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REMARKS ON EXTREMAL FUNCTIONS FOR THE ANISOTROPIC TRUDINGER–MOSER INEQUALITIES INVOLVING L^p NORM

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Abstract. Let $W^{1,n}(\mathbb{R}^n)$ $(n \geqslant 2)$ be the standard Sobolev space, and denote, for $p > n$

$$\gamma_1 = \inf_{u \in W^{1,n}(\mathbb{R}^n),\, u \not\equiv 0} \frac{\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx}{(\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}}},$$

where $F : \mathbb{R}^n \to [0, \infty)$ be a convex function of class $C^2(\mathbb{R}^n \backslash \{0\})$, which is even and positively homogeneous of degree 1. For $\gamma \in [0, \gamma_1)$, we define a norm in $W^{1,n}(\mathbb{R}^n)$ by

$$\|u\|_{F,n,\gamma,p} = \left(\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx - \gamma (\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}} \right)^{\frac{1}{n}}.$$

By performing a blow-up analysis, we prove that for real numbers $0 \leqslant \gamma < \gamma_1$ and $p > n$, the following anisotropic Trudinger-Moser inequality

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda_n |u|^{\frac{n}{n-1}}) dx$$

can be attained by some function $u_0 \in W^{1,n}(\mathbb{R}^n)$ with $\|u_0\|_{F,n,\gamma,p} = 1$, where $\Phi(t) = e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!}$, $\lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$ and κ_n is the volume of the unit Wulff ball. In the case $\gamma = 0$, this is reduced to a result of Zhou-Zhou [19].

1. Introduction

Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. We denote $W_0^{1,n}(\Omega)$ the closure of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W_0^{1,n}(\Omega)} = (\int_\Omega |\nabla u|^n dx)^{1/n}$. The Sobolev embedding theorem asserts that $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leqslant q < \infty$. But the embedding is not valid for $q = \infty$. In this case, the classical Trudinger-Moser inequality [18, 10, 9, 11, 8] claims that

$$\sup_{u \in W_0^{1,n}(\Omega),\, \int_\Omega |\nabla u|^n dx \leqslant 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx < \infty \tag{1}$$

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for any $\alpha \leqslant \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of the unit ball in $\mathbb{R}^n$. The inequality (1) is sharp: for any growth $e^{\alpha|u|^{n/(n-1)}}$ with $\alpha > \alpha_n$, the supremum is infinity. Moreover, when $\alpha \leqslant \alpha_n$, the supremum can be attained by some $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx = 1$, see also [2, 3, 6].

Due to wide range of applications in geometric analysis and partial differential equations, the Trudinger-Moser inequality (1) has been generalized in various ways. Recently, one interesting extension of (1) is the so-called anisotropic Trudinger-Moser inequality, which was originally established by Wang-Xia [14]. Let $F : \mathbb{R}^n \to [0,\infty)$ be a convex function of class $C^2(\mathbb{R}^n \backslash \{0\})$, which is even and positively homogeneous of degree 1. They obtained that the supremum

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega F^n(\nabla u)dx\leqslant 1} \int_\Omega e^{\lambda|u|^{\frac{n}{n-1}}} dx < \infty \tag{2}$$

for $\lambda \leqslant \lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$, here κ_n is the volume of the unit Wulff ball in $\mathbb{R}^n$. Moreover, the constant λ_n is optical in the sense that when $\lambda > \lambda_n$, we can find a sequence v_k such that $\int_\Omega e^{\lambda|v_k|^{n/(n-1)}} dx$ diverges. For the attainability of the supremum in (2), this has been done by Zhou-Zhou [19]. Recently, they also extended (2) to the unbounded domain in [20], which can be described as follows

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(F^n(\nabla u)+|u|^n)dx\leqslant 1} \int_{\mathbb{R}^n} \Psi(\lambda_n|u|^{\frac{n}{n-1}})dx < \infty, \tag{3}$$

where $\Psi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$, and the supremum can be attained by some function $u \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}(F^n(\nabla u) + |u|^n)dx = 1$. Liu [7] obtained the extremal functions for an improved Trudinger-Moser inequality on a smooth bounded domain. More precisely, we denote a norm in $W_0^{1,n}(\Omega)$

$$\|u\|_D = \left(\int_\Omega F^n(\nabla u)dx - \tau\|u\|_p^n\right)^{\frac{1}{n}}$$

for $p > 1$ and $0 \leqslant \tau < \inf_{u\in W_0^{1,n}(\Omega),u\neq 0} \frac{\|F(\nabla u)\|_n^n}{\|u\|_p^n}$. Then there holds

$$\sup_{u\in W_0^{1,n}(\Omega),\|u\|_D\leqslant 1} \int_\Omega e^{\lambda_n|u|^{\frac{n}{n-1}}} dx < \infty \tag{4}$$

and the supremum in (4) can be attained.

2. Main results

In this note, we will consider possible extensions of the anisotropic Trudinger-Moser inequality involving L^p norm for the unbound domain in $\mathbb{R}^n$, and complement the main results in [7, 20]. For any $u \in W^{1,n}(\mathbb{R}^n)$ and $p > n$, denote

$$\gamma_1 = \inf_{u \in W^{1,n}(\mathbb{R}^n), u \not\equiv 0} \frac{\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx}{(\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}}}.$$

For $0 \leqslant \lambda < \lambda_n$, we define

$$\|u\|_{F,n,\gamma,p} = \left(\int_{\mathbb{R}^n} (F^n(\nabla u) + |u|^n) dx - \gamma (\int_{\mathbb{R}^n} |u|^p dx)^{\frac{n}{p}} \right)^{\frac{1}{n}}.$$

Our first result can be stated as follows:

THEOREM 1. *Let $n \geqslant 2$, $p > n$ and $0 \leqslant \gamma < \gamma_1$. Then*
(1) For any $0 \leqslant \gamma < \gamma_1$, there holds

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \, \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda |u|^{\frac{n}{n-1}}) dx < \infty; \tag{5}$$

(2) For any $\lambda > \lambda_n$, the supremum infinity, i.e.

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \, \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda |u|^{\frac{n}{n-1}}) dx = +\infty,$$

where

$$\Phi(t) = e^t - \sum_{j=0}^{n-1} \frac{t^j}{j!}.$$

As an immediate consequence of the preceding theorem, we have

COROLLARY 1. *For any $0 \leqslant \gamma < \gamma_1$, we have*

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Psi(\lambda_n |u|^{\frac{n}{n-1}}) dx < \infty, \tag{6}$$

where $\Psi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$.

For the existence of extremals for (5), we have the following:

THEOREM 2. *Let $n \geqslant 2$, $p > n$, for any $0 \leqslant \gamma < \gamma_1$, there exists $u_0 \in W^{1,n}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ with $\|u_0\|_{F,n,\gamma,p} = 1$ such that*

$$\int_{\mathbb{R}^n} \Phi(\lambda_n |u_0|^{\frac{n}{n-1}}) dx = \sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda_n |u|^{\frac{n}{n-1}}) dx.$$

We mention that Corollary 1 fully extends [20, Theorem 1.2] and [7, Theorem 1.1] for the entire space, while Theorem 2 partially extends [7, Theorem 1.2] because here we study the modified function $\Phi(t)$ which is obtained for $\Psi(t)$ by subtracting the term corresponding to the L^n norm. This helps us to yield the compactness necessary to prove the attainability of the superemum in (5).

Here and throughout this note, let us now denote $F^o(x)$ is the polar function of $F(x)$. Actually, $F^o(x)$ is dual to F in the sense that

$$F^o(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, \xi \rangle}{F^o(\xi)}.$$

We use the notation $\mathscr{W}_\rho := \{x \in \mathbb{R}^n : F^o(x) \leqslant \rho\}$ to represent a Wulff ball of radius ρ with the center at 0 and the same letter C to denote constants.

