Mathematical
Inequalities
& Applications
Volume 28, Number 3 (2025), 463–469 doi:10.7153/mia-2025-28-29

ON A PROPERTY OF BOX–CONVEX FUNCTIONS

RALUCA PAŞCA*, MIRCEA IVAN AND BOGDAN GAVREA

(*Communicated by S. Varošanec*)

Abstract. In this paper, we use α-Bernstein operators to solve an open problem related to box-convex functions. Future targeted applications reside in the area of stochastic optimization in problems such as AUC maximization and stochastic programs with chance constraints.

1. Introduction

Many years ago, Raşa proposed in [13] a conjecture related to the preservation of convexity by the Bernstein-Schnabl operators. It was proved for the first time in [11], by J. Mrowiec, T. Rajba, S. Wąsowicz. Other generalizations connected to it were provided in the last years (see [1], [2], [3], [9], [7], [8]). From [12] it is known that Bernstein polynomials preserve q-monotonicity for all orders $q \geqslant 1$. In [2], Abel and Leviatan proved an important inequality for q-monotone functions which was extended in [10]. Our main goal is to solve the open problem proposed in [10].

In [4], the following generalization of the Bernstein operators depending on a non-negative real parameter was derived. Given a function $f\colon [0,1] \to \mathbb{R}$, for each positive integer n and for any fixed non-negative real number α, the α-Bernstein operator associated with f is the function $T_{n,\alpha}(f)\colon [0,1] \to \mathbb{R}$ defined by

$$T_{n,\alpha}(f)(x) := T_{n,\alpha}(f;x) = \sum_{i=0}^{n} p_{n,i}^{(\alpha)}(x) f\left(\frac{i}{n}\right),$$

where $p_{1,0}^{(\alpha)}(x) = 1-x$, $p_{1,1}^{(\alpha)}(x) = x$, and

$$p_{n,i}^{(\alpha)}(x) = \left(\binom{n-2}{i}(1-\alpha)x + \binom{n-2}{i-2}(1-\alpha)(1-x) + \binom{n}{i}\alpha x(1-x)\right) \times x^{i-1}(1-x)^{n-i-1},$$

for $n \geqslant 2$, $x \in [0,1]$. We convene on the binomial coefficients $\binom{k}{l}$ to have the form:

$$\binom{k}{l} = \begin{cases} \dfrac{k!}{l!(k-l)!}, & \text{if } 0 \leqslant l \leqslant k, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Mathematics subject classification (2020): 41A36, 41A17.
Keywords and phrases: α-Bernstein operators, box-convex functions.
* Corresponding author.

When $\alpha = 1$, the α-Bernstein operator $T_{n,\alpha}(f)$ reduces to the classical Bernstein operator of degree n, i.e.,

$$T_{n,1}(f)(x) = B_n(f)(x) = \sum_{i=0}^{n} p_{n,i}(x) f\left(\frac{i}{n}\right),$$

where $p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$ are the Bernstein basis polynomials of degree n. With the convention (1) and $i, n \in \mathbb{N}$, the Bernstein basis polynomial $p_{n,i}(x)$ is not identically zero if and only if $0 \leqslant i \leqslant n$.

The rate of convergence and a Voronovskaja type theorem for $T_{n,\alpha}$ are given in [4]. The operators $T_{n,\alpha}$ inherit monotonicity and convexity properties from the classical Bernstein operators. More precisely, the operators $T_{n,\alpha}$ preserve monotonicity and convexity ([4], Theorem 3.3 and Theorem 4.1.).

We observe that $p_{n,i}^{(\alpha)}(x)$ can be written in terms of the Bernstein basis as

$$p_{n,i}^{(\alpha)}(x) = (1-\alpha)(1-x)p_{n-2,i}(x) + (1-\alpha)x p_{n-2,i-2}(x) + \alpha p_{n,i}(x), \quad i, n \in \mathbb{N}. \tag{2}$$

Let $B_k(f(at+b);x)$ denote the Bernstein operator of degree k applied to the function $g(t) = at + b$, i.e.,

$$B_k(f(at+b);x) = B_k(g)(x), \quad \text{with} \quad g(t) = at + b.$$

It follows from (2) that the α-Bernstein operator $T_{n,\alpha}$ can be written in the form

$$\begin{aligned} T_{n,\alpha}(f;x) = {} & (1-\alpha)(1-x)B_{n-2}\left(f\left(\frac{n-2}{n}t\right);x\right) \\ & +(1-\alpha)xB_{n-2}\left(f\left(\frac{(n-2)t+2}{n}\right);x\right) + \alpha B_n(f(t);x). \end{aligned} \tag{3}$$

In this paper we will consider only $\alpha \in [0,1]$. Under this assumption, it follows that $T_{n,\alpha}$ are linear positive operators for $n \in \mathbb{N}$.

