22), we deduce from (20) that there exists a constant $D_0>0$ independent of x_0 and r such that

$$\frac{1}{|B(x_0,r)|^{\frac{N}{2}}}\left(\int_{B(x_0,r)}\omega_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{\omega_\beta(x)}\right)^{\frac{N}{2}-1}\leqslant D_0. \tag{25}$$

Second case. Suppose that $B(x_0,r)\cap B(0,r)=\emptyset$. In this case, we have

$$\frac{|x_0|}{2}\leqslant|x|\leqslant 2|x_0|,\forall x\in B(x_0,r).$$

Hence,

$$\frac{1}{|B(x_0,r)|^{\frac{N}{2}}}\left(\int_{B(x_0,r)}w_\beta(x)dx\right)\left(\int_{B(x_0,r)}\frac{dx}{w_\beta(x)}\right)^{\frac{N}{2}-1}$$
$$\leqslant\left(\frac{\sup_{\frac{|x_0|}{2}\leqslant|x|\leqslant2|x_0|}w_\beta(t)}{\inf_{\frac{|x_0|}{2}\leqslant|x|\leqslant2|x_0|}w_\beta(t)}\right)\leqslant\sup_{\tau>0}\left(\frac{\sup_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}{\inf_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}\right). \tag{26}$$

If $4\tau<1$, then

$$\frac{\sup_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}{\inf_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}=\frac{(1-\log\tau)^{\beta(\frac{N}{2}-1)}}{(1-\log(4\tau))^{\beta(\frac{N}{2}-1)}}.$$

Taking into account that

$$\sup_{0<\tau<\frac14}\left(\frac{1-\log\tau}{1-\log(4\tau)}\right)^{\beta(\frac{N}{2}-1)}<+\infty,$$

it follows that

$$\sup_{0<\tau<\frac14}\left(\frac{\sup_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}{\inf_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}\right)<+\infty. \tag{27}$$

If $\frac14\leqslant\tau<1$, then

$$\frac{\sup_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}{\inf_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}\leqslant\frac{\sup_{\frac14\leqslant t\leqslant4}w_\beta(t)}{\inf_{\frac14\leqslant t\leqslant4}w_\beta(t)}<+\infty,$$

and consequently,

$$\sup_{\frac14\leqslant\tau<1}\left(\frac{\sup_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}{\inf_{\tau\leqslant t\leqslant4\tau}w_\beta(t)}\right)<+\infty. \tag{28}$$

If $\tau \geqslant 1$, then it follows

$$\frac{\sup_{\alpha \leqslant t \leqslant 4\alpha} \omega_\beta(t)}{\inf_{\tau \leqslant t \leqslant 4\tau} \omega_\beta(t)} = \frac{\sup_{\tau \leqslant t \leqslant 4\tau} \chi(t)}{\inf_{\tau \leqslant t \leqslant 4\tau} \chi(t)} \leqslant M,$$

and consequently,

$$\sup_{\tau \geqslant 1} \left(\frac{\sup_{\tau \leqslant t \leqslant 4\tau} \omega_\beta(t)}{\inf_{\tau \leqslant t \leqslant 4\tau} \omega_\beta(t)} \right) < +\infty. \tag{29}$$

Combining (27) and (28), we deduce from that there exists a constant $D_1 > 0$ independent of x_0 and r such that

$$\frac{1}{|B(x_0,r)|^{\frac{N}{2}}} \left(\int_{B(x_0,r)} w_\beta(x) dx \right) \left(\int_{B(x_0,r)} \frac{dx}{w_\beta(x)} \right)^{\frac{N}{2}-1} \leqslant D_1. \tag{30}$$

This finish the proof. □

3. Some useful preliminary results

In this section, we will derive several technical lemmas for our use later. First we begin by the radial lemma due to Lions [42]. Let $W^{1,p}(\mathbb{R}^N)$ be the first order Sobolev space and consider the subspace of radial function namely $W^{1,p}_{rad}(\mathbb{R}^N)$. We have

LEMMA 2. [42] *Let $N \geqslant 2$, $1 \leqslant p < +\infty$, $u \in W^{1,p}_{rad}(\mathbb{R}^N)$, then there exists a positive constant $C = C(N,p)$ such that*

$$|u(x)| \leqslant C \frac{1}{|x|^{\frac{N-1}{p}}} |u|_p^{\frac{p-1}{p}} |\nabla u|_p^{\frac{1}{p}} \quad \textit{for} \quad \textit{a.e.}\ x \in \mathbb{R}^N.$$

In particular, for $p = \frac{N}{2}$, we get the followin inequality:

$$|u(x)| \leqslant C \frac{1}{|x|^{\frac{2(N-1)}{N}}} |u|_{\frac{N}{2}}^{\frac{N-2}{N}} |\nabla u|_{\frac{N}{2}}^{\frac{2}{N}} \quad \textit{for} \quad \textit{a.e.}\ x \in \mathbb{R}^N. \tag{31}$$

It follows that, for $N \geqslant 4$, using Young inequality and the fact that $w_\beta(x) \geqslant 1$, we get

$$\begin{aligned} |u(x)| &\leqslant C \frac{N-2}{2} \frac{1}{|x|^{\frac{2(N-1)}{N}}} \left(|u|_{\frac{N}{2}} + |\nabla u|_{\frac{N}{2}} \right) \quad \text{for} \quad \text{a.e.}\ x \in \mathbb{R}^N \\ &\leqslant C \frac{N-2}{2} \frac{1}{|x|^{\frac{2(N-1)}{N}}} \|u\| \quad \text{for} \quad \text{a.e.}\ x \in \mathbb{R}^N \\ &\leqslant C \frac{1}{|x|^{\frac{2(N-1)}{N}}} \|u\| \quad \text{for} \quad \text{a.e.}\ x \in \mathbb{R}^N. \end{aligned} \tag{32}$$

Now, we give the following Strauss compactness lemma [46].

LEMMA 3. *Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of continuous functions: $\mathbb{R}^n \to \mathbb{R}$. For $c>0$, let $y(c)=\sup\{|t| : t=P_n(s)$ for some n and s such that $|Q_n(s)|<c|P_n(s)|\}$.*
Assume the following conditions:

i) $y(c)<\infty$ for all $c>0$. (In other words, $\dfrac{P_n}{Q_n}\to 0$ uniformly as $P_n\to\infty$.)

ii) $(u_n)_n$ is a sequence of measurable functions: $\mathbb{R}^N\to\mathbb{R}$ such that $\sup_n \int_\Omega |Q_n(u_n(x))|\,dx<\infty$ for all bounded sets Ω.

iii) $P_n(u_n(x))\to v(x)$ for almost every $x\in\mathbb{R}^N$.

Then:

a) $\int_\Omega |P_n(u_n)-v|\,dx\to 0$.

b) Assume in addition that

iv) $P_n(s)=o(Q_n(s))$ as $s\to 0$ uniformly in n.
v) $u_n(x)\to 0$ as $|x|\to\infty$ uniformly in x and n.

Then $\int_{\mathbb{R}^N} |P_n(u_n)-v|\,dx\to 0$.

We denote by B the unit ball of $\mathbb{R}^N$ and consider the subspace

$$W^{2,\frac{N}{2}}_{0,rad}(B,w)=closure\left\{u\in C^\infty_{0,rad}(B)\;\;\Big|\;\;\int_B \log\left(\frac{e}{|x|}\right)^{\beta(\frac{N}{2}-1)}|\Delta u|^{\frac{N}{2}}dx<\infty\right\}.$$

We have the following results.

LEMMA 4. *Let u be a radially symmetric function in $C^\infty_{0,rad}(B)$. Then, we have*

(i) [55]

$$|u(x)|\leqslant\left(\frac{N}{\alpha_\beta}\left(\left|\log\left(\frac{e}{|x|}\right)\right|-1\right)\right)^{\frac{1}{\gamma}}\left(\int_B w_\beta(x)|\Delta u|^{\frac{N}{2}}dx\right)^{\frac{2}{N}}$$
$$\leqslant\left(\frac{N}{\alpha_\beta}\left(\left|\log\left(\frac{e}{|x|}\right)\right|-1\right)\right)^{\frac{1}{\gamma}}\|u\|.$$

(ii) $\int_B e^{|u(x)|^\gamma}dx<+\infty,\;\;\forall\;\;u\in W^{2,\frac{N}{2}}_{0,rad}(B)$.

(iii) The following embedding is continuous

$$E\hookrightarrow L^p(\mathbb{R}^N)\;\;\text{for all}\;\;p\geqslant\frac{N}{2}.$$

(vi) *E is compactly embedded in* $L^q(\mathbb{R}^N)$ *for all* $q \geqslant \frac{N}{2}$.

Proof. (i) see [48].

(ii) From (i) and using the identity $\log(\frac{e}{|x|}) - |\log(|x|)| = 1 \ \ \forall x \in B$, we get

$$|u(x)|^\gamma \leqslant \frac{1}{\alpha_\beta}\left|\log\left(\frac{e}{|x|}\right)\right| \|u\|^\gamma \leqslant \frac{N}{\alpha_\beta}\left(1+\left|\log(|x|)\right|\right) \ \|u\|^\gamma.$$

Hence, using the fact that the function $r \mapsto r^{N-1} e^{\frac{\|u\|^\gamma(1+|\log r|)}{\alpha_\beta}}$ is increasing, we get

$$\int_{|x|<1} e^{|u|^\gamma} dx \leqslant NV_N \int_0^1 r^{N-1} e^{\frac{N\|u\|^\gamma(1+|\log r|)}{\alpha_\beta}} dr \leqslant NV_N e^{\frac{N\|u\|^\gamma}{\alpha_\beta}} < +\infty.$$

Then (ii) follows by density.

(iii) Since $w_\beta(x) \geqslant 1$, then by Sobolev theorem, the following embedding are continuous

$$E \hookrightarrow W^{2,\frac{N}{2}}_{rad}(\mathbb{R}^N, w_\beta) \hookrightarrow W^{2,\frac{N}{2}}_{rad}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \ \ \forall q \geqslant \frac{N}{2}.$$

We assert that the embedding $E \longrightarrow L^q(\mathbb{R}^N)$ is compact. To do this, set $Q(s) = |s|^q$ and $P(s) = |s|^{q+\varepsilon_0} + |s|^{q-\varepsilon_0}$, where $0 < \varepsilon_0 < q - \frac{N}{2}$.

Clearly, $\dfrac{Q(s)}{P(s)} \longrightarrow 0$ as $|s| \longrightarrow +\infty$, and $\dfrac{Q(s)}{P(s)} \longrightarrow 0$ as $|s| \longrightarrow 0$. Let $(u_n)_n \in E$ be such that $u_n \hookrightarrow 0$ weakly in E and $u_n(x) \longrightarrow 0$ a.e. $x \in \mathbb{R}^N$. By the continuity of the embedding $E \hookrightarrow L^{q+\varepsilon_0}(\mathbb{R}^N)$ and $E \hookrightarrow L^{q-\varepsilon_0}(\mathbb{R}^N)$, we obtain that

$$\sup_n \int_{\mathbb{R}^N} |P(u_n)| < +\infty.$$

On the other hand, by (32), $u_n(x) \longrightarrow 0$ as $|x| \longrightarrow +\infty$, uniformly in $n \in \mathbb{N}$. Therefore, we can apply the compactness Strauss Lemma 3 to deduce that $Q(u_n) \longrightarrow 0$ strongly in $L^1(\mathbb{R}^N)$.