Recall that, for a measurable function u on $\Omega \subset \mathbb{R}^n$, the one-dimensional decreasing rearrangement of u is

$$u^\star(t) = \sup\{s \geqslant 0 : |\{x \in \Omega : |u(x)| > s\}| > t\}$$

for $t \in \mathbb{R}$. The convex symmetrization of u with respect to F is defined by

$$u^*(x) = u^\star(\kappa_n F^o(x)^n), \quad x \in \Omega^*.$$

Here Ω^* is the homothetic Wulff ball centered at the origin having the same measure as Ω. Other results about convex symmetrization may be found in [1].

The remaining part of this note is organized as follows: In section 3, we prove point (2) of Theorem 1. We use the blow-up analysis to prove point (1) of Theorem 1 and Theorem 2. In section 4 , we obtain the existence of the subcritical maximizers. In section 5, we analyze the convergence of maximizers sequence and its blow-up behavior. In section 6, a sequence of functions is constructed to reach a contraction, which completes the proof of point (1) of Theorem 1 and Theorem 2.

3. Test functions computations

In order to prove point (2) of Theorem 1, we consider the sequence defined, for $k \in \mathbb{N}$, as

$$w_k(x) = \frac{1}{\sqrt[n]{n\kappa_n}} \begin{cases} (\log k)^{\frac{n-1}{n}}, & \text{if} \quad 0 \leqslant F^o(x) < \frac{L_k}{k}, \\ \frac{\log\left(\frac{L_k}{F^o(x)}\right)}{\sqrt[n]{\log k}}, & \text{if} \quad \frac{L_k}{k} \leqslant F^o(x) < L_k, \\ 0, & \text{if} \quad F^o(x) \geqslant L_k, \end{cases}$$

where $L_k = \frac{(\log k)^{\frac{1}{2np}}}{\log(\log k)}$. Obviously, $\{w_k\} \subset W^{1,n}(\mathbb{R}^n)$ be a sequence consisting of radial symmetric functions with respect to $F^o(x)$ and $L_k \to 0$ as $k \to \infty$. Moreover, we have by straightforward calculation

$$\int_{\mathbb{R}^n} F^n(\nabla w_k) dx = \frac{1}{\log k} \int_{\frac{L_k}{k}}^{L_k} \frac{1}{t} dt = 1,$$

and

$$\int_{\mathscr{W}_{\frac{L_k}{k}}} |w_k|^n dx = (\log k)^{n-1} \int_0^{\frac{L_k}{k}} t^{n-1} dt = \frac{L_k^n (\log k)^{n-1}}{nk^n} = o_k(1).$$

Integration by parts, it follows that

$$\begin{aligned}\int_{\mathscr{W}_{L_k} \setminus \mathscr{W}_{\frac{L_k}{k}}} |w_k|^n dx &= \frac{L_k^n}{\log k} \int_{\frac{L_k}{k}}^{k} \left(\log \frac{L_k}{t}\right)^n t^{n-1} dt \\ &= \frac{L_k^n}{\log k} \frac{(n-1)!}{n^{n-2}} \int_{\frac{1}{k}}^{1} \log\left(\frac{1}{s}\right) s^{n-1} ds + o_k(1) \\ &= \frac{L_k^n}{\log k} \frac{(n-1)!}{n^n} \left(1 - \frac{1}{k^n}\right) + o_k(1) = o_k(1).\end{aligned}$$

Similarly, we also yield

$$\int_{\mathbb{R}^n} |w_k|^p dx = \frac{L_k^n}{k^n} \frac{(\log k)^{\frac{n-1}{n}p}}{n^{\frac{p}{n}} \kappa_n^{\frac{p}{n}-1}} + \frac{L_k^n}{(\log k)^{\frac{n}{p}}} \frac{p!(n\kappa_n)^{1-\frac{n}{p}}}{n^{p+1}} \left(1 - \frac{1}{k^n}\right) + o_k(1) = o_k(1).$$

In view of the above estimates, we obtain

$$\|w_k\|_{F,n,\gamma,p}^n = 1 + o_k(1).$$

Considering $\widetilde{w}_k = w_k / \|w_k\|_{F,n,\gamma,p}$, we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(\lambda |\widetilde{w}_k|^{\frac{n}{n-1}}) dx &\geqslant \int_{\mathscr{W}_{\frac{L_k}{k}}} \left(e^{\lambda |\widetilde{w}_k|^{\frac{n}{n-1}}} - \sum_{j=0}^{n-1} \frac{\lambda^j |\widetilde{w}_k|^{\frac{n}{n-1}}}{j!}\right) dx \\ &\geqslant \left(k^{\frac{\lambda}{(n\kappa_n)^{\frac{1}{n-1}}}} e^{O(1)} + O((\log k)^{n-1})\right) \frac{\kappa_n L_k^n}{k^n}.\end{aligned}$$

The last term on the right hand side goes to infinity as $k \to \infty$, thanks to $\lambda > \lambda_n$. Thus point (2) of Theorem 1 is finished.

4. The subcritical functionals

For notation convenience, we set

$$FTM := \sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\lambda_n |u|^{\frac{n}{n-1}}) dx$$

and also write

$$FTM_\varepsilon(u) := \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} |u|^{\frac{n}{n-1}}) dx,$$

where $\lambda_{n,\varepsilon} = \lambda_n - \varepsilon$ for $0 < \varepsilon < \lambda_n$. We have the following lemma.

LEMMA 1. *Let* $p > n \geqslant 2$, $0 \leqslant \gamma < \gamma_1$. *Then for any* $0 < \varepsilon < \lambda_n$, *there exists some function* $u_\varepsilon \in W^{1,n}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ *such that* $\|u_\varepsilon\|_{F,n,\gamma,p} = 1$ *and*

$$FTM_\varepsilon(u_\varepsilon) = \sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} FTM_\varepsilon(u).$$

Moreover, u_ε *can be chosen to be nonnegative, radially symmetric and radially decreasing with respect to* $F^o(x)$.

Proof. For any $u \in W^{1,n}(\mathbb{R}^n)$, let u^* be the convex symmetrization of u with respect to $F^o(x)$. It is known that $\|F(\nabla u^*)\|_{L^n(\mathbb{R}^n)} \leqslant \|F(\nabla u)\|_{L^n(\mathbb{R}^n)}$, $\|u^*\|_{L^n(\mathbb{R}^n)} = \|u\|_{L^n(\mathbb{R}^n)}$, $\|u^*\|_{L^p(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}$, and

$$FTM_\varepsilon(u^*) \geqslant FTM_\varepsilon(u), \tag{7}$$

On the other hand,

$$FTM_\varepsilon(u^*) \leqslant \sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} FTM_\varepsilon(u),$$

which together with (7) implies that

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{F,n,\gamma,p} \leqslant 1} FTM_\varepsilon(u) = \sup_{u \in \Im, \|u\|_{F,n,\gamma,p} \leqslant 1} FTM_\varepsilon(u),$$

where $\Im$ is a set consisting of all nonnegative radially symmetric functions with respect to $F^o(x)$. Without of generality, we choose a sequence $\{v_i\} \subset \Im$ with $\|v_i\|_{F,n,\gamma,p} = 1$, such that

$$FTM_\varepsilon(v_i) \to \sup_{u \in \Im, \|u\|_{F,n,\gamma,p} \leqslant 1} FTM_\varepsilon(u) \text{ as } i \to \infty. \tag{8}$$

Since v_i is bounded in $W^{1,n}(\mathbb{R}^n)$, we can assume up to a subsequence that

$$\begin{cases} v_i \rightharpoonup u_\varepsilon \text{ weakly in } W^{1,n}(\mathbb{R}^n), \\ v_i \to u_\varepsilon \text{ strongly in } L^s_{\text{loc}}(\mathbb{R}^n), \ \forall s > 1, \\ v_i \to u_\varepsilon \text{ a.e. in } \mathbb{R}^n. \end{cases}$$