Given g defined on a suitable interval of the real axis and $h > 0$, the classical finite difference operator Δ_h is defined by

$$\Delta_h^1 g(x) := \Delta_h g(x) = \begin{cases} g(x+h) - g(x), & x, x+h \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

We recall now the definition of box-convexity [6]. A function $f \in C([0,1] \times [0,1])$ is called box-convex of order (q,s), if for any distinct points $x_0, x_1, \ldots, x_q \in [0,1]$ and any distinct points $y_0, y_1, \ldots, y_s \in [0,1]$

$$\begin{bmatrix} x_0, x_1, \ldots, x_q \\ y_0, y_1, \ldots, y_s \end{bmatrix}; f \geqslant 0,$$

where

$$\begin{bmatrix} x_0, \ldots, x_q \\ y_0, \ldots, y_s \end{bmatrix}; f = \left[x_0, \ldots x_q; [y_0, \ldots, y_s; f(x,y)]_y\right]_x = \left[y_0, \ldots, y_s; [x_0, \ldots, x_q; f(x,y)]_x\right]_y.$$

In [10], the following open problem is proposed:

PROBLEM 1. ([10]) *Let q,s be two natural numbers, $q,s \geqslant 2$ and let $x_k,t_k \in [0,1]$, $k=1,\dots,q$, such that $x_k \neq t_k$ and $y_i,z_i \in [0,1]$, $i=1,\dots,s$, be such that $y_i \neq z_i$. If $g \in C([0,1]\times[0,1])$ is a (q,s)–box convex function, prove that*

$$\operatorname{sgn}\left(\left(\prod_{k=1}^{q}(x_k-t_k)\right)\left(\prod_{r=1}^{s}(y_r-z_r)\right)\right) \tag{4}$$
$$\times \sum_{i_1,\dots,i_q=0}^{n}\ \sum_{j_1,\dots,j_s=0}^{m}\left(\prod_{k=1}^{q}\left(p_{n,i_k}^{(\alpha)}(x_k)-p_{n,i_k}^{(\alpha)}(t_k)\right)\right)\left(\prod_{r=1}^{s}\left(p_{m,j_r}^{(\beta)}(y_r)-p_{m,s_r}^{(\beta)}(z_r)\right)\right)$$
$$\times g\left(\frac{i_1+\ldots+i_q}{nq},\frac{j_1+\ldots+j_s}{ms}\right) \geqslant 0.$$

REMARK 1. For $\alpha=\beta=1$ the assertion is true, [9]. For $\alpha=\beta=1$, $s=0$, $m=n$, and $g(x,y)=\int_0^1 f\left(\frac{nx+\alpha t}{qn+\alpha}\right)\mathrm{d}t$, (4) is equivalent to the inequality from [2, Theorem A].

Our main goal is to generalize inequality (4) for a class $\mathscr{T}$ of positive linear operators $T_{n,\alpha}$.

2. Main results

Let $f\colon [0,1]\to\mathbb{R}$, and e_k denote the k–th order monomial, i.e., $e_k(x)=x^k$, $x\in[0,1]$. Consider the family of discrete operators $\mathscr{T}=\{U_n\}_{n\in\mathbb{N}}$, such that

$$U_n(f)(x)=\sum_{i=0}^{n}u_{n,i}(x)f\left(\frac{i}{n}\right),\quad u_{n,i}\in C^1([0,1]),\quad U_n(e_0)\quad \textit{is constant},$$

and

$$(U_nf)'(x)=\sum_{i=0}^{n-1}c_{n,i}(x)\Delta_{\frac{1}{n}}f\left(\frac{i}{n}\right),\quad c_{n,i}\geqslant 0,\quad c_{n,i}\in C([0,1]), \tag{5}$$

where $u_{n,i}$ and $c_{n,i}$ are independent of f.