This concludes the lemma. □

REMARK 1. By Lemma 4 (ii) and (32), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u(x)|^p dx &= \int_B |u(x)|^p dx + \int_{\mathbb{R}^N\setminus B} |u(x)|^p dx \\ &\leqslant NV_N\|u\|^p \int_0^1 r^{N-1}(1+|\log r|)^{\frac{p}{\gamma}} dr + CNV_N\|u\|^p \int_1^\infty r^{N-1-2\frac{p(N-1)}{N}} dr \\ &\leqslant V_N\|u\|^p + CNV_N\|u\|^p \int_1^\infty r^{N-1-2\frac{p(N-1)}{N}} dr.\end{aligned}$$

The last integral is finite provided $p > \frac{N^2}{2(N-1)} > \frac{N}{2}$. The result of the previous lemma is thus partially found.

LEMMA 5. [31] *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ a continuous function. Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x,u_n)$ and $f(x,u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x,u_n)u_n|dx \leqslant C,$$

where C is a positive constant, then

$$f(x,u_n) \to f(x,u) \ \text{ in } \ L^1(\Omega).$$

4. Proof of Theorem 6

We begin by proving the first statement of Theorem 5. We have for all $u \in E$,

$$\int_{\mathbb{R}^N} (e^{|u|^\gamma} - 1)dx = \int_{|x|\geqslant 1} (e^{|u|^\gamma} - 1)dx + \int_{|x|<1} (e^{|u|^\gamma} - 1)dx. \tag{33}$$

On the one hand,

$$\int_{|x|\geqslant 1} (e^{|u|^\gamma} - 1)dx = \sum_{k=1}^{+\infty} \frac{1}{k!} \int_{|x|\geqslant 1} |u|^{\gamma k} dx. \tag{34}$$

From Lemma 2, we get

$$\begin{aligned} \int_{|x|\geqslant 1} |u|^{\gamma k} dx \leqslant NV_N \|u\|^{\gamma k} \int_1^{+\infty} \frac{1}{r^{1-N+2\gamma k \frac{N-1}{N}}} dr &= NV_N \|u\|^{\gamma k} \frac{1}{-N + 2\gamma k \frac{N-1}{N}} \\ &\leqslant NV_N \|u\|^{\gamma k} \frac{1}{-N + 2\gamma \frac{N-1}{N}}, \end{aligned} \tag{35}$$

$$\text{for all } \ k \geqslant 1; \ \ \frac{N^2 - 4N + 2}{N(N-2)} < \beta < 1.$$

Combining (34) and (35), we have

$$\int_{|x|\geqslant 1} (e^{|u|^\gamma} - 1)dx \leqslant \frac{NV_N}{-N + 2\gamma \frac{N-1}{N}} \sum_{k=1}^{+\infty} \frac{\|u\|^{\gamma k}}{k!} = \frac{NV_N}{-N + 2\gamma \frac{N-1}{N}} e^{\|u\|^\gamma} < +\infty. \tag{36}$$

Now we are going to estimate the second integral in (33). Set

$$v(x) = \begin{cases} u(x) - u(e_1), & 0 \leqslant |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases} \tag{37}$$

where $e_1 = (1,0,0,0,\dots,0) \in \mathbb{R}^N$. Clearly $v \in W^{2,\frac{N}{2}}_{0,rad}(B, w_\beta)$.

For all $\varepsilon > 0$, we have

$$|u|^\gamma = |v + u(e_1)|^\gamma \leqslant (1+\varepsilon)|v|^\gamma + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u(e_1)|^\gamma.$$

Then, from Lemma 4 (ii), we have

$$\begin{aligned}\int_{|x|<1} e^{|u|^\gamma}dx &\leqslant \int_{|x|<1} e^{(1+\varepsilon)|v|^\gamma} e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}|u(e_1)|^\gamma} dx \\ &\leqslant e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}|u(e_1)|^\gamma} \int_{|x|<1} e^{(1+\varepsilon)|v|^\gamma}dx < +\infty.\end{aligned} \tag{38}$$

Combining (33), (36), (38), (32) and Theorem 4, we conclude that

$$\int_{\mathbb{R}^N} (e^{|u|^\gamma} - 1)dx < +\infty, \quad \text{for all } u \in E.$$

This ends the proof of the first item.

By (36) we have

$$\sup_{u\in E,\|u\|\leqslant 1} \int_{|x|\geqslant 1} (e^{\alpha|u|^\gamma} - 1)dx \leqslant \sup_{u\in E,\|u\|\leqslant 1} \frac{NV_N}{-N+2\gamma\frac{N-1}{N}} e^{\|u\|^\gamma} \leqslant \frac{NV_N}{-N+2\gamma\frac{N-1}{N}} e. \tag{39}$$

On the other hand, by (38), (32) and using the radial lemma 4 (i), we get

$$\begin{aligned}\sup_{u\in E,\|u\|\leqslant 1} \int_{|x|\leqslant 1} (e^{\alpha|u|^\gamma} - 1)dx &\leqslant e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}|u(e_1)|^\gamma} \sup_{u\in E,\|u\|\leqslant 1} \int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^\gamma}dx \\ &\leqslant e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}(C\|u\|)^\gamma} \sup_{u\in E,\|u\|\leqslant 1} \int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^\gamma}dx \\ &\leqslant e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}(C)^\gamma} \sup_{u\in E,\|u\|\leqslant 1} \int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^\gamma}dx.\end{aligned} \tag{40}$$

Let $\alpha < \alpha_\beta$. Clearly, there exists $\varepsilon > 0$ such that $\alpha(1+\varepsilon) < \alpha_\beta$.

Do not forgot that

$$\begin{aligned}\|v\|^{\frac{N}{2}}_{W^{2,\frac{N}{2}}_{0,rad}(B)} &= \int_B |\Delta v|^{\frac{N}{2}} \left(\log\left(\frac{e}{|x|}\right)\right)^{\beta(\frac{N}{2}-1)} dx \\ &= \int_B |\Delta u|^{\frac{N}{2}} w_\beta(x)dx \leqslant \|u\|^{\frac{N}{2}} \leqslant 1.\end{aligned} \tag{41}$$

Then,

$$\begin{aligned}&\sup_{u\in E,\|u\|\leqslant 1} \int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^\gamma}dx \\ &\leqslant \sup\left\{\int_{|x|<1} e^{\alpha(1+\varepsilon)|v|^\gamma}dx, \;\; v \in W^{2,\frac{N}{2}}_{0,rad}(w,B), \;\; \|v\|_{W^{2,\frac{N}{2}}_{0,rad}(B)} \leqslant 1\right\}.\end{aligned}$$

So by (40), there exists $C(\beta) > 0$ such that

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{|x|<1} e^{\alpha|u|^\gamma}dx \leqslant e^{\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{1-\gamma}}}\right)^{1-\gamma}(C)^\gamma} C(\beta). \tag{42}$$

Combining (40) and (41), we get

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{|x|<1} (e^{\alpha|u|^\gamma}-1)dx < +\infty.$$

Furthermore

$$\int_{|x|\geqslant 1} (e^{\alpha|u|^\gamma}-1)dx = \sum_{k=1}^{+\infty}\frac{\alpha^k}{k!}\int_{|x|\geqslant 1}|u|^{\gamma k}dx. \tag{43}$$

Combining (35) and (43), we infer

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{|x|\geqslant 1} (e^{\alpha|u|^\gamma}-1)dx < +\infty. \tag{44}$$

It follows from (40) and (44) that

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{\mathbb{R}^N} (e^{\alpha|u|^\gamma}-1)dx < +\infty, \quad \text{for all } \alpha<\alpha_\beta.$$

Let's look at the case of $\alpha=\alpha_\beta$. It is clear that (39) is valid for $\alpha=\alpha_\beta$. So, we get

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{|x|\geqslant 1} (e^{\alpha_\beta|u|^\gamma}-1)dx < +\infty. \tag{45}$$

We are going to show that

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{|x|<1} (e^{\alpha_\beta|u|^\gamma}-1)dx < +\infty. \tag{46}$$

For this, we consider $u\in E$, $u\neq 0$ such that $\|u\|\leqslant 1$ and $\varepsilon>0$ such that

$$(1+\varepsilon)^{\frac{N}{2\gamma}} = \frac{1}{\left(\int_{|x|<1}|\Delta u|^{\frac{N}{2}}w_\beta(x)dx+\int_{|x|<1}|\nabla u|^{\frac{N}{2}}dx+\int_{|x|<1}|v|^{\frac{N}{2}}dx\right)}.$$

Moreover, we have a similar inequality to (38) that is

$$\begin{aligned}\int_{|x|<1} e^{\alpha_\beta|u|^\gamma}dx &\leqslant \int_{|x|<1} e^{\alpha_\beta(1+\varepsilon)|v|^\gamma} e^{\alpha_\beta\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma}|u(e_1)|^\gamma}dx \\ &\leqslant e^{\alpha_\beta\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma}|u(e_1)|^\gamma}\int_{|x|<1} e^{(1+\varepsilon)\alpha_\beta|v|^\gamma}dx,\end{aligned} \tag{47}$$

where v is given by (37). On the other hand, we have from radial Lemma 2,

$$
\begin{aligned}
|u(e_1)|^\gamma &\leqslant C_4\Big(\int_{|x|\geqslant 1}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u+u(e_1)-u(e_1)|^{\frac{N}{2}}\big)dx\Big)^{\frac{2\gamma}{N}}\\
&\leqslant C_4\Big(\int_{\mathbb{R}^N}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u|^{\frac{N}{2}}\big)dx\\
&\quad -\int_{|x|<1}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u-u(e_1)+u(e_1)|^{\frac{N}{2}}\big)dx\Big)^{\frac{2\gamma}{N}}+C_3\\
&\leqslant C_4\Big(1-\int_{|x|<1}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u-u(e_1)|^{\frac{N}{2}}+|u(e_1)|^{\frac{N}{2}}\big)dx\Big)^{\frac{2\gamma}{N}}\\
&\leqslant C_4\Big(1-C_5-\int_{|x|<1}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u-u(e_1)|^{\frac{N}{2}}\big)dx\Big)^{\frac{2\gamma}{N}}\\
&\leqslant C_4\Big(1-\int_{|x|<1}\big(w_\beta|\Delta u|^{\frac{N}{2}}+|\nabla u|^{\frac{N}{2}}+|u-u(e_1)|^{\frac{N}{2}}\big)dx\Big)^{\frac{2\gamma}{N}}\\
&\leqslant C_4\left(1-\frac{1}{(1+\varepsilon)^{\frac{N}{2\gamma}}}\right)^{\frac{2\gamma}{N}}. \qquad (48)
\end{aligned}
$$

Also,

$$\int_{|x|<1}|(1+\varepsilon)^\gamma\Delta v|^{\frac{N}{2}}w_\beta(x)dx+\int_{|x|<1}|(1+\varepsilon)^\gamma\nabla v|^{\frac{N}{2}}dx+\int_{|x|<1}|(1+\varepsilon)^\gamma v|^{\frac{N}{2}}dx=1.$$

Then, by Theorem 4, there exists $C>0$ such that

$$\int_{|x|<1}e^{(1+\varepsilon)\alpha_\beta|v|^\gamma}dx<C. \qquad (49)$$

Using (48), we get,

$$\int_{|x|<1}e^{\alpha_\beta|u|^\gamma}dx\leqslant C\exp\left(\alpha_\beta\left(1-\frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma}\left(1-\frac{1}{(1+\varepsilon)^{\frac{N}{2\gamma}}}\right)^{\frac{2\gamma}{N}}\right).$$

But the function $\xi: t\to\left(1-\frac{1}{t^{\frac{1}{\gamma-1}}}\right)^{1-\gamma}\left(1-\frac{1}{t^{\frac{N}{2\gamma}}}\right)^{\frac{2\gamma}{N}}$ defined on $(1,+\infty)$ is decreasing and verifies $\lim_{t\to+\infty}\xi(t)=1$. Hence, ξ is bounded and therefore we get

$$\int_{|x|<1}e^{\alpha_\beta|u|^\gamma}dx<+\infty. \qquad (50)$$

So, (46) holds.