We can easily get that $u_\varepsilon \in \Im$. From the weak convergence of v_i in $W^{1,n}(\mathbb{R}^n)$, we see $\|u_\varepsilon\|_{F,n,\gamma,p} \leqslant \limsup_{i\to\infty} \|v_i\|_{F,n,\gamma,p} \leqslant 1$. Since $u \in \Im$, $u^n(\rho)|\mathscr{W}_\rho| \leqslant \int_{\mathscr{W}_\rho} u^n dx \leqslant \frac{\gamma_1}{\gamma_1 - \gamma}$, and so

$$u(x) \leqslant u(\rho) \leqslant \frac{\|u\|_{L^n(\mathbb{R}^n)}}{\sqrt[n]{\kappa_n}\rho} \leqslant \frac{C_{n,\gamma}}{\rho}, \quad x \in \mathscr{W}^c_\rho. \tag{9}$$

In view of (9), we deduce

$$\begin{aligned} \int_{\mathscr{W}^c_R} \Phi(\lambda_{n,\varepsilon} u^{\frac{n}{n-1}}) dx &= \sum_{j=n}^{\infty} \int_{\mathscr{W}^c_R} \frac{\lambda^j_{n,\varepsilon}}{j!} u^{\frac{nj}{n-1}} dx \\ &\leqslant \sum_{j=n}^{\infty} \frac{\kappa_n(n-1)}{j-n+1} \frac{\lambda^j_{n,\varepsilon}}{j!} \frac{C^{\frac{n}{n-1}j}_{n,\gamma}}{R^{\frac{n}{n-1}j-n}}. \end{aligned}$$

Hence we can choose $R > 0$ sufficiently large such that

$$\int_{\mathscr{W}_R^c} \Phi(\lambda_{n,\varepsilon} u^{\frac{n}{n-1}}) dx < \nu \tag{10}$$

for any $\nu > 0$. On the other hand, we have by the mean value theorem

$$\begin{aligned} \Phi(\lambda_{n,\varepsilon} v_j^{\frac{n}{n-1}}) - \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) &= \Phi'(\vartheta)\lambda_{n,\varepsilon}(v_i^{\frac{n}{n-1}} - u_\varepsilon^{\frac{n}{n-1}}) \\ &\leqslant \max\{\Phi'(\lambda_{n,\varepsilon} v_i^{\frac{n}{n-1}}), \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})\} \\ &\quad \times \lambda_{n,\varepsilon}(v_i^{\frac{n}{n-1}} - u_\varepsilon^{\frac{n}{n-1}}), \end{aligned} \tag{11}$$

where ϑ lies between $\lambda_{n,\varepsilon} v_i^{\frac{n}{n-1}}$ and $\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}$. A simple modification of the argument in [16] (Lemma 2.1) yields the following estimate: for $s \geqslant 1$, $t \geqslant 0$, there holds

$$\Phi(t)^s \leqslant \Phi(st). \tag{12}$$

It follows that

$$\begin{aligned} \int_{\mathscr{W}_R} \Phi'(\vartheta)^s dx &\leqslant \int_{\mathscr{W}_R} \Phi'(\lambda_{n,\varepsilon} s v_i^{\frac{n}{n-1}}) dx + \int_{\mathscr{W}_R} \Phi'(\lambda_{n,\varepsilon} s u_\varepsilon^{\frac{n}{n-1}}) dx \\ &\leqslant \int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s v_i^{\frac{n}{n-1}}} dx + \int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s u_\varepsilon^{\frac{n}{n-1}}} dx + C_1 \\ &\leqslant \int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s v_i^{\frac{n}{n-1}}} dx + C. \end{aligned}$$

We now estimate the first integral. Taking $v_{i,R} = v_i(x) - v_i(R)$, one can derive that

$$v_i^{\frac{n}{n-1}}(x) \leqslant (1+\delta) v_{i,R}^{\frac{n}{n-1}} + C_\delta v_i^{\frac{n}{n-1}}(R)$$

for each $\delta > 0$ and $v_{i,R} \in W_0^{1,n}(\mathscr{W}_R)$. Furthermore, the Hölder inequality implies

$$\int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s v_i^{\frac{n}{n-1}}(x)} dx \leqslant \left(\int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s s_1 (1+\delta) v_{i,R}^{\frac{n}{n-1}}(x)} dx \right)^{\frac{1}{s_1}} \left(\int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon} s s_2 C_\delta v_i^{\frac{n}{n-1}}(R)} dx \right)^{\frac{1}{s_2}}.$$

Choosing $s > 1$ and $s_1 > 1$ sufficiently close to 1 and $\delta > 0$ sufficiently small such that $\lambda_{n,\varepsilon}(1+\delta) s s_1 < \lambda_n$, noting Turdinger-Moser inequality (4), one can see that $e^{\lambda_{n,\varepsilon} s v_i^{n/(n-1)}}$ is bounded in $L^1(\mathscr{W}_R)$. We employ this fact, thereby obtaining

$$\int_{\mathscr{W}_R} \Phi'(\vartheta)^s dx \leqslant C.$$

This inequality together with (11) and the fact that $v_i \to u_\varepsilon$ in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for any $q > 0$, gives

$$\lim_{i \to \infty} \int_{\mathscr{W}_R} \Phi(\lambda_{n,\varepsilon} v_i^{\frac{n}{n-1}}) dx = \int_{\mathscr{W}_R} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx.$$

Combining now (8) and (10), we obtain

$$\lim_{i\to\infty} FTM_\varepsilon(v_i) = FTM_\varepsilon(u_\varepsilon) = \sup_{u\in\Im, \|u\|_{F,n,\gamma,p}\leqslant 1} FTM_\varepsilon(u).$$

It is easy to check that $u_\varepsilon \not\equiv 0$. Also, we must have $\|u_\varepsilon\|_{F,n,\gamma,p} = 1$. Suppose this is not true. That is, $0 < \|u_\varepsilon\|_{F,n,\gamma,p} < 1$. It follows that

$$FTM_\varepsilon(u_\varepsilon/\|u_\varepsilon\|_{F,n,\gamma,p}) > FTM_\varepsilon(u_\varepsilon) = \sup_{u\in\Im, \|u\|_{F,n,\gamma,p}\leqslant 1} FTM_\varepsilon(u),$$

which is impossible. Moreover, by a straightforward calculation, we derive the Euler-Lagrange equation of u_ε as follows:

$$\begin{cases} -Q_n u_\varepsilon = \dfrac{u_\varepsilon^{\frac{1}{n-1}}}{\alpha_\varepsilon}\Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) - u_\varepsilon^{n-1} + \gamma \|u_\varepsilon\|_p^{n-p} u_\varepsilon^{p-1} \text{ in } \mathbb{R}^n, \\ u_\varepsilon \geqslant 0, \ \ \|u_\varepsilon\|_{F,n,\gamma,p} = 1 \text{ in } \mathbb{R}^n, \\ \alpha_\varepsilon = \int_{\mathbb{R}^n} u_\varepsilon^{\frac{n}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx, \end{cases} \tag{13}$$

where $Q_n u = \sum_{j=1}^n \frac{\partial}{\partial x_j}(F^{n-1}(\nabla u) F_{\xi_j}(\nabla u))$ is a Finsler Laplacian operator. Applying the standard elliptic estimate to (13), we have $u_\varepsilon \in C^1(\mathbb{R}^n)$. This completes the proof of the lemma. □

5. Blow-up analysis

In this section, we use the method of blow-up analysis to describe the asymptotic behavior of the maximizers u_ε, the proof is inspired by the works [4, 5, 7, 15, 17, 20].