The following theorem is the main result of the paper:

THEOREM 1. *Let $U_n\in\mathscr{T}$, and q,s be two natural numbers, $q,s\geqslant 2$, and let $x_k,t_k\in[0,1]$, $k=1,\dots,q$ such that $x_k\neq t_k$ and $y_r,z_r\in[0,1]$, $r=1,\dots,s$ be such that $y_r\neq z_r$. If $g\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a (q,s)–box convex function, then the following inequality is satisfied:*

$$\operatorname{sgn}\left(\left(\prod_{k=1}^{q}(x_k-t_k)\right)\left(\prod_{r=1}^{s}(y_r-z_r)\right)\right) \tag{6}$$
$$\times \sum_{i_1,\dots,i_q=0}^{n}\ \sum_{j_1,\dots,j_s=0}^{m}\left(\prod_{k=1}^{q}\left(u_{n,i_k}(x_k)-u_{n,i_k}(t_k)\right)\right)\left(\prod_{r=1}^{s}\left(u_{m,j_r}(y_r)-u_{m,s_r}(z_r)\right)\right)$$
$$\times g\left(\frac{i_1+\ldots+i_q}{nq},\frac{j_1+\ldots+j_s}{ms}\right) \geqslant 0.$$

Proof. In proving this result, we will use the following construction:

Let $\Delta\colon\, a = x_0 < x_1 < \ldots < x_n = b$ be a partition of the interval $[a,b]$. Define the piecewise linear continuous nondecreasing functions $u_k\colon\, [a,b] \to \mathbb{R}$,

$$u_k(x) := \frac{|x - x_{k-1}| - |x - x_k| + x_k - x_{k-1}}{2}, \quad k = 1,2,\ldots,n.$$

Let $f\colon\, [a,b] \to \mathbb{R}$. One can check that the piecewise linear continuous function $S_\Delta f \equiv S_\Delta(f)\colon\, [a,b] \to \mathbb{R}$,

$$S_\Delta(f)(x) := f(x_0) + \sum_{\nu=1}^{n} u_\nu(x)\,[x_{\nu-1}, x_\nu; f],$$

interpolates f at x_k, $k = 0,1,\ldots,n$.

In general, if $L\colon\, C[a,b] \to C^1[a,b]$ is a discretely defined linear operator, i.e.,

$$L(f) := \sum_{\nu=1}^{n} \alpha_\nu\, f(x_\nu),$$

then

$$L(f) = L(S_\Delta f) = L(e_0) f(x_0) + \sum_{\nu=1}^{n} L(u_\nu)\,[x_{\nu-1}, x_\nu; f].$$

If L preserves monotonicity, and $L(e_0)$ is a constant, it follows that $(L(u_k))' \geqslant 0$, $k = 1,2,\ldots,n$, and $(L(e_0))' = 0$.

We obtain

$$(L(f))' = \sum_{\nu=1}^{n} (L(u_\nu))'\,[x_{\nu-1}, x_\nu; f].$$

We prove the equality

$$\sum_{i_1,\ldots,i_q=0}^{n} \left(\prod_{k=1}^{q} u'_{n,i_k}(x_k) \right) g_1\left(\frac{i_1 + \ldots + i_q}{nq} \right) \tag{7}$$
$$= \frac{q!}{n^q q^q} \sum_{i_1,\ldots,i_q=0}^{n} \left(\prod_{k=1}^{q} c_{n,i_k}(x_k) \right) \left[\frac{i_1 + \ldots + i_q}{nq}, \ldots, \frac{i_1 + \ldots + i_q + q}{nq}; g_1 \right],$$

for $g_1\colon [0,1]\to\mathbb{R}$. We start with equality (5). By using this multiple times we get

$$\begin{aligned}
&\sum_{i_1,\ldots,i_q=0}^{n}\prod_{k=1}^{q}u'_{n,i_k}(x_k)g_1\left(\frac{i_1+\ldots+i_q}{nq}\right)\\
&=\sum_{i_1,\ldots,i_q=0}^{n}\left(\prod_{k=1}^{q-1}u'_{n,i_k}(x_k)\right)\left(U'_n\,g_1\left(\frac{i_1+\ldots+i_q-1}{nq}+\frac{\cdot}{q}\right)\right)(x_q)\\
&=\sum_{i_q=0}^{n}c_{n,i_q}(x_q)\sum_{i_1,\ldots,i_{q-2}=0}^{n}\left(\prod_{k=1}^{n}u'_{n,i_k}(x_k)\right)\\
&\quad\times\left(U'_n\left(\Delta_{\frac{1}{nq}}\left(g_1\left(\frac{i_1+\ldots+i_{q-2}}{qn}+\frac{\cdot}{q}+\frac{i_q}{nq}\right)\right)\right)\right)(x_{q-1})\\
&=\sum_{i_{q-1}=0}^{n}\sum_{i_q=0}^{n}c_{n,i_q}(x_q)c_{n,i_{q-1}}(x_{q-1})\\
&\quad\times\sum_{i_1,\ldots,i_{q-2}=0}^{n}\left(\prod_{k=1}^{q-2}u'_{n,i_k}(x_k)\right)\Delta^2_{\frac{1}{nq}}g_1\left(\frac{i_1+\ldots+i_q}{nq}\right)\\
&=\ldots=\sum_{i_1,\ldots,i_q=0}^{n}\left(\prod_{k=1}^{q}c_{n,i_k}(x_k)\right)\Delta^q_{\frac{1}{nq}}g_1\left(\frac{i_1+\ldots+i_q}{nq}\right).
\end{aligned}$$