In the next step, we show that if $\alpha > \alpha_\beta$, then the supremum is infinite. Now, we will use particular functions [48], namely the Adams' functions. We consider the sequence defined for all $n \geqslant 3$ by

$$w_n(x) = \begin{cases} \left(\dfrac{\log n}{\alpha_\beta}\right)^{\frac{1}{\gamma}} - \dfrac{N n^{\frac{2(1-\beta)}{N}} |x|^{2(1-\beta)}}{2(\alpha_\beta)^{\frac{1}{\gamma}} (\log n)^{\frac{2+(N-2)\beta}{N}}} \\ \quad + \dfrac{N}{2(\alpha_\beta)^{\frac{1}{\gamma}} (\log n)^{\frac{2+(N-2)\beta}{N}}} & \text{if } 0 \leqslant |x| \leqslant \frac{e}{\sqrt[N]{n}} \\ \dfrac{N^{1-\beta}}{\alpha_\beta^{\frac{1}{\gamma}} (\log(n))^{\frac{2(1-\beta)}{N}}} \left(\log\left(\dfrac{e}{|x|}\right)\right)^{1-\beta} & \text{if } \frac{e}{\sqrt[N]{n}} \leqslant |x| \leqslant \frac{1}{2} \\ \zeta_n & \text{if } |x| \geqslant \frac{1}{2} \end{cases} \tag{51}$$

where $\zeta_n \in C^\infty_{0,rad}(B)$ is such that

$$\zeta_n\left(\frac{1}{2}\right) = \frac{N^{1-\beta}}{\alpha_\beta^{\frac{1}{\gamma}} (\log n)^{\frac{2(1-\beta)}{N}}} (\log 2e)^{1-\beta},$$

$$\frac{\partial \zeta_n}{\partial r}\left(\frac{1}{2}\right) = \frac{-2(1-\beta)N^{1-\beta}}{\alpha_\beta^{\frac{1}{\gamma}} (\log n)^{\frac{2(1-\beta)}{N}}} (\log(2e))^{-\beta}$$

$$\zeta_n(1) = \frac{\partial \zeta_n}{\partial r}(1) = 0$$

and ξ_n, $\nabla \xi_n$, $\Delta \xi_n$ are all $o\left(\dfrac{1}{[\log n]^{\frac{1}{\gamma}}}\right)$. Here, $\dfrac{\partial \zeta_n}{\partial r}$ denotes the first derivative of ζ_n in the radial variable $r = |x|$.

Let $v_n(x) = \dfrac{w_n}{\|w_n\|}$. We have, $v_n \in E$, $\|v_n\|^{\frac{N}{2}} = 1$.

We compute $\Delta w_n(x)$, we get

$$\Delta w_n(x) = \begin{cases} \dfrac{-N(1-\beta)(4-2\beta)(n\sqrt[N]{n})^{\frac{2(1-\beta)}{N}} |x|^{-2\beta}}{\alpha_\beta^{\frac{1}{\gamma}} (\log n)^{\frac{(N-2)\beta+2}{N}}} & \text{if } 0 \leqslant |x| \leqslant \frac{e}{\sqrt[N]{n}} \\ \dfrac{-(1-\beta)N^{(1-\beta)}}{|x|^2} \dfrac{\left(\log(\frac{e}{|x|})\right)^{-\beta} \left((N-2) + \beta\left(\log \frac{e}{|x|}\right)^{-1}\right)}{\alpha_\beta^{\frac{1}{\gamma}} (\log(n))^{\frac{2(1-\beta)}{N}}} & \text{if } \frac{e}{\sqrt[N]{n}} \leqslant |x| \leqslant \frac{1}{2} \\ \Delta \zeta_n & \text{if } |x| \geqslant \frac{1}{2}. \end{cases}$$

So,

$$\|\Delta w_n\|_{\frac{N}{2},w}^{\frac{N}{2}} = \underbrace{NV_N \int_0^{\frac{e}{\sqrt[N]{n}}} r^{N-1} |\Delta w_n(x)|^{\frac{N}{2}} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)} dr}_{I_1}$$

$$+ \underbrace{NV_N \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{N-1} |\Delta w_n(x)|^{\frac{N}{2}} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)} dr}_{I_2}$$

$$+ \underbrace{NV_N \int_{\frac{1}{2}}^{1} r^{N-1} |\Delta \zeta_n(x)|^{\frac{N}{2}} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)} dr + NV_N \int_1^{+\infty} |\triangle \zeta_n|^{\frac{N}{2}} \chi(r) r^{N-1} dr}_{I_3}.$$

By using integration by parts, we obtain,

$$\begin{aligned}
I_1 &= NV_N \frac{(\sqrt[N]{n})^{1-\beta} N^{\frac{N}{2}} (1-\beta)^{\frac{N}{2}} (4-2\beta)^{\frac{N}{2}}}{N(1-\beta)\big(\alpha_\beta\big)^{\frac{N}{2\gamma}} \big(\log(n)\big)^{\frac{(N-2)\beta+2}{2}}} \left[r^{N(1-\beta)} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)} \right]_0^{\frac{e}{\sqrt[N]{n}}} \\
&\quad + NV_N \frac{(\sqrt[N]{n})^{1-\beta} \beta(\frac{N}{2}-1) N^{\frac{N}{2}} (1-\beta)^{\frac{N}{2}} (4-2\beta)^{\frac{N}{2}}}{2\big(\alpha_\beta\big)^{\frac{N}{2\gamma}} (\log(n))^{\frac{(N-2)\beta+2}{2}}} \\
&\quad \times \int_0^{\frac{e}{\sqrt[N]{n}}} r^{N(1-\beta)-1} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)-1} dr \\
&= o\Big(\frac{1}{\log(n)}\Big).
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 =& NV_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{\big(\alpha_\beta\big)^{\frac{N}{2\gamma}} \big(\log(e\ \sqrt[N]{n})\big)^{1-\beta}} \\
& \times \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} \frac{1}{r} \Big(\log \frac{e}{r}\Big)^{-\frac{\beta N}{2}} \Big((N-2) + \beta \Big(\log \frac{e}{r}\Big)^{-1}\Big)^{\frac{N}{2}} \Big(\log \frac{e}{r}\Big)^{\beta(\frac{N}{2}-1)} dr \\
=& NV_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{\big(\alpha_\beta\big)^{\frac{N}{2\gamma}} \big(\log(n)\big)^{1-\beta}} (N-2)^{\frac{N}{2}} \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} \frac{1}{r} \Big(\log \frac{e}{r}\Big)^{-\beta} \Big(1 + o\Big(\log \frac{e}{r}\Big)^{-1}\Big)^{\frac{N}{2}} dr \\
=& NV_N \frac{(1-\beta)^{\frac{N}{2}} N^{\frac{N(1-\beta)}{2}}}{\big(\alpha_\beta\big)^{\frac{N}{2\gamma}} \big(\log(n)\big)^{1-\beta}} (N-2)^{\frac{N}{2}} \\
& \times \left(\int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} \frac{1}{r} \Big(\log \frac{e}{r}\Big)^{-\beta} dr + \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} \frac{1}{r} \Big(\log \frac{e}{r}\Big)^{-\beta} o\Big(\log \frac{e}{r}\Big)^{-1} dr \right)
\end{aligned}$$

$$
\begin{aligned}
=&NV_N\frac{(1-\beta)^{\frac{N}{2}}N^{\frac{N(1-\beta)}{2}}}{\left(\alpha_\beta\right)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}}(N-2)^{\frac{N}{2}}\left[\frac{1}{1-\beta}\left(\log\frac{e}{r}\right)^{1-\beta}\right]_{\frac{1}{2}}^{\frac{e}{\sqrt[N]{n}}}\\
&-NV_N\frac{(1-\beta)^{\frac{N}{2}}N^{\frac{N(1-\beta)}{2}}}{\left(\alpha_\beta\right)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}}(N-2)^{\frac{N}{2}}\left(\int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}}\frac{1}{r}\left(\log\frac{e}{r}\right)^{-\beta}o\left(\log\frac{e}{r}\right)^{-1}dr\right)\\
=&1+o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{1-\beta}}\right)
\end{aligned}
$$

and $I_3=o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{\frac{2}{\gamma}}}\right)$. Then $\|\Delta w_n\|^{\frac{N}{2}}_{\frac{N}{2},w}=1+o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{\frac{N}{\gamma}}}\right)$.

In the sequel we prove the following key lemma.

LEMMA 6. *The Adams' function given by* (51) *verifies* $\lim_{n\to+\infty}\|w_n\|^{\frac{N}{2}}=1$.