We now assert that

$$\liminf_{\varepsilon\to 0} \alpha_\varepsilon > 0.$$

To see this, assume by contradiction that $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. By the inequality $\Phi(t) \leqslant t\Phi'(t)$ for $t \geqslant 0$, we have

$$\int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx \leqslant \lambda_{n,\varepsilon} \int_{\mathbb{R}^n} u_\varepsilon^{\frac{n}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx. \tag{14}$$

But we deduce upon sending $\varepsilon \to 0$ in (14) that $FTM_\varepsilon(u_\varepsilon) = 0$. It is impossible.

Recalling $\|u_\varepsilon\|_{F,n,p,\gamma} = 1$, we thereby obtain u_ε is bounded in $W^{1,n}(\mathbb{R}^n)$. We may assume $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,n}(\mathbb{R}^n)$, $u_\varepsilon \to u_0$ strongly in $L^q_{\text{loc}}(\mathbb{R}^n)$ for $q > 1$. In particular, it is worth remarking that u_ε converges strongly to u_0 in $L^s(\mathbb{R}^n)$ for $s \geqslant n$. In fact, let $\eta_1 \in C_0^\infty(\mathbb{R}^n, [0,1])$ such that $|\nabla \eta_1| \leqslant C/R$ and

$$\eta_1(x) = \begin{cases} 0, & \text{if } x \in \mathscr{W}_R, \\ 1, & \text{if } x \in \mathbb{R}^n \setminus \mathscr{W}_{2R}. \end{cases}$$

Multiply (13) by $\eta_1 u_\varepsilon$ and integrate on $\mathbb{R}^n$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}^n} F^n(\nabla u_\varepsilon)\eta_1 dx + \int_{\mathbb{R}^n} u_\varepsilon F^{n-1}(\nabla u_\varepsilon)F_\xi(\nabla u_\varepsilon)\nabla\eta_1 dx + \int_{\mathbb{R}^n} \eta_1 u_\varepsilon^n dx\\ &= \frac{1}{\alpha_\varepsilon}\int_{\mathbb{R}^n} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})\eta_1 u_\varepsilon^{\frac{n}{n-1}} dx + \gamma\|u_\varepsilon\|_p^{n-p}\int_{\mathbb{R}^n}\eta_1 u_\varepsilon^p dx. \end{aligned} \tag{15}$$

Since $u_\varepsilon \in \mathfrak{I}$, one has

$$\int_{\mathscr{W}_R^c} u_\varepsilon^{\frac{n}{n-1}(j+1)} dx \leqslant \frac{\kappa_n(n-1)}{j+2-n}\frac{C_{n,\gamma}^{\frac{n}{n-1}(j+1)}}{R^{\frac{n}{n-1}(j+1)-n}} \leqslant \frac{C}{R}$$

thanks to (9), for $x \in \mathscr{W}_R^c$, $j \geqslant n-1$ and $R > 1$. Noting also that $\sum_{j=n-1}^{\infty}\frac{\lambda_{n,\varepsilon}^j}{j!}$ converges, we find

$$\int_{\mathbb{R}^n} \eta_1 u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx \leqslant \sum_{j=n-1}^{\infty}\frac{\lambda_{n,\varepsilon}^j}{j!}\int_{\mathscr{W}_R^c} u_\varepsilon^{\frac{n(j+1)}{n-1}} dx \leqslant \frac{C}{R}.$$

We use the Hölder inequality to discover that

$$\int_{\mathbb{R}^n} u_\varepsilon F^{n-1}(\nabla u_\varepsilon)F_\xi(\nabla u_\varepsilon)\nabla\eta_1 dx \leqslant \frac{C}{R}\|F(\nabla u_\varepsilon)\|_n^{n-1}\|u_\varepsilon\|_n^n \leqslant \frac{C}{R}.$$

Also

$$\|u_\varepsilon\|_p^{n-p}\int_{\mathbb{R}^n}\eta_1 u_\varepsilon^p dx \leqslant \frac{C}{R^{p-n}}.$$

Inserting the above estimates into (15), we have

$$\int_{\mathscr{W}_R^c} u_\varepsilon^n dx \leqslant \frac{C}{R} + \frac{C}{R^{p-n}}.$$

Thus we find that for any $\nu > 0$, there exists $R_1 > 0$ sufficiently large such that

$$\int_{\mathscr{W}_{R_1}^c} u_\varepsilon^n dx \leqslant \frac{\nu}{3}.$$

By the absolute integrability of u_0, there exists $R_2 > 0$, satisfying

$$\int_{\mathscr{W}_{R_2}^c} u_0^n dx \leqslant \frac{\nu}{3}.$$

Choosing $R_0 = \max\{R_1, R_2\}$, we have

$$\int_{\mathscr{W}_{R_0}^c} |u_\varepsilon^n - u_0^n| dx < \frac{\nu}{3}.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u_\varepsilon^n dx = \int_{\mathbb{R}^n} u_0^n dx.$$

In addition, for $s>n$, taking $R_3>0$, such that $u_\varepsilon<1$ if $F^o(x)>R_3$, then we have $\int_{\mathscr{W}^c_{\widetilde{R}}} u_\varepsilon^s dx \leqslant \int_{\mathscr{W}^c_{\widetilde{R}}} u_\varepsilon^n dx$ where $\widetilde{R}=\max\{R_1,R_3\}$. Similarly as above, we get for $s>n$, u_ε converges strongly to u_0 in $L^s(\mathbb{R}^n)$.

Denote

$$c_\varepsilon = u_\varepsilon(0) = \max_{x\in\mathbb{R}^n} u_\varepsilon(x).$$

For the remainder of this section, we will suppose that $c_\varepsilon \to +\infty$ as $\varepsilon\to 0$. Along this way we will need the following result.

LEMMA 2. *Let $c_\varepsilon\to+\infty$ as $\varepsilon\to 0$. Then u_ε has two elementary properties: (i) $u_0\equiv 0$; (ii) $F^n(\nabla u_\varepsilon)dx \rightharpoonup \delta_0$ weakly in the sense of measure, δ_0 denoting the Dirac measure on giving unit mass to the point 0.*

Proof. Assume the result (ii) does not hold. Then there exists $\overline{R}>0$ such that

$$\limsup_{\varepsilon\to 0}\int_{\mathscr{W}_{\overline{R}}} F^n(\nabla u_\varepsilon)dx < 1-\mu$$

for $0<\mu<1$. We set $\overline{u}_\varepsilon(x)=u_\varepsilon(x)-u_\varepsilon(\overline{R})$ for $x\in\mathscr{W}_{\overline{R}}$ and thus $\overline{u}_\varepsilon(x)\in W_0^{1,n}(\mathscr{W}_{\overline{R}})$. Accordingly $\|F(\nabla\overline{u}_\varepsilon)\|^n_{L^n(\mathscr{W}_{\overline{R}})} = \|F(\nabla u_\varepsilon)\|^n_{L^n(\mathscr{W}_{\overline{R}})} < 1-\mu$. Recall the fundamental inequality (12), we have by the Hölder inequality

$$\begin{aligned}
\int_{\mathscr{W}_{\overline{R}}}\left(\frac{u_\varepsilon^{\frac{1}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})}{\alpha_\varepsilon}\right)^s dx &\leqslant \frac{1}{\alpha_\varepsilon^s}\int_{\mathscr{W}_{\overline{R}}}\left(u_\varepsilon^{\frac{s}{n-1}}\Phi'(\lambda_{n,\varepsilon}su_\varepsilon^{\frac{n}{n-1}})\right)dx\\
&\leqslant \frac{1}{\alpha_\varepsilon^s}\left(\int_{\mathscr{W}_{\overline{R}}} u_\varepsilon^{\frac{ss_1}{n-1}}dx\right)^{\frac{1}{s_1}}\left(\int_{\mathscr{W}_{\overline{R}}}\Phi'(\lambda_{n,\varepsilon}ss_2u_\varepsilon^{\frac{n}{n-1}})dx\right)^{\frac{1}{s_2}}\\
&\leqslant \frac{1}{\alpha_\varepsilon^s}\left(\int_{\mathscr{W}_{\overline{R}}} u_\varepsilon^{\frac{ss_1}{n-1}}dx\right)^{\frac{1}{s_1}}\left(\int_{\mathscr{W}_{\overline{R}}} e^{\lambda_{n,\varepsilon}ss_2u_\varepsilon^{\frac{n}{n-1}}}dx\right)^{\frac{1}{s_2}}, \qquad (16)
\end{aligned}$$