Now with $x=\frac{i_1+\ldots+i_q}{nq}$ and $h=\frac{1}{nq}$ in the identity $[x,x+h,\ldots,x+qh]=\frac{1}{q!}\cdot\frac{1}{h^q}\Delta_h^q f(x)$ we obtain (7). Using the definition of the double divided difference and (7), for $H\colon [0,1]\times[0,1]\to\mathbb{R}$ we are led to:

$$\begin{aligned}
&\sum_{i_1,\ldots,i_q=0}^{n}\sum_{j_1,\ldots,j_s=0}^{m}\left(\prod_{k=1}^{q}u'_{n,i_k}(x_k)\right)\left(\prod_{r=1}^{s}u'_{m,j_r}(z_r)\right)\\
&\times H\left(\frac{i_1+\ldots+i_q}{nq},\frac{j_1+\ldots+j_s}{ms}\right)\\
=&\frac{q!s!}{q^qs^sn^qm^s}\sum_{i_1,\ldots,i_q=0}^{n}\sum_{j_1,\ldots,j_s=0}^{m}\left(\prod_{k=1}^{q}c_{n,i_k}(x_k)\prod_{r=1}^{s}c_{m,j_r}(z_r)\right)D^{i_1,\ldots,i_q}_{j_1,\ldots,j_s}(H),
\end{aligned}$$

where

$$D^{i_1,\ldots,i_q}_{j_1,\ldots,j_s}(H)=\left[\begin{matrix}\frac{i_1+\ldots+i_q}{nq},\ \ldots,\ \frac{i_1+\ldots+i_q+q}{nq}\\ \frac{j_1+\ldots+j_s}{ms},\ \ldots,\ \frac{j_1+\ldots+j_s+s}{ms}\end{matrix};\ H\right]$$

Assume that

$$\operatorname{sgn}\left(\left(\prod_{k=1}^{q}(x_k-t_k)\right)\left(\prod_{r=1}^{s}(y_r-z_r)\right)\right)>0.$$

We will show that the remaining part in (6) is non-negative. We have

$$\sum_{i_1,\dots,i_q=0}^{n}\sum_{j_1,\dots,j_s=0}^{m}\left(\prod_{k=1}^{q}\left(u_{n,i_k}(x_k)-u_{n,i_k}(t_k)\right)\right)\left(\prod_{r=1}^{s}\left(u_{m,j_r}(y_r)-u_{m,s_r}(z_r)\right)\right)$$
$$\times g\left(\frac{i_1+\ldots+i_q}{nq},\frac{j_1+\ldots+j_s}{ms}\right)$$
$$=\int_{t_1}^{x_1}\cdots\int_{t_q}^{x_q}\int_{z_1}^{y_1}\cdots\int_{z_s}^{y_s}\sum_{i_1,\dots,i_q=0}^{n}\prod_{k=1}^{q}u'_{n,i_k}(\tau_k)\sum_{j_1,\dots,j_s=0}^{m}\prod_{r=1}^{s}u'_{m,j_r}(\gamma_r)$$
$$\times g\left(\frac{i_1+\ldots+i_q}{nq},\frac{j_1+\ldots+j_s}{ms}\right)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_q\,\mathrm{d}\gamma_1\ldots\mathrm{d}\gamma_s$$
$$=\frac{q!}{q^q}\frac{s!}{s^s}\frac{1}{n^q m^s}\int_{t_1}^{x_1}\cdots\int_{t_q}^{x_q}\int_{z_1}^{y_1}\cdots\int_{z_s}^{y_s}\sum_{i_1,\dots,i_q=0}^{n}\sum_{j_1,\dots,j_s=0}^{m}\prod_{k=1}^{q}c_{n,i_k}(\tau_k)\prod_{r=1}^{s}c_{m,j_r}(\gamma_r)$$
$$\times\left[\begin{matrix}\frac{i_1+\ldots+i_q}{nq},\ \ldots,\ \frac{i_1+\ldots+i_q+q}{nq}\\ \frac{j_1+\ldots+j_s}{ms},\ \ldots,\ \frac{j_1+\ldots+j_s+s}{ms}\end{matrix};\ g\right]\mathrm{d}\tau_1\ldots\mathrm{d}\tau_q\,\mathrm{d}\gamma_1\ldots\mathrm{d}\gamma_s\geqslant 0. \quad\square$$

REMARK 2. By setting in Theorem 1 the operator $U_n=T_{n,\alpha}$, we obtain the proof of Problem 1.