Proof. We have

$$
\begin{aligned}
\|w_n\|^{\frac{N}{2}}=&\int_{\mathbb{R}^N}w_\beta(x)|\Delta w_n|^{\frac{N}{2}}dx+\int_{\mathbb{R}^N}|\nabla w_n|^{\frac{N}{2}}dx+\int_{\mathbb{R}^N}|w_n|^{\frac{N}{2}}dx\\
=&1+o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{\frac{2}{\gamma}}}\right)+\int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}}|w|_n^{\frac{N}{2}}dx+\int_{\frac{e}{\sqrt[N]{n}}\leqslant|x|\leqslant\frac{1}{2}}|w_n|^{\frac{N}{2}}dx\\
&+\int_{|x|\geqslant\frac{1}{2}}|\zeta|_n^{\frac{N}{2}}dx\\
&+\underbrace{\int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}}|\nabla w_n|^{\frac{N}{2}}dx}_{I_1'}+\underbrace{\int_{\frac{e}{\sqrt[N]{n}}\leqslant|x|\leqslant\frac{1}{2}}|\nabla w_n|^{\frac{N}{2}}dx}_{I_2'}+\underbrace{\int_{|x|\geqslant\frac{1}{2}}|\nabla\zeta_n|^{\frac{N}{2}}dx}_{I_3'}.
\end{aligned}
$$

We have,

$$
\begin{aligned}
I_1'=&NV_N\frac{N^{\frac{N}{2}}(1-\beta)^{\frac{N}{2}}}{\alpha_\beta^{\frac{N}{2\gamma}}\left(\log(n)\right)^{\frac{2+(N-2)\beta)}{2}}}\int_0^{\frac{e}{n\sqrt[N]{n}}}r^{N(\frac{3}{2}-\beta)-1}dr\\
=&NV_N\frac{N^{\frac{N}{2}}(1-\beta)^{\frac{N}{2}}}{\alpha_\beta^{\frac{N}{2\gamma}}\left(\log(n)\right)^{\frac{2+(N-2)\beta)}{2}}}\left[\frac{r^{N(\frac{3}{2}-\beta)}}{N(\frac{3}{2}-\beta)}\right]_0^{\frac{e}{n\sqrt[N]{n}}}\\
=&V_N\frac{N^{\frac{N}{2}}(1-\beta)^{\frac{N}{2}}}{\alpha_\beta^{\frac{N}{2\gamma}}n^{(N+1)(\frac{3}{2}-\beta)}(\frac{3}{2}-\beta)\log(n))^{\frac{N(\gamma-1)}{2\gamma}}}\\
=&o\left(\frac{1}{n^{(\frac{3}{2}-\beta)(N+1)}\log n}\right).
\end{aligned}
$$

Also, using the fact that the function $r\mapsto r^{\frac{N}{2}-1}\left(\log\frac{e}{r}\right)^{-\frac{N}{2}\beta}$ is increasing on $[0,1]$, we get

$$I_2' = NV_N N^{(1-\beta)\frac{N}{2}} \frac{(1-\beta)^{\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}} \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{\frac{N}{2}-1}\left(\log\frac{e}{r}\right)^{-\frac{N}{2}\beta} dr$$
$$\leqslant NV_N N^{(1-\beta)\frac{N}{2}} \frac{(1-\beta)^{\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}} \left(\frac{1}{2}\right)^{\frac{N}{2}-1}\left(\log 2e\right)^{-\frac{N}{2}\beta}$$
$$= o\left(\frac{1}{[\log(n)]^{1-\beta}}\right)$$

and $I_3' = o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{\frac{2}{\gamma}}}\right)$.

For $|x| \leqslant \frac{e}{\sqrt[N]{n}}$, we have

$$|w_n|^{\frac{N}{2}} \leqslant \left(\frac{\log(n)}{\alpha_\beta}\right)^{\frac{1}{\gamma}} + \left(\frac{N}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\log n)^{\frac{2+(N-2)\beta}{N}}}\right)^{\frac{N}{2}}.$$

Then,

$$\int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}} |w_n|^{\frac{N}{2}} dx$$
$$\leqslant \left(\left(\frac{\log(n)}{\alpha_\beta}\right)^{\frac{1}{\gamma}} + \frac{N}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\log n)^{\frac{2+(N-2)\beta}{N}}}\right)^{\frac{N}{2}} NV_N \int_0^{\frac{e}{\sqrt[N]{n}}} r^{N-1} dr = o_n(1).$$

Also,

$$\int_{\frac{e}{\sqrt[N]{n}}\leqslant|x|\leqslant\frac{1}{2}} |w_n|^{\frac{N}{2}} dx = \frac{NV_N N^{(1-\beta)\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}} \int_{\frac{e}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{N-1}\left(\log\left(\frac{e}{r}\right)\right)^{\frac{N(1-\beta)}{2}} dr.$$

Using the fact that the function $r \mapsto r^{N-1}\left(\log\frac{e}{r}\right)^{\frac{N(1-\beta)}{2}}$ is increasing on $[0,1]$, we obtain

$$\int_{\frac{e}{\sqrt[N]{n}}\leqslant|x|\leqslant\frac{1}{2}} |w_n|^{\frac{N}{2}} dx \leqslant \frac{NV_N N^{(1-\beta)\frac{N}{2}}}{(\alpha_\beta)^{\frac{N}{2\gamma}}\left(\log(n)\right)^{1-\beta}} \frac{1}{2^{N-1}}\left(\log(2e)\right)^{\frac{2(1-\beta)}{N}} = o_n(1).$$

Finaly,

$$\int_{|x|\geqslant\frac{1}{2}} |w_n|^{\frac{N}{2}} dx = \int_{|x|\geqslant\frac{1}{2}} |\zeta_n|^{\frac{N}{2}} dx = o_n(1).$$

Then, $\| w_n\|^{\frac{N}{2}} = 1 + o\left(\frac{1}{(\log e\ \sqrt[N]{n})^{\frac{2}{\gamma}}}\right)$. The Lemma is proved. □

Now, let $v_n(x) = \dfrac{w_n}{\|w_n\|}$. From the definition of w_n, it is easy to see that

$$-\frac{Nn^{\frac{2(1-\beta)}{N}}|x|^{2(1-\beta)}}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\log n)^{\frac{2+(N-2)\beta}{N}}} + \frac{N}{2(\alpha_\beta)^{\frac{1}{\gamma}}(\log n)^{\frac{2+(N-2)\beta}{N}}} \geqslant 0 \text{ for all } 0 \leqslant |x| \leqslant \frac{1}{n\sqrt[N]{n}}.$$

Then, for all $0 \leqslant |x| \leqslant \frac{e}{\sqrt[N]{n}}$, $|w_n|^{\frac{N}{2}} \geqslant \left(\dfrac{\log(n)}{\alpha_\beta}\right)^{\frac{N}{2\gamma}}$. Let $\overline{\alpha} = \frac{\alpha}{\alpha_\beta}$, we have

$$\begin{aligned}
\sup_{u\in E,\|u\|\leqslant 1}\int_{\mathbb{R}^N}(e^{\alpha|u|^\gamma}-1)dx &\geqslant \lim_{n\longrightarrow+\infty}\int_{|x|\leqslant\frac{e}{\sqrt[N]{n}}}(e^{\alpha|v_n|^\gamma}-1)dx\\
&\geqslant \lim_{n\longrightarrow+\infty} NV_N\int_0^{\frac{e}{\sqrt[N]{n}}}\left(r^{N-1}e^{\overline{\alpha}\log(n)} - r^{N-1}\right)dr\\
&\geqslant \lim_{n\longrightarrow\infty} e^N V_N\left(e^{\log n(\overline{\alpha}-1)}\right) = +\infty \text{ if } \overline{\alpha} > 1.
\end{aligned}$$

Then,

$$\sup_{u\in E,\|u\|\leqslant 1}\int_{\mathbb{R}^N}(e^{\alpha|u|^\gamma}-1)dx = +\infty \;\; \forall \;\; \alpha > \alpha_\beta.$$

5. A Lions-type compactness concentration Lemma

In the sequel, we prove a concentration compactness result of Lions type.

LEMMA 7. *Let $(u_k)_k$ be a sequence in E. Suppose that, $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in E, $u_k(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, $\nabla u_k(x) \to \nabla u(x)$ a.e. $x \in \mathbb{R}^N$, $\Delta u_k(x) \to \Delta u(x)$ a.e. $x \in \mathbb{R}^N$ and $u \not\equiv 0$. Then*

$$\sup_k \int_{\mathbb{R}^N}(e^{p\;\alpha_\beta|u_k|^\gamma}-1)dx < +\infty, \;\; \textit{where} \;\; \alpha_\beta = N[(N-2)NV_N]^{\frac{2}{(N-2)(1-\beta)}}(1-\beta)^{\frac{1}{(1-\beta)}},$$

for all $1 < p < U(u)$, where $U(u)$ is given by:

$$U(u) := \begin{cases} \dfrac{1}{(1-\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}} & \text{if} \;\; \|u\| < 1 \\ +\infty & \text{if} \;\; \|u\| = 1. \end{cases}$$

Moreover, the last inequality is sharp in the sense that there exist a sequence $(u_k) \subset E$ and a function $u \in E\setminus\{0\}$ such that $\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in E and

$$\sup_k \int_{\mathbb{R}^N}(e^{p\;\alpha_\beta|u_k|^\gamma}-1)dx = +\infty \;\; \textit{for all} \quad p > U(u).$$

Proof. For $a,\, b \in \mathbb{R}$, $q > 1$. If q' its conjugate i.e. $\frac{1}{q}+\frac{1}{q'} = 1$ we have, by Young inequality, that

$$(e^{a+b}-1) \leqslant \frac{1}{q}(e^{qa}-1) + \frac{1}{q'}(e^{q'b}-1).$$

Also, we have

$$(1+a)^q \leqslant (1+\varepsilon)a^q + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}}\right)^{1-q}, \quad \forall a \geqslant 0, \ \forall \varepsilon > 0 \ \forall q > 1. \tag{52}$$

So, we get

$$\begin{aligned} |u_k|^\gamma &= |u_k - u + u|^\gamma \\ &\leqslant (|u_k - u| + |u|)^\gamma \\ &\leqslant (1+\varepsilon)|u_k - u|^\gamma + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u|^\gamma \end{aligned}$$

which implies that

$$\begin{aligned} \int_{\mathbb{R}^N} \left(e^{p\ \alpha_\beta |u_k|^\gamma} - 1\right) dx \leqslant \frac{1}{q} \int_{\mathbb{R}^N} \left(e^{pq\ \alpha_\beta (1+\varepsilon)|u_k - u|^\gamma} - 1\right) dx \\ + \frac{1}{q'} \int_{\mathbb{R}^N} \left(e^{pq'\ \alpha_\beta \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u|^\gamma} - 1\right) dx, \end{aligned}$$

for any $p > 1$. From Theorem 5 (i), the last integral is finite.

To finish the proof, we need to prove that for all p such that $1 < p < U(u)$,

$$\sup_k \int_{\mathbb{R}^N} \left(e^{pq\ \alpha_\beta (1+\varepsilon)|u_k - u|^\gamma} - 1\right) dx < +\infty, \tag{53}$$

for some $\varepsilon > 0$ and $q > 1$.

In what follows, we assume that $\|u\| < 1$ and in the case that $\|u\| = 1$, the proof is similar.