where $s,s_1,s_2>1$ and $\frac{1}{s_1}+\frac{1}{s_2}=1$. Meanwhile, for any $\nu>0$, there exists some constant C_0 depending on n and ν, such that for all $x\in\mathscr{W}_{\overline{R}}$,

$$u_\varepsilon^{\frac{n}{n-1}} \leqslant (1+\nu)\overline{u}_\varepsilon^{\frac{n}{n-1}} + C_0u_\varepsilon^{\frac{n}{n-1}}(\overline{R}) \leqslant (1+\nu)\overline{u}_\varepsilon^{\frac{n}{n-1}} + C\overline{R}^{\frac{n}{n-1}}. \qquad (17)$$

Here we used (9). Choosing $\nu>0$ sufficiently small and $s,s_2>1$ sufficiently close to 1, such that

$$ss_2(1+\nu)\|F(\nabla\overline{u}_\varepsilon)\|^{\frac{n}{n-1}}_{L^n(\mathscr{W}_{\overline{R}})} < 1.$$

Inserting (17) into (16), and noting that u_ε is bounded in $L^q(\mathscr{W}_{\overline{R}})$ for $q>1$, one can see from (2) that

$$\int_{\mathscr{W}_{\overline{R}}}\left(\alpha_\varepsilon^{-1}u_\varepsilon^{\frac{1}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})\right)^s dx \leqslant C \qquad (18)$$

for some $s>1$. Also $\|u_\varepsilon\|_p^{n-p}u_\varepsilon^{p-1}$ is bounded in $L^{\frac{p}{p-1}}(\mathscr{W}_{\overline{R}})$. Applying the standard elliptic estimate to (13), we get u_ε is uniformly bounded in $\mathscr{W}_{\overline{R}/2}$. This result contradicts $c_\varepsilon \to +\infty$ as $\varepsilon\to 0$. This confirms that $F^n(\nabla u_\varepsilon)dx \rightharpoonup \delta_0$ weakly in the sense of measure.

Now according to $\|u_\varepsilon\|_{F,n,\gamma,p}=1$ and $F^n(\nabla u_\varepsilon)dx \rightharpoonup \delta_0$, we get $\int_{\mathbb{R}^n} u_\varepsilon^n dx = o_\varepsilon(1)$, $\int_{\mathbb{R}^n} u_\varepsilon^p dx = o_\varepsilon(1)$ for $0<\gamma<\gamma_1$. Then we have

$$\int_{\mathbb{R}^n} u_0^n dx \leqslant \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n} u_\varepsilon^n dx = 0.$$

It follows that $u_0\equiv 0$. □

LEMMA 3. *Let $r_\varepsilon^n = \alpha_\varepsilon c_\varepsilon^{-\frac{n}{n-1}} e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}}$. Then for any $\sigma<\frac{\lambda_n}{n}$, we have*

$$\lim_{\varepsilon\to 0} r_\varepsilon^n e^{n\sigma c_\varepsilon^{\frac{n}{n-1}}} = 0.$$

Proof. By definition of r_ε, we obtain

$$r_\varepsilon^n e^{n\sigma c_\varepsilon^{\frac{n}{n-1}}} = \frac{e^{(n\sigma-\lambda_{n,\varepsilon})c_\varepsilon^{\frac{n}{n-1}}}}{c_\varepsilon^{\frac{n}{n-1}}}\int_{\mathbb{R}^n} u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx. \tag{19}$$

Note that, for any $R>0$

$$\int_{\mathscr{W}_R^c} u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx = \sum_{j=n-1}^{\infty}\frac{\lambda_{n,\varepsilon}^j}{j!}\int_{\mathscr{W}_R^c} u_\varepsilon^{\frac{n(j+1)}{n-1}}dx \leqslant C(R),$$

and therefore

$$\lim_{\varepsilon\to 0}\frac{e^{(n\sigma-\lambda_{n,\varepsilon})c_\varepsilon^{\frac{n}{n-1}}}}{c_\varepsilon^{\frac{n}{n-1}}}\int_{\mathscr{W}_R^c} u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx = 0. \tag{20}$$

On the other hand, using the fact

$$-(\lambda_{n,\varepsilon}-n\sigma)c_\varepsilon^{\frac{n}{n-1}} \leqslant -(\lambda_{n,\varepsilon}-n\sigma)u_\varepsilon^{\frac{n}{n-1}}$$

and proving in a similar manner as in (18), we get

$$e^{(n\sigma-\lambda_{n,\varepsilon})c_\varepsilon^{\frac{n}{n-1}}}\int_{\mathscr{W}_R} u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx \leqslant \int_{\mathscr{W}_R} u_\varepsilon^{\frac{n}{n-1}} e^{n\sigma u_\varepsilon^{\frac{n}{n-1}}}dx \leqslant C(R)$$

and thus

$$\lim_{\varepsilon\to 0}\frac{e^{(n\sigma-\lambda_{n,\varepsilon})c_\varepsilon^{\frac{n}{n-1}}}}{c_\varepsilon^{\frac{n}{n-1}}}\int_{\mathscr{W}_R} u_\varepsilon^{\frac{n}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx = 0. \tag{21}$$

The desire result follows from (19)–(21). □

In order to derive the asymptotic of u_ε near the blow-up point, we first define

$$v_\varepsilon(x) = c_\varepsilon^{-1} u_\varepsilon(r_\varepsilon x) \tag{22}$$

and

$$w_\varepsilon(x) = c_\varepsilon^{\frac{1}{n-1}} (u_\varepsilon(r_\varepsilon x) - c_\varepsilon). \tag{23}$$

LEMMA 4. *Suppose $v_\varepsilon(x)$ and $w_\varepsilon(x)$ be defined as in (22) and (23). Then $v_\varepsilon(x) \to 1$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w_\varepsilon(x) \to w_0(x)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, w_0 satisfies*

$$-Q_n w_0(x) = e^{\frac{n}{n-1}\lambda_n w_0} \quad \text{in} \quad \mathbb{R}^n \tag{24}$$

in the distributional sense.