3. Conclusions and future work

In this paper we used α-Bernstein operators to prove the open problem proposed in [10]. Future work is concerned with the use of α-Bernstein operators in the context of stochastic optimization. We are interested in two types of stochastic optimization problems: AUC (Area Under the Curve) maximization problems, [5], and stochastic optimization problems with chance constraints, [14]. We see the advantage of using α-Bernstein vs classical Bernstein operators in the fact that the parameter α gives us an additional opportunity in tuning the resulting algorithms, while we are still able to preserve monotonicity and convexity.

Acknowledgements. Special thanks to the anonymous reviewer for their many insightful suggestions and comments.

REFERENCES

[1] U. ABEL, *An inequality involving Bernstein polynomials and convex functions*, Journal of Approximation Theory **222** (2017), 1–7, https://doi.org/10.1016/j.jat.2017.05.006.

[2] U. ABEL, D. LEVIATAN, *An extension of Raşa's conjecture to q-monotone functions*, Results Math. **75** (4) (2020), 181–193, https://doi.org/10.1007/s00025-020-01308-y.

[3] U. ABEL, I. RAŞA, *A sharpening of a problem on Bernstein polynomials and convex functions*, Math. Inequal. Appl. **21** (3) (2018), 773–777, https://dx.doi.org/10.7153/mia-2018-21-55.

[4] X. CHEN, J. TAN, Z. LIU, J. XIE, *Approximation of functions by a new family of generalized Bernstein operators*, J. Math. Anal. Appl. **450** (2017), 244–261, https://doi.org/10.1016/j.jmaa.2016.12.075.

[5] C. CORTES AND M. MOHRI, *AUC optimization vs. error rate minimization*, in: Advances in Neural Information Processing Systems, (2004), 313–320.

[6] S. GAL, *Shape Preserving Approximation by Real and Complex Polynomials*, Berkhauser, Boston, 2008.

[7] B. GAVREA, *On a convexity problem in connection with some linear operators*, J. Math. Anal. Appl. **461** (1) (2018), 319–332.

[8] B. GAVREA, *On a convexity problem with applications to Mastroianni type operators*, Math. Inequal. Appl. **23** (2020), 765–773.

[9] B. GAVREA, I. GAVREA, *An inequality involving some linear positive operators and box-convex functions*, Results Math **77** (33) (2022).

[10] B. GAVREA, I. GAVREA, D. IANOŞI, *On q-monotonicity of α-Bernstein operators*, Math. Inequal. and Appl. **25** (2) (2022), 611–619.

[11] J. MROWIEC, T. RAJBA, S. WĄSOWICZ, *A solution to the problem of Raşa conected with Bernstein polynomials*, J. Math. Anal. Appl. **446** (1) (2017), 864–878.

[12] T. POPOVICIU, *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*, Thèses de l'entre-deux-guerres, Faculté des Sciences de Paris, 1933.

[13] I. RAŞA, *2. Problem, p. 164. Ann. Univ. Paedagog. Crac. Stud. Math. 13, 139–169*, in: Report of Meeting Conference on Ulam's Type Stability, Rytro, Poland, June 2–6 (2014).

[14] Z. ZHAO AND M. KUMAR, *A split-Bernstein approach to chance constrained programs*, 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, 2014.

(Received September 6, 2024)

Raluca Paşca
Department of Mathematics
Technical University of Cluj-Napoca
Str. Memorandumului nr. 28, Cluj-Napoca, 400114, Romania
e-mail: Raluca.Pasca@math.utcluj.ro

Mircea Ivan
Department of Mathematics
Technical University of Cluj-Napoca
Str. Memorandumului nr. 28, Cluj-Napoca, 400114, Romania
e-mail: Mircea.Ivan@math.utcluj.ro

Bogdan Gavrea
School of Mathematical and Natural Sciences
Arizona State University
Glendale, AZ, U.S.A.
e-mail: Bogdan.Gavrea@asu.edu