When

$$\|u\| < 1$$

and

$$p < \frac{1}{(1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}(1+\nu) < 1.$$

On the other hand, by Brezis-Lieb's Lemma [9] we have

$$\|u_k - u\|^{\frac{N}{2}} = \|u_k\|^{\frac{N}{2}} - \|u\|^{\frac{N}{2}} + o(1) \ \text{ where } \ o(1) \to 0 \ \text{ as } \ k \to +\infty.$$

Then,

$$\|u_k - u\|^{\frac{N}{2}} = 1 - \|u\|^{\frac{N}{2}} + o(1),$$

and so

$$\lim_{k\to+\infty}\|u_k-u\|^{\gamma}=(1-\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}.$$

Therefore, for every $\varepsilon>0$, there exists $k_\varepsilon\geqslant 1$ such that

$$\|u_k-u\|^{\gamma}\leqslant(1+\varepsilon)(1-\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}},\ \ \forall\ \ k\geqslant k_\varepsilon.$$

If we take $q=1+\varepsilon$ with $\varepsilon=\sqrt[3]{1+\nu}-1$, then $\forall k\geqslant k_\varepsilon$, we have

$$pq(1+\varepsilon)\|u_k-u\|^{\gamma}\leqslant 1.$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}^N}\left(e^{pq\ \alpha_\beta(1+\varepsilon)|u_k-u|^{\gamma}}-1\right)dx &\leqslant \int_{\mathbb{R}^N}\left(e^{(1+\varepsilon)pq\ \alpha_\beta(\frac{|u_k-u|}{\|u_k-u\|})^{\gamma}\|u_k-u\|^{\gamma}}-1\right)dx\\ &\leqslant \int_{\mathbb{R}^N}\left(e^{\ \alpha_\beta(\frac{|u_k-u|}{\|u_k-u\|})^{\gamma}}-1\right)dx\\ &\leqslant \sup_{\|u\|\leqslant 1}\int_{\mathbb{R}^N}\left(e^{\ \alpha_\beta|u|^{\gamma}}-1\right)dx<+\infty.\end{aligned}$$

Now, (53) follows from (15). This complete the proof of the first statement.

Now, let the sequence w_n given by (51). Let also $u\in C_0^\infty(\mathbb{R}^N$ be a radial function such that $u(x)=0$, for all $|x|\leqslant\frac{1}{2}$ or $|x|\geqslant 2$ and $\|u\|<1$. Set $u_n=w_n+u$. Since Δw_n and Δu have disjoint supports and ∇w_n and ∇ have also disjoint support then we get

$$\begin{aligned}\|u_n\|^{\frac{N}{2}}=&\int_{\mathbb{R}^N}w_\beta|\Delta w_n|^{\frac{N}{2}}+\int_{\mathbb{R}^N}w_\beta|\Delta u|^{\frac{N}{2}}+\int_{\mathbb{R}^N}|\nabla w_n|^{\frac{N}{2}}\\ &+\int_{\mathbb{R}^N}|\nabla u|^{\frac{N}{2}}+\int_{\mathbb{R}^N}|w_n|^{\frac{N}{2}}+\int_{\mathbb{R}^N}|u_n|^{\frac{N}{2}}.\end{aligned}$$

From Lemma 6 we know that $\lim\limits_{n\to+\infty}\|w_n\|^{\frac{N}{2}}=1$. Then

$$\|u_n\|^{\frac{N}{2}}\to 1+\|u\|^{\frac{N}{2}}$$

and

$$\|u_n\|^{\gamma}\to\left(1+\|u\|^{\frac{N}{2}}\right)^{\frac{2\gamma}{N}}.$$

It's obvious that $w_n\hookrightarrow 0$ weakly in E. Let $\alpha\geqslant\alpha_\beta U(u)$. we have

$$\begin{aligned}\int_{\mathbb{R}^N}(e^{\frac{|u_n|^{\gamma}}{\|u_n\|^{\gamma}}}-1)dx &\geqslant \int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}}(e^{\frac{|u_n|^{\gamma}}{\|u_n\|^{\gamma}}}-1)dx\\ &\geqslant\left(\exp\left(\alpha\left(\frac{\log(n)}{\alpha_\beta\|u_n\|^{\gamma}}\right)\right)-1\right)NV_N\int_0^{\frac{e}{\sqrt[N]{n}}}r^{N-1}dr\\ &\geqslant V_N\frac{1}{n}\left(\exp\left(\alpha\left(\frac{\log(n)}{\alpha_\beta\|u_n\|^{\gamma}}\right)\right)-1\right).\end{aligned}\tag{54}$$

Also,

$$\frac{\alpha}{\alpha_\beta \|u_n\|^\gamma} \to \frac{\alpha}{\alpha_\beta (1+\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}}$$

and

$$U(u) = \frac{1}{(1-\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}} > (1+\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}.$$

Having in mind that $\alpha \geqslant \alpha_\beta U(u)$, we get

$$\frac{\alpha}{\alpha_\beta (1+\|u\|^{\frac{N}{2}})^{\frac{2\gamma}{N}}} > 1. \tag{55}$$

Therefore, passing to the limit in (54) and using (55), we get,

$$V_N \frac{1}{n}\left(\exp\left(\alpha\left(\frac{\log(n)}{\alpha_\beta \|u_n\|^\gamma}\right)\right) - 1\right) = e^{\log n\left(\frac{\alpha}{\alpha_\beta \|u_n\|^\gamma}-1\right)} - \frac{V_N}{n} \to +\infty \text{ as } n \to +\infty.$$

This closes the proof of the lemma.

6. The variational formulation for the problem (16)

Note that, by the hypothesis (H_3), for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$|f(t)| \leqslant |t|^{\frac{N}{2}-1}, \ \forall\, 0 < |t| \leqslant \delta_0. \tag{56}$$

Moreover, since f is critical at infinity, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall t \geqslant C_\varepsilon \ \ |f(t)| \leqslant \varepsilon \exp(\, a|t|^\gamma - 1) \ \text{ with } \ a > \alpha_0. \tag{57}$$

In particular, we obtain for $q \geqslant 2$,

$$|f(t)| \leqslant \frac{\varepsilon}{C_\varepsilon^{q-1}} |t|^{q-1} \exp(a\,|t|^\gamma - 1) \ \text{ with } \ a > \alpha_0. \tag{58}$$

Hence, using (56), (57), (58) and the continuity of f, for every $\varepsilon > 0$, for every $q > N$, there exists a positive constant C such that

$$|f(t)| \leqslant \varepsilon |t|^{\frac{N}{2}-1} + C|t|^{q-1}\left(e^{a\,|t|^\gamma} - 1\right), \qquad \forall\, t \in \mathbb{R}, \ \forall\, a > \alpha_0. \tag{59}$$

It follows from (59) and (H_2), that for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(t) \leqslant \varepsilon |t|^{\frac{N}{2}} + C|t|^q\left(e^{a\,|t|^\gamma} - 1\right), \qquad \text{for all } t, \ \forall\, a > \alpha_0. \tag{60}$$

So, by (15) and (60) the functional $\mathscr{J}$ given by (19), is well defined. Moreover, by standard arguments, $\mathscr{J} \in C^1(E,\mathbb{R})$. $\square$

6.1. The mountain pass geometry of the energy

In the sequel, we prove that the functional $\mathscr{J}$ has a mountain pass geometry.

PROPOSITION 1. *Assume that the hypothesis* $(H_1),(H_2)$ *and* (H_3) *hold. Then,*

(i) there exist ρ, $\beta_0>0$ *such that* $\mathscr{J}(u)\geqslant\beta_0$ *for all* $u\in E$ *with* $\|u\|=\rho$.

(ii) Let $\phi_1\in E\backslash\{0\}$. *Then,* $\mathscr{J}(t\phi_1)\to-\infty$, *as* $t\to+\infty$.

Proof. From (59), for all $\varepsilon>0$, there exists $C>0$ such that

$$F(t)\leqslant\varepsilon|t|^{\frac{N}{2}}+C|t|^q\big(e^{a\,t^\gamma}-1\big),\qquad\text{for all }\ t\in\mathbb{R}.$$

Then, using the last inequality, we get

$$\mathscr{J}(u)\geqslant\frac{2}{N}\|u\|^{\frac{N}{2}}-\varepsilon\int_{\mathbb{R}^N}|u|^{\frac{N}{2}}dx-C\int_{\mathbb{R}^N}|u|^q\big(e^{a\,u^\gamma}-1\big)\,dx.$$

From the Hölder inequality and using the following inequality

$$\big(e^s-1\big)^\nu\leqslant e^{\nu s}-1,\ \ \forall\ \ s\geqslant0\ \ \forall\nu\geqslant1,$$

we obtain

$$\mathscr{J}(u)\geqslant\frac{2}{N}\|u\|^{\frac{N}{2}}-\varepsilon\int_{\mathbb{R}^N}|u|^{\frac{N}{2}}dx-C\Big(\int_{\mathbb{R}^N}\big(e^{2a\,|u|^\gamma}-1\big)dx\Big)^{\frac{1}{2}}\|u\|_{2q}^q. \tag{61}$$

From the Theorem 1, if we choose $u\in E$ such that

$$2a\|u\|^\gamma\leqslant\alpha_\beta, \tag{62}$$

we get

$$\int_{\mathbb{R}^N}\big(e^{2a\,|u|^\gamma}-1\big)dx=\int_{\mathbb{R}^N}\big(e^{2a\,\|u\|^\gamma(\frac{|u|}{\|u\|})^\gamma}\big)-1\big)dx<+\infty.$$

On the other hand from Sobolev embedding Lemma 4, there exist constants $C_1>0$ and $C_2>0$ such that $\|u\|_{2q}\leqslant C_2\|u\|$ and $\|u\|_{\frac{N}{2}}^{\frac{N}{2}}\leqslant C_1\|u\|^{\frac{N}{2}}$. So,

$$\mathscr{J}(u)\geqslant\frac{2}{N}\|u\|^{\frac{N}{2}}-\varepsilon C_1\|u\|^{\frac{N}{2}}-C\|u\|^q=\|u\|^{\frac{N}{2}}\Big(\frac{2}{N}-\varepsilon C_1-C_2\|u\|^{q-\frac{N}{2}}\Big),$$

for all $u\in E$ satisfying (62). Since $q>N$, we can choose $\rho=\|u\|\leqslant(\frac{\alpha_\beta}{2a})^{\frac{1}{\gamma}}$ and for ε such that $\dfrac{2}{NC_1}>\varepsilon$, there exists $\beta_0=\rho^{\frac{N}{2}}\Big(\dfrac{2}{N}-\varepsilon C_1-C_2\rho^{q-\frac{N}{2}}\Big)>0$ with $\mathscr{J}(u)\geqslant\beta_0>0$.

(ii) Let $\phi_1\in E\backslash\{0\}$, $\|\phi_1\|=1$. We define the function

$$\varphi(t)=\mathscr{J}(t\phi_1)=\frac{2}{N}t^{\frac{N}{2}}\|\phi_1\|^{\frac{N}{2}}-\int_{\mathbb{R}^4}F(t\phi_1)dx.$$

By (H_2) and (H_3) there exist two positive constants C_1 and C_2 such that

$$F(t) \geqslant C_1 t^{\theta} - C_2 t^{\frac{N}{2}}, \ \forall\, t \in \mathbb{R}.$$

Hence, since $\theta > N$,

$$\varphi(t) = \mathscr{J}(t\phi_1) \leqslant \frac{2}{N} t^{\frac{N}{2}} \|\phi_1\|^{\frac{N}{2}} - C_1 |t|^{\theta} \|\phi_1\|_{\theta} + C_2 |t|^{\frac{N}{2}} \|\phi_1\|_{\frac{N}{2}}^{\frac{N}{2}} \to -\infty, \ \text{as} \ t \to +\infty.$$

We take $e = \bar{t}\phi_1$, for some $\bar{t} > 0$ large enough. □

7. Proof of Theorem 6 and Theorem 7

7.1. Palais-Smale sequence

We begin by the Palais-Smale sequence that is

LEMMA 8. *Assume that (H_1), (H_2), (H_3) and (H_4). If $(u_n) \subset E$ is a (PS) sequence and $u \in E$ is a weak limit, then*

(i) (u_n) is bounded in E.