Proof. For equation (13), we can compute

$$-Q_n v_\varepsilon = c_\varepsilon^{-n} e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}} v_\varepsilon^{\frac{1}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon x)) - r_\varepsilon^n v_\varepsilon^{n-1} + \gamma c_\varepsilon^{p-n} r_\varepsilon^n \|u_\varepsilon\|_p^{n-p} v_\varepsilon^{p-1} \tag{25}$$

and

$$-Q_n w_\varepsilon = e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}} v_\varepsilon^{\frac{1}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon x)) - r_\varepsilon^n c_\varepsilon^n v_\varepsilon^{n-1} + \gamma c_\varepsilon^p r_\varepsilon^n \|u_\varepsilon\|_p^{n-p} v_\varepsilon^{p-1}. \tag{26}$$

Utilizing the fact $|v_\varepsilon| \leqslant 1$ and the decay estimate of r_ε, we infer that

$$\left\| c_\varepsilon^p r_\varepsilon^n \|u_\varepsilon\|_p^{n-p} v_\varepsilon^{p-1} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R}^n)} = c_\varepsilon r_\varepsilon^{\frac{n}{p}} \|u_\varepsilon\|_p^{n-1} = o_\varepsilon(1).$$

In addition,

$$\begin{aligned} h_\varepsilon(x) :&= c_\varepsilon^{-n} v_\varepsilon^{\frac{1}{n-1}} e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon x)) \\ &= c_\varepsilon^{-n} v_\varepsilon^{\frac{1}{n-1}} e^{\lambda_{n,\varepsilon}(u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon x) - c_\varepsilon^{\frac{n}{n-1}})} - c_\varepsilon^{-n} v_\varepsilon^{\frac{1}{n-1}} e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}} \sum_{j=0}^{n-2} \frac{\lambda_{n,\varepsilon}^j u_\varepsilon^{\frac{jn}{n-1}}(r_\varepsilon x)}{j!}. \end{aligned}$$

It follows that $h_\varepsilon(x)$ is uniformly bounded in $L^\infty(\mathscr{W}_R)$ for fixed $R > 0$. We can apply Theorem 1 in [13] to equation (25) and hence infer $v_\varepsilon \to v_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, here v_0 satisfies

$$-Q_n v_0 = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Since $v_0(0) = 1$, the Liouville theorem leads to $v_0 \equiv 1$ in $\mathbb{R}^n$.

For simplicity, all terms on the right side of (26) are marked as $g_\varepsilon(x)$. Clearly, $g_\varepsilon(x)$ is bounded in $L^q(\mathscr{W}_R)$ for some $q > 1$. Also $-w_\varepsilon \geqslant 0$, so that by Theorems 6 and 8 in [12], we can obtain w_ε is uniformly bounded in $\mathscr{W}_{R/2}$ and consequently we have $-Q_n w_\varepsilon = O(1)$ in $\mathscr{W}_R$. Then Theorem 1 in [13] together with Ascoli-Arzele's theorem implies there exists w_0, such that $w_\varepsilon \to w_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. A direct computation similar as [4], the details of which we omit, verifies

$$v_\varepsilon^{\frac{1}{n-1}} e^{-\lambda_{n,\varepsilon} c_\varepsilon^{\frac{n}{n-1}}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon x)) = (1 + o_\varepsilon(1)) e^{\frac{n}{n-1}\lambda_n w_0} + o_\varepsilon(1).$$

Also we have $r_\varepsilon^n c_\varepsilon^n v_\varepsilon^{n-1} = o_\varepsilon(1)$ and $c_\varepsilon^p r_\varepsilon^n \|u_\varepsilon\|_p^{n-p} v_\varepsilon^{p-1} = o_\varepsilon(1)$. Therefore w_0 satisfies (24) with $w_0(0) = 0 = \max_{x\in\mathbb{R}^n} w_\varepsilon(x)$. $\square$

We can proceed as in [20] that

$$w_0(x) = -\frac{n-1}{\lambda_n}\log\left(1+\kappa_n^{\frac{1}{n-1}} F^o(x)^{\frac{n}{n-1}}\right).$$

Integration by parts, we obtain

$$\int_{\mathbb{R}^n} e^{\lambda_n \frac{n}{n-1} w_0} dx = n\kappa_n \int_0^\infty \frac{r^{n-1}}{\left(1+\kappa_n^{\frac{1}{n-1}} r^{\frac{n}{n-1}}\right)^n} dr = 1. \tag{27}$$

Next we shall be concerned with the convergence of u_ε away from 0. Following [5], define

$$u_{\varepsilon,\beta} = \min\{u_\varepsilon, \beta c_\varepsilon\}.$$

Then we establish the following result.

LEMMA 5. *For each* $0<\beta<1$, *there holds*

$$\lim_{\varepsilon\to 0} \|F(\nabla u_{\varepsilon,\beta})\|_{F,n,\gamma,p}^n = \beta.$$

Proof. Since

$$|\{x|u_\varepsilon \geqslant \beta c_\varepsilon\}|(\beta c_\varepsilon)^n \leqslant \int_{u_\varepsilon \geqslant \beta c_\varepsilon} u_\varepsilon^n dx \leqslant \frac{\gamma_1}{\gamma_1-\gamma},$$

then we can choose a sequence ρ_ε which converges zero such that $\{x|u_\varepsilon \geqslant \beta c_\varepsilon\} \subset \mathscr{W}_{\rho_\varepsilon}$. We have first, by the fact u_ε converges in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for $q>1$

$$\lim_{\varepsilon\to 0}\int_{u_\varepsilon \geqslant \beta c_\varepsilon} u_{\varepsilon,\beta}^q dx \leqslant \lim_{\varepsilon\to 0}\int_{u_\varepsilon \geqslant \beta c_\varepsilon} u_\varepsilon^q dx = 0 \tag{28}$$

and secondly,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} u_\varepsilon^q (u_\varepsilon - \beta c_\varepsilon)^+ dx = 0. \tag{29}$$

Now we multiply (13) by $(u_\varepsilon - \beta c_\varepsilon)^+$ and take the integral over all $x\in\mathbb{R}^n$

$$\begin{aligned}
&\int_{\mathbb{R}^n} F^n(\nabla(u_\varepsilon - \beta c_\varepsilon)^+)dx\\
&= -\int_{\mathbb{R}^n} u_\varepsilon^{n-1}(u_\varepsilon-\beta c_\varepsilon)^+ dx + \gamma\|u_\varepsilon\|_p^{n-p}\int_{\mathbb{R}^n} u_\varepsilon^{p-1}(u_\varepsilon - \beta c_\varepsilon)^+ dx\\
&\quad + \int_{\mathbb{R}^n} \frac{u_\varepsilon^{\frac{1}{n-1}}(u_\varepsilon-\beta c_\varepsilon)^+}{\alpha_\varepsilon}\Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx\\
&\geqslant \int_{\mathscr{W}_{R\varepsilon}} \frac{u_\varepsilon^{\frac{1}{n-1}}(u_\varepsilon-\beta c_\varepsilon)^+}{\alpha_\varepsilon}\Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx + o_\varepsilon(1)\\
&= (1+o_\varepsilon(1))(1-\beta)\int_{\mathscr{W}_R} e^{\lambda_{n,\varepsilon}(u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon y) - c_\varepsilon^{\frac{n}{n-1}})} dy + o_\varepsilon(1),
\end{aligned}$$

according to (28) and (29). Sending $\varepsilon \to 0$ first and then $R \to +\infty$ shows that

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^n} F^n(\nabla (u_\varepsilon - \beta c_\varepsilon)^+)dx \geqslant 1-\beta. \tag{30}$$

We choose $u_{\varepsilon,\beta}$ as a test function being computed as in the proof of (30) and obtain

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^n} F^n(\nabla u_{\varepsilon,\beta})dx \geqslant \beta. \tag{31}$$

Noting that

$$\int_{\mathbb{R}^n} F^n(\nabla u_{\varepsilon,\beta})dx + \int_{\mathbb{R}^n} F^n(\nabla (u_\varepsilon - \beta c_\varepsilon)^+)dx = \int_{\mathbb{R}^n} F^n(\nabla u_\varepsilon)dx = 1 + o_\varepsilon(1) \tag{32}$$

Combining (30)–(32), we get the result as desired. □

LEMMA 6. *Let $c_\varepsilon \to +\infty$, then*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx = \limsup_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{c_\varepsilon^{\frac{n}{n-1}}}$$

and consequently $\alpha_\varepsilon / c_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Proof. Since $\Phi'(t) = \frac{t^{n-1}}{(n-1)!} + \Phi(t)$, we have

$$\begin{aligned}\alpha_\varepsilon &= \int_{\mathbb{R}^n} u_\varepsilon^{\frac{n}{n-1}} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx + \frac{\lambda_{n,\varepsilon}^{n-1}}{(n-1)!} \int_{\mathbb{R}^n} u_\varepsilon^{\frac{n^2}{n-1}} dx \\ &\leqslant c_\varepsilon^{\frac{n}{n-1}} \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx + o_\varepsilon(1)\end{aligned}$$

and therefore

$$\limsup_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{c_\varepsilon^{\frac{n}{n-1}}} \leqslant \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx. \tag{33}$$