(ii)

$$\int_{\mathbb{R}^N} F(u_n)dx \to \int_{\mathbb{R}^N} F(u)dx.$$

Proof. For the first item, we can see the proof in the proposition 2 below.
Now, we claim that

$$f(u_n) \to f(u) \ \text{ in } \ L^1(B_R), \ \text{ for any } \ R > 0. \tag{63}$$

Since $u_n \to u$ in $L^{\frac{N}{2}}(\mathbb{R}^N)$, then $u_n \to u$ in $L^{\frac{N}{2}}(B_R)$. Furthermore, $\int_{B_R} f(u_n)u_n dx \leqslant C$. It follows from Lemma 5 that (63) holds.

Now from (63), we deduce that, for any $R > 0$

$$\int_{B_R} F(u_n)dx \to \int_{B_R} F(u)dx. \tag{64}$$

Indeed by (H_4) we have

$$0 < F(u_n) \leqslant M_0 |f(u_n)| \ \text{ a.e. } \ \{x \in \mathbb{R}^N \mid |u_n| \geqslant t_0\}$$

and from (H_2)

$$0 < F(u_n) \leqslant \frac{t_0}{\theta} |f(u_n)| \ \text{ a.e. } \ \{x \in \mathbb{R}^N \mid |u_n| < t_0\}.$$

Hence, applying the generalized Lebesgue dominated convergence theorem, we can conclude that (64) holds for any $R > 0$.

Now we claim that for any $\varepsilon > 0$, there exists $R > 1$ such that

$$\left| \int_{\mathbb{R}^N \setminus B_R} (F(u_n) - F(u))dx \right| \leqslant \varepsilon, \forall\, n \geqslant 1. \tag{65}$$

In order to prove our claim, it's sufficient to see that for any $0 < \varepsilon < 1$, there exists $R > 1$ such that

$$\int_{\mathbb{R}^N \setminus B_R} F(u_n)dx \leqslant C\varepsilon \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R} F(u)dx \leqslant C\varepsilon. \tag{66}$$

Let $R > 1$ arbitrarily fixed. By (60), and for $q = \frac{N}{2}$ there exists a positive constant C_3 such that

$$\int_{\mathbb{R}^N \setminus B_R} (F(u_n)dx \leqslant \varepsilon \int_{\mathbb{R}^N \setminus B_R} |u_n|^{\frac{N}{2}} dx + C_3 \int_{\mathbb{R}^N \setminus B_R} |u|_n^{\frac{N}{2}} \, (e^{a\, |u_n|^\gamma} - 1)dx, \;\; \forall\, n \geqslant 1,\; a > \alpha_0.$$

Using the power series expansion of the exponential function and estimating the single terms with the radial lemma 2, the fact that (u_n) is bounded in E, we get for any $n \geqslant 1$

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_R} |u_n|^{\frac{N}{2}} \, (e^{a\, |u_n|^\gamma} - 1)dx &= \sum_{k=1}^{+\infty} \frac{a^k}{k!} \int_{\mathbb{R}^N \setminus B_R} |u_n|^{\gamma k + \frac{N}{2}} dx \\
&\leqslant C_4 \sum_{k=1}^{+\infty} \frac{a^k}{k!} (C)^{\gamma k} \|u_n\|^{\gamma k + \frac{N}{2}} \frac{R^{-2\frac{(N-1)(\gamma k)}{N} + 1}}{\frac{2(N-1)(\gamma k)}{N} - 1} \\
&\leqslant C_4 \frac{\|u_n\|^{\frac{N}{2}}}{R} \sum_{k=1}^{+\infty} \frac{a^k}{k!} (C\|u_n\|)^{\gamma k} \frac{1}{\frac{2(N-1)(\gamma)}{N} - 1} \qquad (67) \\
&\leqslant \frac{C_4}{R} \frac{\|u_n\|^{\frac{N}{2}}}{\frac{2(N-1)(\gamma)}{N} - 1} \sum_{k=1}^{+\infty} \frac{(C\, a\, \|u_n\|^\gamma)^k}{k!} \\
&= \frac{C_4}{R} \frac{\|u_n\|^{\frac{N}{2}}}{\frac{2(N-1)(\gamma)}{N} - 1} e^{a\, C\, \|u_n\|^\gamma}, \;\; \forall n \\
&\leqslant \frac{C_5(a,\varepsilon)}{R}. \qquad (68)
\end{aligned}$$

On the other hand, using Sobolev embedding and the fact that (u_n) is bounded in E, there exists C_1 such that

$$\varepsilon \int_{\mathbb{R}^N \setminus B_R} |u_n|^{\frac{N}{2}} dx \leqslant \varepsilon C \|u_n\|^{\frac{N}{2}} \leqslant \varepsilon C_1.$$

It follows that,

$$\int_{\mathbb{R}^N \setminus B_R} (F(u_n)dx \leqslant \varepsilon C_1 + \frac{C_5(a,\varepsilon)}{R}.$$

We can assume without loss of generality that $\frac{C_5(a,\varepsilon)}{\varepsilon} > 1$. Taking

$$R = \frac{C_5(a,\varepsilon)}{\varepsilon} > 1,$$

we get

$$\int_{\mathbb{R}^N \setminus B_R} F(u_n)dx \leqslant \varepsilon(C_1+1), \ \forall \ n \geqslant 1.$$

In the same way,

$$\int_{\mathbb{R}^N \setminus B_R} F(u)dx \leqslant \varepsilon(C_1+1).$$

Then,

$$\left| \int_{\mathbb{R}^N \setminus B_R} (F(u_n) - F(u))dx \right| \leqslant \int_{\mathbb{R}^N \setminus B_R} F(u_n) + \int_{\mathbb{R}^N \setminus B_R} F(u)dx \leqslant 2\varepsilon(C_1+1)$$

and (65) holds. □

7.2. Estimate of the mountain pass level

LEMMA 9. *Assume that f verifies the conditions (H_1), (H_2), (H_4) and (H_5). Then, for the sequence (v_n) given by (51), there exists $n \geqslant 1$ such that*

$$\max_{t \geqslant 0} \mathscr{J}(tv_n) < \frac{2}{N} \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{N}{2\gamma}}. \tag{69}$$

Proof. By contradiction, suppose that for all $n \geqslant 1$,

$$\max_{t \geqslant 0} \mathscr{J}(tv_n) \geqslant \frac{2}{N} \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{N}{2\gamma}}.$$

By contradiction, suppose that for all $n \geqslant 1$,

$$\max_{t \geqslant 0} \mathscr{J}(tv_n) \geqslant \frac{2}{N} \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{N}{2\gamma}}.$$

Therefore, for any $n \geqslant 1$, there exists $t_n > 0$ such that

$$\max_{t \geqslant 0} \mathscr{J}(tv_n) = \mathscr{J}(t_n v_n) \geqslant \frac{2}{N} \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{N}{2\gamma}}$$

and so,

$$\frac{2}{N} t_n^{\frac{N}{2}} - \int_{\mathbb{R}^N} F(x, t_n v_n)dx \geqslant \frac{2}{N} \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{N}{2\gamma}}.$$

Then, by using (H_1)

$$t_n^{\frac{N}{2}} \geqslant \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}. \tag{70}$$

On the other hand,

$$\frac{d}{dt}\mathscr{I}(tv_n)\big|_{t=t_n} = t_n^{\frac{N}{2}-1} - \int_{\mathbb{R}^N} f(x,t_n v_n)v_n dx = 0,$$

then,

$$t_n^{\frac{N}{2}} = \int_{\mathbb{R}^N} f(x,t_n v_n)t_n v_n dx. \tag{71}$$

Now, we claim that the sequence (t_n) is bounded in $(0,+\infty)$. Indeed, it follows from (H_4) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(t)t \geqslant (\gamma_0 - \varepsilon)e^{\alpha_0 t^\gamma} \quad \forall |t| \geqslant t_\varepsilon. \tag{72}$$

Using (71), we get

$$t_n^{\frac{N}{2}} = \int_{\mathbb{R}^N} f(x,t_n v_n)t_n v_n dx \geqslant \int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}} f(x,t_n v_n)t_n v_n dx.$$

We have for all $0 \leqslant |x| \leqslant \frac{e}{\sqrt[N]{n}}$, $w_n^\gamma \geqslant \left(\frac{\log(n)}{\alpha_\beta}\right)$. From (70) and the result of Lemma 6,

$$t_n v_n \geqslant \frac{t_n}{\|w_n\|}\left(\frac{\log(n)}{\alpha_\beta}\right)^{\frac{1}{\gamma}} \to \infty \text{ as } n \to +\infty.$$

Hence, it follows from (72) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geqslant n_0$

$$t_n^{\frac{N}{2}} \geqslant (\gamma_0 - \varepsilon)\int_{0\leqslant|x|\leqslant\frac{e}{\sqrt[N]{n}}} e^{\alpha_0 t_n^\gamma |v|_n^\gamma} dx$$

and

$$t_n^{\frac{N}{2}} \geqslant NV_N(\gamma_0 - \varepsilon)\int_0^{\frac{1}{\sqrt[N]{n}}} r^{N-1} e^{\alpha_0 t_n^\gamma\left(\frac{\log(n)}{\|w_n\|^\gamma \alpha_\beta}\right)} dr. \tag{73}$$

Hence,

$$1 \geqslant NV_N(\gamma_0 - \varepsilon)\ e^{\alpha_0 t_n^\gamma\left(\frac{\log(n)}{\|w_n\|^\gamma \alpha_\beta}\right) - \log Nn - \frac{N}{2}\log t_n}.$$

Therefore (t_n) is bounded. Also, we have from the formula (71) that

$$\lim_{n\to+\infty} t_n^{\frac{N}{2}} \geqslant \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}.$$

Now, suppose that

$$\lim_{n\to+\infty} t_n^{\frac{N}{2}} > \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}},$$

then for n large enough, there exists some $\delta > 0$ such that $t_n^{\gamma} \geqslant \frac{\alpha_\beta}{\alpha_0} + \delta$. Consequently the right hand side of (73) tends to infinity and this contradicts the boudness of (t_n). Since (t_n) is bounded, we get

$$\lim_{n\to+\infty} t_n^{\frac{N}{2}} = \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}. \tag{74}$$

Let consider the unit ball B of $\mathbb{R}^N$ and the sets

$$\mathscr{A}_n = \{x \in B \mid t_n v_n \geqslant t_\varepsilon\} \quad \text{and} \quad \mathscr{C}_n = B \setminus \mathscr{A}_n.$$

We have,

$$\begin{aligned} t_n^{\frac{N}{2}} &= \int_{\mathbb{R}^N} f(t_n v_n) t_n v_n dx \geqslant \int_B f(t_n v_n) t_n v_n dx = \int_{\mathscr{A}_n} f(t_n v_n) t_n v_n dx + \int_{\mathscr{C}_n} f(t_n v_n) t_n v_n \\ &\geqslant (\gamma_0 - \varepsilon) \int_{\mathscr{A}_n} e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx + \int_{\mathscr{C}_n} f(t_n v_n) t_n v_n dx \\ &= (\gamma_0 - \varepsilon) \int_B e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx - (\gamma_0 - \varepsilon) \int_{\mathscr{C}_n} e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx + \int_{\mathscr{C}_n} f(t_n v_n) t_n v_n dx. \end{aligned}$$

Since $v_n \to 0$ a.e. in B, $\chi_{\mathscr{C}_n} \to 1$ a.e. in B, therefore using the dominated convergence theorem, we get

$$\int_{\mathscr{C}_n} f(x, t_n v_n) t_n v_n dx \to 0 \quad \text{and} \quad \int_{\mathscr{C}_n} e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx \to N V_N.$$

Then,

$$\lim_{n\to+\infty} t_n^{\frac{N}{2}} = \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}} \geqslant (\gamma_0 - \varepsilon) \lim_{n\to+\infty} \int_B e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx - (\gamma_0 - \varepsilon) N V_N.$$

On the other hand,

$$\int_B e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx \geqslant \int_{\frac{1}{\sqrt[N]{n}} \leqslant |x| \leqslant \frac{1}{2}} e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx + \int_{\mathscr{C}_n} e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx.$$

Then, using (70)

$$\begin{aligned} \lim_{n\to+\infty} t_n^{\frac{N}{2}} &\geqslant \lim_{n\to+\infty} (\gamma_0 - \varepsilon) \int_B e^{\alpha_0 t_n^{\gamma} v_n^{\gamma}} dx \\ &\geqslant \lim_{n\to+\infty} (\gamma_0 - \varepsilon) N V_N \int_{\frac{1}{\sqrt[N]{n}}}^{\frac{1}{2}} r^{N-1} e^{\frac{(\log \frac{e}{r})^{\frac{N}{N-2}}}{(\log(e\sqrt[N]{n}))^{\frac{2}{N-2}} \|w_n\|^{\gamma}}} dr. \end{aligned}$$

Therefore, making the change of variable

$$s = \frac{(\log \frac{e}{r})}{(\log(e\sqrt[N]{n}))^{\frac{2}{N-2}} \|w_n\|^{\gamma}} = P \frac{(\log \frac{e}{r})}{\|w_n\|^{\gamma}}, \quad \text{with} \quad P = \frac{1}{(\log(e\sqrt[N]{n}))^{\frac{2}{N-2}}}$$

we get

$$\begin{aligned}
\lim_{n\to+\infty} t_n^{\frac{N}{2}} &\geqslant \lim_{n\to+\infty} (\gamma_0-\varepsilon)\int_B e^{\alpha_0 t_n^\gamma v_n^\gamma}dx \\
&\geqslant \lim_{n\to+\infty} NV_N(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma}{P}\int_{\frac{P\log(2e)}{\|w_n\|^\gamma}}^{\frac{P\log(e\sqrt[N]{n})}{\|w_n\|^\gamma}} e^{N(1-\frac{s\|w_n\|^\gamma}{P})+\frac{\|w_n\|^{\frac{2\gamma}{N-2}}}{P^{\frac{N}{N-2}}}s^{\frac{N}{N-2}}}ds \\
&\geqslant \lim_{n\to+\infty} NV_N(\gamma_0-\varepsilon)\frac{\|w_n\|^\gamma}{P}e^N\int_{\frac{P\log(2e)}{\|w_n\|^\gamma}}^{\frac{P\log(e\sqrt[N]{n})}{\|w_n\|^\gamma}} e^{-\frac{N}{P}\|w_n\|^\gamma s}ds \\
&= \lim_{n\to+\infty}(\gamma_0-\varepsilon)NV_N\frac{e^N}{N}\big(-e^{-N\log(e\sqrt[N]{n})}+e^{-N\log(2e)}\big) \\
&= (\gamma_0-\varepsilon)V_N e^{N(1-\log(2e))}.
\end{aligned}$$

It follows that

$$\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}} \geqslant (\gamma_0-\varepsilon)V_N e^{N(1-\log(2e))}$$

for all $\varepsilon>0$. So,

$$\gamma_0 \leqslant \frac{\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}}{V_N e^{N(1-\log(2e))}},$$

which is in contradiction with the condition (H_5). □

7.3. The compactness level of the energy

The primary challenge within the variational approach to the critical growth problem arises due to the absence of compactness. Specifically, the global Palais-Smale condition doesn't hold. However, a partial Palais-Smale condition is retained under a specific threshold. In the subsequent proposition, we pinpoint the initial level at which the energy exhibits non-compactness.

PROPOSITION 2. *Let $\mathscr{J}$ be the energy associated to the problem* (16) *defined by* (19), *and suppose that the conditions (H_1), (H_2) and (H_4) are satisfied.*

(i) If the function $f(t)$ satisfies the condition (17) *for some $\alpha_0>0$, then the functional $\mathscr{J}$ satisfies the Palais-Smale condition $(PS)_c$ for any*

$$c<\frac{2}{N}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}.$$

(ii) If f is subcritical at $+\infty$, then the functional $\mathscr{J}$ satisfies the Palais-Smale condition $(PS)_c$ for any $c\in\mathbb{R}$.

Proof. (i) Consider a $(PS)_c$ sequence (u_n) in E, for some $c \in \mathbb{R}$, that is

$$\mathscr{J}(u_n) = \frac{2}{N}\|u_n\|^{\frac{N}{2}} - \int_{\mathbb{R}^N} F(x,u_n)dx \to c, \;\; n \to +\infty \tag{75}$$

and

$$\begin{aligned} \mathscr{J}'(u_n)\varphi = & \Big| \int_{\mathbb{R}^N} \left(w_\beta(x)|\Delta u_n|^{\frac{N}{2}-2}\Delta u_n \Delta\varphi + |\nabla u_n|^{\frac{N}{2}-2}\nabla u_n.\nabla\varphi + V(x)|u_n|^{\frac{N}{2}-2}u_n\varphi \right) dx \\ & - \int_{\mathbb{R}^N} f(x,u_n)\ \varphi\ dx \Big| \\ & \leqslant \varepsilon_n\|\varphi\|, \end{aligned} \tag{76}$$

for all $\varphi \in E$, where $\varepsilon_n \to 0$, as $n \to +\infty$.

For n large enough, there exists a constant $C > 0$ such that

$$\frac{2}{N}\|u_n\|^{\frac{N}{2}} \leqslant C + \int_{\mathbb{R}^N} F(x,u_n)dx.$$

From (H_2), it follows that

$$\int_{\mathbb{R}^N} F(u_n)dx \leqslant \frac{1}{\theta}\int_{\mathbb{R}^N} f(u_n)u_n dx.$$

Using (76) with $\varphi = u_n$, we obtain

$$\int_{\mathbb{R}^N} f(u_n)u_n dx \leqslant \varepsilon_n\|u_n\| + \|u_n\|^{\frac{N}{2}}.$$

Therefore,

$$\frac{2}{N}\|u_n\|^{\frac{N}{2}} \leqslant C_1 + \frac{\varepsilon_n}{\theta}\|u_n\| + \frac{1}{\theta}\|u_n\|^{\frac{N}{2}}.$$

Since, $\theta > \frac{N}{2}$, we get

$$0 < \left(\frac{2}{N} - \frac{1}{\theta}\right)\|u_n\|^{\frac{N}{2}} \leqslant C + \frac{\varepsilon_n}{\theta}\|u_n\|.$$

We deduce that the sequence (u_n) is bounded in E. As consequence, there exists $u \in E$ such that, up to subsequence, $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^q(B)$, for all $q \geqslant \frac{N}{2}$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Also, we can follow [24] to prove that $\nabla u_n(x) \to \nabla u(x)$ a.e. $x \in \mathbb{R}^N$ and $\Delta u_n(x) \to \Delta u(x)$ a.e. $x \in \mathbb{R}^N$.

Furthermore, we have, from (75) and (76), that

$$0 < \int_{\mathbb{R}^N} f(x,u_n)u_n \leqslant C, \tag{77}$$

and

$$0 < \int_{\mathbb{R}^N} F(x,u_n) \leqslant C. \tag{78}$$

By Lemma 8, we have

$$F(x,u_n) \to F(x,u) \ \text{in} \ L^1(\mathbb{R}^N) \quad \text{as} \quad n \to +\infty. \tag{79}$$

Then, from (75), we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(x,u_n)u_n dx = \frac{N}{2}(c + \int_{\mathbb{R}^N} F(x,u)dx) \tag{80}$$

and from (76), we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f(x,u_n)u_n dx = \frac{N}{2}(c + \int_{\mathbb{R}^N} F(x,u)dx). \tag{81}$$

It follows from (H_2) and (76), that

$$\lim_{n\to+\infty} \frac{N}{2} \int_{\mathbb{R}^N} F(x,u_n)dx \leqslant \lim_{n\to+\infty} \int_{\mathbb{R}^N} f(x,u_n)u_n dx = \frac{N}{2}(c + \int_{\mathbb{R}^N} F(x,u)dx). \tag{82}$$

Then, passing to the limit in (76) and using (81),we obtain that u is a weak solution of the problem (16) that is

$$\int_{\mathbb{R}^N} \left(w_\beta(x)\,|\Delta u|^{\frac{N}{2}-2}\,\Delta u\ \Delta\varphi + |\nabla u|^{\frac{N}{2}-2}\nabla u.\nabla\varphi + V(x)|u|^{\frac{N}{2}-2}u\varphi\right) dx = \int_{\mathbb{R}^N} f(x,u)\,\varphi\,dx,$$

for all $\varphi \in E$.

Taking $\varphi = u$ as a test function, we get

$$\begin{aligned} \|u\|^{\frac{N}{2}} &= \int_{\mathbb{R}^N} w_\beta(x)|\Delta u|^{\frac{N}{2}} dx + \int_{\mathbb{R}^N} |\nabla u|^{\frac{N}{2}} dx + \int_{\mathbb{R}^N} |u|^{\frac{N}{2}} dx \\ &= \int_{\mathbb{R}^N} f(x,u)u dx \geqslant \frac{N}{2} \int_{\mathbb{R}^N} F(x,u)dx. \end{aligned}$$

Hence $\mathcal{J}(u) \geqslant 0$. We also have by the Fatou's lemma and (79)

$$0 \leqslant \mathcal{J}(u) \leqslant \frac{2}{N} \liminf_{n\to\infty} \|u_n\|^{\frac{N}{2}} - \int_{\mathbb{R}^N} F(x,u)dx = c.$$

So, we will finish the proof by considering three cases for the level c.

Case 1. $c = 0$. In this case

$$0 \leqslant \mathcal{J}(u) \leqslant \liminf_{n\to+\infty} \mathcal{J}(u_n) = 0.$$

So,

$$\mathcal{J}(u) = 0$$

and then by (79)

$$\lim_{n\to+\infty} \frac{2}{N}\|u_n\|^{\frac{N}{2}} = \int_{\mathbb{R}^N} F(x,u)dx = \frac{2}{N}\|u\|^{\frac{N}{2}}.$$

By Brezis-Lieb's Lemma [9], it follows that $u_n \to u$ in E.

Case 2. $c>0$ and $u=0$. We will prove that this is not possible.