By Lemma 2 and Lemma 5, we have $\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^n} (F^n(\nabla u_{\varepsilon,\beta}) + u_{\varepsilon,\beta}^n)dx = \beta$. Using the mean value theorem and the Hölder inequality, we first note that

$$\begin{aligned}\int_{u_\varepsilon \leqslant \beta c_\varepsilon} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}})dx &\leqslant \lambda_{n,\varepsilon} \int_{\mathbb{R}^n} u_{\varepsilon,\beta}^{\frac{n}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_{\varepsilon,\beta}^{\frac{n}{n-1}})dx \\ &= \lambda_{n,\varepsilon} \int_{\mathbb{R}^n} u_{\varepsilon,\beta}^{\frac{n}{n-1}} \Phi(\lambda_{n,\varepsilon} u_{\varepsilon,\beta}^{\frac{n}{n-1}})dx + o_\varepsilon(1) \\ &\leqslant \lambda_{n,\varepsilon} \left(\int_{\mathbb{R}^n} u_{\varepsilon,\beta}^{\frac{np_1}{n-1}} dx\right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} p_2 u_{\varepsilon,\beta}^{\frac{n}{n-1}})dx\right)^{\frac{1}{p_2}} + o_\varepsilon(1) \\ &\leqslant \lambda_{n,\varepsilon} \left(\int_{\mathbb{R}^n} u_{\varepsilon,\beta}^{\frac{np_1}{n-1}} dx\right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^n} \Psi(\lambda_{n,\varepsilon} p_2 u_{\varepsilon,\beta}^{\frac{n}{n-1}})dx\right)^{\frac{1}{p_2}} + o_\varepsilon(1).\end{aligned}$$

Let $1 < p_2 < \frac{1}{\beta}$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. According to (3) and the estimate

$$\int_{\mathbb{R}^n} u_{\varepsilon,\beta}^q dx \leqslant \int_{\mathbb{R}^n} u_\varepsilon^q dx = o_\varepsilon(1)$$

for $q > 1$. Thus we may continue to write

$$\int_{u_\varepsilon \leqslant \beta c_\varepsilon} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx = o_\varepsilon(1). \tag{34}$$

On the other hand, we have

$$\begin{aligned}\int_{u_\varepsilon > \beta c_\varepsilon} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx &\leqslant \frac{1}{(\beta c_\varepsilon)^{\frac{n}{n-1}}} \int_{u_\varepsilon > \beta c_\varepsilon} u_\varepsilon^{\frac{n}{n-1}} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx \\ &= \frac{1}{(\beta c_\varepsilon)^{\frac{n}{n-1}}} \left(\int_{u_\varepsilon > \beta c_\varepsilon} u_\varepsilon^{\frac{n}{n-1}} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx + o_\varepsilon(1) \right) \\ &= \frac{\alpha_\varepsilon}{(\beta c_\varepsilon)^{\frac{n}{n-1}}} + o_\varepsilon(1). \end{aligned} \tag{35}$$

Thus (34) and (35) imply

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx \leqslant \limsup_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{(\beta c_\varepsilon)^{\frac{n}{n-1}}}.$$

Let $\beta \to 1$. This inequality and (33) complete the proof.

If $\alpha_\varepsilon / c_\varepsilon$ is bounded. Then there exists some constant $C > 0$ such that $\alpha_\varepsilon / c_\varepsilon \leqslant C$. Consequently, we yield $\frac{\alpha_\varepsilon}{c_\varepsilon^{n/(n-1)}} \to 0$ which leads to the following contradiction

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) dx = 0$$

and so the second assertion of the lemma follows. □

There is no problem in showing that for any $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{c_\varepsilon u_\varepsilon^{\frac{1}{n-1}}}{\alpha_\varepsilon} \Phi'(\lambda_{n,\varepsilon} u_\varepsilon^{\frac{n}{n-1}}) \varphi(x) dx = \varphi(0). \tag{36}$$

The reader can see [20] for more details. We turn our attention next to the properties of function sequence $c_\varepsilon^{\frac{1}{n-1}} u_\varepsilon$.

LEMMA 7. *$c_\varepsilon^{\frac{1}{n-1}} u_\varepsilon \to G$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ and weakly in $W^{1,q}(\mathbb{R}^n)$ for any $1 < q < n$, where G is a distributional solution to*

$$-Q_n G = \delta_0 - G^{n-1} + \gamma \|G\|_p^{n-p} G^{p-1}. \tag{37}$$

Moreover, $G \in W^{1,n}(\mathbb{R}^n \setminus \mathscr{W}_r)$ for any $r > 0$ and G takes the form

$$G = -\frac{n}{\lambda_n} \log r + C_G + o_r(1),$$

where C_G is a constant and $r = F^o(x)$.

Proof. Multiplying both sides of (13) by c_ε, we find

$$-Q_n(c_\varepsilon^{\frac{1}{n-1}}u_\varepsilon)=\frac{c_\varepsilon u_\varepsilon^{\frac{1}{n-1}}}{\alpha_\varepsilon}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})-(c_\varepsilon^{\frac{1}{n-1}}u_\varepsilon)^{n-1}+\gamma\|c_\varepsilon^{\frac{1}{n-1}}u_\varepsilon\|_p^{n-p}(c_\varepsilon^{\frac{1}{n-1}}u_\varepsilon)^{p-1}. \tag{38}$$

For convenience in writing, we set $\rho_\varepsilon=c_\varepsilon^{\frac{1}{n-1}}u_\varepsilon$. Then we can rewrite (38) in the form

$$-Q_n\rho_\varepsilon=\frac{c_\varepsilon u_\varepsilon^{\frac{1}{n-1}}}{\alpha_\varepsilon}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})-\rho_\varepsilon^{n-1}+\gamma\|\rho_\varepsilon\|_p^{n-p}\rho_\varepsilon^{p-1}. \tag{39}$$

We now claim that $\|\rho_\varepsilon\|_p$ is bounded. Suppose this is not true; that is, $\|\rho_\varepsilon\|_p\to\infty$ as $\varepsilon\to 0$. Writing $\widetilde{\rho}_\varepsilon=\frac{\rho_\varepsilon}{\|\rho_\varepsilon\|_p}$, we have $\|\widetilde{\rho}_\varepsilon\|_p=1$ and also obtain from (39) that $\widetilde{\rho}_\varepsilon$ satisfies

$$-Q_n\widetilde{\rho}_\varepsilon=\frac{c_\varepsilon u_\varepsilon^{\frac{1}{n-1}}\Phi'(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})}{\alpha_\varepsilon\|\rho_\varepsilon\|_p^{n-1}}-\widetilde{\rho}_\varepsilon^{n-1}+\gamma\widetilde{\rho}_\varepsilon^{p-1} \tag{40}$$

which together with (36) implies that $-Q_n\widetilde{\rho}_\varepsilon$ is bound in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. As a similar progress of Lemma 4.6 in [20], we conclude that $\widetilde{\rho}_\varepsilon$ is bound in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$ for $1<q<n$. Assume $\widetilde{\rho}_\varepsilon\rightharpoonup\rho_0$ weakly in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n)$. Testing (40) with $\phi\in C_0^\infty(\mathbb{R}^n)$ and letting $\varepsilon\to 0$, we obtain

$$\int_{\mathbb{R}^n}F^{n-1}(\nabla\rho_0)F_\xi(\nabla\rho_0)\nabla\phi dx=-\int_{\mathbb{R}^n}\rho_0^{n-1}\phi dx+\gamma\int_{\mathbb{R}^n}\rho_0^{p-1}\phi dx,$$

which forces $\rho_0=0$ since $0<\gamma<\gamma_1$. This contradicts to $\|\rho_0\|_p=1$. Therefore our claim is proved.