We will show in the following that

$$\mathscr{J}_n=\int_{\mathbb{R}^N} f(u_n)u_n dx \to 0 \text{ as } n\to+\infty. \tag{83}$$

In fact, if (83) holds, then taking $\varphi=u_n$ in (76) we get

$$\|u_n\|^{\frac{N}{2}} \leqslant c\varepsilon_n+\mathscr{J}_n \to 0 \text{ as } n\to 0$$

which gives $\lim\limits_{n\to+\infty}\|u_n\|^{\frac{N}{2}}=0$. This contradicts the fact that $c\neq 0$ because from (75) and (76), we have

$$\lim_{n\to+\infty}\|u_n\|^{\frac{N}{2}}=\frac{N}{2}c.$$

It therefore remains to prove that (83) is valid. Let $a>\alpha_0$ and $q\geqslant 1$. By (59) and again the boundedness of (u_n) in E, we obtain for all $\varepsilon>0$,

$$\mathscr{J}_n \leqslant C_1\varepsilon + C(a,q,\varepsilon)\int_{\mathbb{R}^N}|u|_n^q\,(e^{a\,|u_n|^\gamma}-1)dx \ \forall\ n\geqslant 1.$$

Applying Hölder's inequality with $p,p'>1$ and $\frac{1}{p}+\frac{1}{p'}=1$, we get

$$\mathscr{J}_n(u)\leqslant C\|u_n\|^q_{p'q}\Big(\int_{\mathbb{R}^N}\,(e^{p\,a\,|u_n|^\gamma}-1)dx\Big)^{\frac{1}{p}}.$$

Since $(\frac{N}{2}c)^{\frac{2\gamma}{N}}<\left(\frac{\alpha_\beta}{\alpha_0}\right)$, there exists $\eta\in(0,\frac{1}{2})$ such that $(\frac{N}{2}c)^{\frac{2\gamma}{N}}=(1-2\eta)\left(\frac{\alpha_\beta}{\alpha_0}\right)$. On the other hand, $\|u_n\|^\gamma\to(\frac{N}{2}c)^{\frac{2\gamma}{N}}$, so there exists $n_\eta>0$ such that for all $n\geqslant n_\eta$, we get $\|u_n\|^\gamma\leqslant(1-\eta)\frac{\alpha_\beta}{\alpha_0}$. Therefore, if we choose $a=(1+\frac{\eta}{2})\alpha_0$, $p=(1+\frac{\eta}{2})$ we get

$$pa\left(\frac{|u_n|}{\|u_n\|}\right)^\gamma\|u_n\|^\gamma\leqslant\alpha_0\left(1+\frac{\eta}{2}\right)^2\left(\frac{|u_n|}{\|u_n\|}\right)^\gamma(1-\eta)\leqslant\alpha_\beta\left(\frac{|u_n|}{\|u_n\|}\right)^\gamma.$$

Therefore, the integral is bounded in view of (17). On the other hand, choosing $q>2$, so $p'q>2$ and therefore $u_n\to 0$ $L^{qp'}(\mathbb{R}^N)$. Then $\mathscr{J}_n\to 0$ as $n\to+\infty$.

Case 3. $c>0$ and $u\neq 0$. In this case, we claim that $\mathscr{J}(u)=c$ and therefore, we get

$$\lim_{n\to+\infty}\|u_n\|^{\frac{N}{2}}=\frac{N}{2}\big(c+\int_{\mathbb{R}^N}F(x,u)dx\big)=\big(\mathscr{J}(u)+\int_{\mathbb{R}^N}F(x,u)dx\big)=\|u\|^{\frac{N}{2}}.$$

Do not forgot that

$$\mathscr{J}(u)\leqslant\frac{2}{N}\liminf_{n\to+\infty}\|u_n\|^{\frac{N}{2}}-\int_{\mathbb{R}^N}F(x,u)dx=c.$$

We argue by contradiction and suppose that $\mathscr{J}(u) < c$. Then,

$$\|u\|^{\frac{N}{2}} < \Big(\frac{N}{2}\Big(c+\int_{\mathbb{R}^N} F(x,u)dx\Big)\Big)^{\frac{2}{N}}. \tag{84}$$

Set

$$v_n = \frac{u_n}{\|u_n\|}$$

and

$$v = \frac{u}{(\frac{N}{2}(c+\int_{\mathbb{R}^N} F(x,u)dx))^{\frac{2}{N}}}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in E, $\nabla v_n(x) \to \nabla v(x)$ a.e. $x \in \mathbb{R}^N$, $\Delta v_n(x) \to \Delta v(x)$ a.e. $x \in \mathbb{R}^N$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 7, we get

$$\sup_n \int_{\mathbb{R}^N} \big(e^{p\alpha_\beta |v_n|^\gamma} - 1\big)dx < \infty, \ \text{ for } \ 1 < p < U(v) = (1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}. \tag{85}$$

Since $u_n \hookrightarrow u$ in E, it suffice to prove that

$$\mathscr{J}'(u_n)(u_n - u) \to 0 \quad \text{as} \quad n \to +\infty,$$

and that's the case when

$$\int_{\mathbb{R}^N} f(u_n)(u_n-u)dx \to 0. \tag{86}$$

Arguing as in Case 1, we can thus reduce the proof of (86) to showing the existence of $a > \alpha_0$ and $q \geqslant 1$ such that

$$\mathscr{I}_n := \int_{\mathbb{R}^N} |u_n|^{q-1}|u_n-u|(e^{a\,|u_n|^\gamma} - 1)dx \to 0 \ \text{ as } \ n \to +\infty.$$

We apply Hölder's inequality twice with $p, p', t, t' > 1$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{t} + \frac{1}{t'} = 1$, we get

$$\begin{aligned}\mathscr{I}_n &\leqslant C(a,\varepsilon)\|u_n\|^{q-1}_{p't'(q-1)}\|u_n-u\|_{p't}\Big(\int_{\mathbb{R}^N} (e^{p\,a\,|u_n|^\gamma} - 1)dx\Big)^{\frac{1}{p}} \\ &\leqslant \Big(\int_{\mathbb{R}^N} (e^{\tau\,a\,|u_n|^\gamma} - 1)dx\Big)^{\frac{1}{p}}\end{aligned}$$

for any $\tau > p$.

From (85), it follows that

$$\sup_n \int_{\mathbb{R}^N} \big(e^{\tau\,a\,|u_n|^\gamma} - 1\big)dx = \sup_n \int_{\mathbb{R}^N} \big(e^{\tau\,a\,|v_n|^\gamma \|u_n\|^\gamma} - 1\big)dx < \infty$$

provided $a > \alpha_0, p > 1$ and $\tau > p$ can be chosen so that $a\,\tau\|u_n\|^\gamma < (1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}\alpha_\beta$ and $1 < p < U(v) = (1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}$.

We have

$$(1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}} = \left(\frac{(\frac{N}{2}(c+\int_B F(x,u)dx)}{(\frac{N}{2}(c+\int_B F(x,u)dx)) - \|u\|^{\frac{N}{2}})}\right)^{\frac{2\gamma}{N}}$$
$$= \left(\frac{c+\int_B F(x,u)dx}{c-\mathscr{J}(u)}\right)^{\frac{2\gamma}{N}}. \tag{87}$$

On the other hand,

$$\lim_{n\to+\infty}\|u_n\|^{\gamma} = \left(\frac{N}{2}\left(c+\int_{\mathbb{R}^N} F(x,u)dx\right)\right)^{\frac{2\gamma}{N}},$$

then, for all ε such that $0<\varepsilon<1$ and for n large enough

$$\alpha_0(1+\varepsilon)\|u_n\|^{\gamma} \leqslant \alpha_0(1+2\varepsilon)\left(\frac{N}{2}\left(c+\int_{\mathbb{R}^N} F(x,u)dx\right)\right)^{\frac{2\gamma}{N}}.$$

Taking $a=(1+\varepsilon)\alpha_0$, $\tau=(1+\varepsilon)p$ and using (87), we get

$$a\|u_n\|^{\gamma}\tau \leqslant \alpha_0(1+7\varepsilon)\left(\frac{N}{2}\left(c+\int_{\mathbb{R}^N} F(x,u)dx\right)\right)^{\frac{2\gamma}{N}}(1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}$$
$$\leqslant p\alpha_0(1+7\varepsilon)2^{\frac{2\gamma}{N}}\left(c-\mathscr{J}(u)\right)^{\frac{2\gamma}{N}}.$$

But $\mathscr{J}(u)\geqslant 0$ and $c<\frac{2}{N}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}$, then there exists $\eta\in(0,1)$ such that $c^{\frac{2\gamma}{N}}=(1-\eta)(\frac{2}{N})^{\frac{2\gamma}{N}}\frac{\alpha_\beta}{\alpha_0}$.

If we choose $\varepsilon=\frac{\eta}{7}$, we get,

$$a\|u_n\|^{\gamma}\tau \leqslant (1+\eta)(1-\eta)p\alpha_\beta \leqslant p\alpha_\beta < p\ \ (1-\|v\|^{\frac{N}{2}})^{\frac{-2\gamma}{N}}.$$

So, with this choice of $\tau>p>1$ and $a>\alpha_0$, we have

$$\mathscr{I}_n \leqslant C(a,\alpha_0)\|u_n\|^{q-1}_{p't'(q-1)}\|u_n-u\|_{p't}$$

where $C(a,\alpha_0)$ is a positive constant depending only on a and α_0. Now, since $(q-1)p't'>q-1$ and $p't>t$, choosing $q\geqslant 3$ and $t\geqslant 2$ we have that (u_n) is bounded in $L^{((q-1)p't'}(\mathbb{R}^N)$, so $\mathscr{I}_n\to 0$ as $n\to+\infty$.

Hence,

$$\lim_{n\to+\infty}\|u_n\|^{\frac{N}{2}} = \frac{N}{2}\left(c+\int_{\mathbb{R}^N} F(x,u)dx\right) = \|u\|^{\frac{N}{2}}$$

and this contradicts (84). So, $\mathscr{J}(u)=c$ and consequently, $u_n\to u$.

Proof of (ii) follows from (i). Indeed, since (u_n) is bounded in E, there exists a positive constant $M>0$ such that $\|u_n\| \leqslant M$. As f is subcritical at infinity, then if we choose $a>0$ such that $a \leqslant \frac{\alpha_\beta}{p\, M^\gamma}$, the integral

$$\int_{\mathbb{R}^N} (e^{p\ a\ |u_n|^\gamma} - 1)dx$$

is finite for all $p \geqslant 1$. So, arguing as in (i) we get that

$$\mathscr{I}_n \to 0 \ \text{ as } \ n \to +\infty.$$

Now according to the Proposition 2, the functional $\mathscr{J}$ satisfies the $(PS)_c$ condition at a level $c < \frac{2}{N}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{N}{2\gamma}}$, in the critical case and at all level c, in the subcritical case. Moreover, Proposition 1 confirms that the functional $\mathscr{J}$ exhibits a mountain pass structure. Consequently, by the Ambrosetti and Rabinowitz Theorem [5], $\mathscr{J}$ possesses a non-zero critical point u within the space E. This leads to the proof of Theorem 6 and Theorem 7. □

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