The remaining part of the proof is completely analogous to that of ([20], Lemma 4.6 and Lemma 4.7), we omit the details but refer the reader to [20]. □

We quote the following Carleson-Change's type estimate, which is shown in [19], provides the essential step to get an upper bound for FTM. More precisely

LEMMA 8. *Let $\phi_\varepsilon\in W_0^{1,n}(\mathscr{W}_1)$ with $\int_{\mathscr{W}_1}F^n(\nabla\phi_\varepsilon)dx=1$. Suppose $\phi_\varepsilon\rightharpoonup 0$ weakly in $W_0^{1,n}(\mathscr{W}_1)$ and $\int_{\mathscr{W}_1\setminus\mathscr{W}_\rho}F^n(\nabla\phi_\varepsilon)dx=0$ for $0<\rho<1$, then*

$$\limsup_{\varepsilon\to 0}\int_{\mathscr{W}_1}(e^{\lambda_n|\phi_\varepsilon|^{\frac{n}{n-1}}}-1)dx\leqslant\kappa_n e^{\sum_{k=1}^{n-1}\frac{1}{k}}. \tag{41}$$

Now by (37), we compute

$$\int_{\mathscr{W}_\delta^c}(F^n(\nabla G)+G^n)dx=-\frac{n}{\lambda_n}\log\delta+C_G+\gamma\|G\|_p^n+o_\delta(1)$$

for any fixed $\delta>0$. Hence we get

$$\begin{aligned}\int_{\mathscr{W}_\delta}F^n(\nabla u_\varepsilon)dx&=1-\frac{1}{c_\varepsilon^{\frac{n}{n-1}}}\left(\int_{\mathscr{W}_\delta^c}(F^n(\nabla G)+G^n)dx-\int_{\mathscr{W}_\delta}G^n dx+\gamma\left(\int_{\mathbb{R}^n}G^p dx\right)^{\frac{n}{p}}\right)\\&=1-\frac{\frac{n}{\lambda_n}\log\frac{1}{\delta}+C_G+o_\delta(1)+o_\varepsilon(1)}{c_\varepsilon^{\frac{n}{n-1}}}.\end{aligned}$$

Here we use $\|u_\varepsilon\|_{F,n,\gamma,p}=1$. Writing $\overline{u}_\varepsilon=(u_\varepsilon-u_\varepsilon(\delta))^+$, then $\overline{u}_\varepsilon\in W_0^{1,n}(\mathscr{W}_\delta)$ and $\overline{u}_\varepsilon\rightharpoonup 0$ weakly in $W_0^{1,n}(\mathscr{W}_\delta)$. Furthermore

$$\tau_\delta:=\int_{\mathscr{W}_\delta}F^n(\nabla\overline{u}_\varepsilon)dx\leqslant 1-\frac{\frac{n}{\lambda_n}\log\frac{1}{\delta}+C_G+o_\delta(1)+o_\varepsilon(1)}{c_\varepsilon^{\frac{n}{n-1}}}. \tag{42}$$

By Lemma 8, we infer the estimate

$$\limsup_{\varepsilon\to 0}\int_{\mathscr{W}_\delta}(e^{\lambda_n(\overline{u}_\varepsilon/\sqrt[n]{\tau_\delta})^{\frac{n}{n-1}}}-1)dx\leqslant\kappa_n\delta^n e^{\sum_{k=1}^{n-1}\frac{1}{k}}. \tag{43}$$

Hence by inequality (42)

$$\begin{aligned}\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}}&\leqslant\lambda_n(\overline{u}_\varepsilon+u_\varepsilon(\delta))^{\frac{n}{n-1}}\\&\leqslant\lambda_n\overline{u}_\varepsilon^{\frac{n}{n-1}}+\frac{n}{n-1}\lambda_n u_\varepsilon(\delta)\overline{u}_\varepsilon^{\frac{1}{n-1}}+o_\varepsilon(1)\\&\leqslant\lambda_n(\overline{u}_\varepsilon/\sqrt[n]{\tau_\delta})^{\frac{n}{n-1}}-n\log\delta+\lambda_n C_G+o(1)\end{aligned}$$

and owing to (43), we get

$$\begin{aligned}\int_{\mathscr{W}_{Rr_\varepsilon}}\Phi(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx&=\delta^{-n}e^{\lambda_n C_G+o(1)}\int_{\mathscr{W}_{Rr_\varepsilon}}(e^{\lambda_n(\overline{u}_\varepsilon/\sqrt[n]{\tau_\delta})^{\frac{n}{n-1}}}-1)dx+o_\varepsilon(1)\\&\leqslant\delta^{-n}e^{\lambda_n C_G+o(1)}\int_{\mathscr{W}_\delta}(e^{\lambda_n(\overline{u}_\varepsilon/\sqrt[n]{\tau_\delta})^{\frac{n}{n-1}}}-1)dx+o_\varepsilon(1)\\&\leqslant\kappa_n e^{\lambda_n C_G+\sum_{k=1}^{n-1}\frac{1}{k}+o(1)}+o(1).\end{aligned}$$

Then

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\mathscr{W}_{Rr_\varepsilon}}\Phi(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx\leqslant\kappa_n e^{\lambda_n C_G+\sum_{k=1}^{n-1}\frac{1}{k}}. \tag{44}$$

We take a change of variable $x=r_\varepsilon y$ and recall (27), to discover

$$\begin{aligned}\int_{\mathscr{W}_{Rr_\varepsilon}}\Phi(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx&=r_\varepsilon^n\int_{\mathscr{W}_R}e^{\lambda_{n,\varepsilon}(u_\varepsilon^{\frac{n}{n-1}}(r_\varepsilon y)-c_\varepsilon^{\frac{n}{n-1}})}dy+o_\varepsilon(1)\\&=\frac{\alpha_\varepsilon}{c_\varepsilon^{\frac{n}{n-1}}}(\int_{\mathscr{W}_R}e^{\lambda_n\frac{n}{n-1}w_0}dx+o_\varepsilon(1))+o_\varepsilon(1)\\&=\frac{\alpha_\varepsilon}{c_\varepsilon^{\frac{n}{n-1}}}(1+o_\varepsilon(1)+o_R(1)).\end{aligned}$$

Due to Lemma 6 and (44), we immediately obtain

$$FTM=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\Phi(\lambda_{n,\varepsilon}u_\varepsilon^{\frac{n}{n-1}})dx\leqslant\kappa_n e^{\lambda_n C_G+\sum_{k=1}^{n-1}\frac{1}{k}}. \tag{45}$$

6. Proof of main Theorems

If c_ε is a bounded sequence, then applying the standard elliptic estimate to (13), we derive that $u_\varepsilon \to u_0$ in $C^1_{\text{loc}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \Phi(\lambda_n |u_0|^{\frac{n}{n-1}})dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon}|u_\varepsilon|^{\frac{n}{n-1}})dx = FTM, \tag{46}$$

where $\|u_0\|_{F,n,\gamma,p} = 1$. Therefore, u_0 is an extremal function for FTM.

If c_ε is not bounded, the blow-up phenomenon occurs. We have got an upper bound shown in (45), we now construct a family of test function $\psi_\varepsilon \in W^{1,n}(\mathbb{R}^n)$ with $\|\psi_\varepsilon\|_{F,n,\gamma,p} = 1$ and

$$\int_{\mathbb{R}^n} \Phi(\lambda_{n,\varepsilon}|\psi_\varepsilon|^{\frac{n}{n-1}})dx > \kappa_n e^{\lambda_n C_G + \sum_{k=1}^{n-1}\frac{1}{k}} \tag{47}$$

provided ε is sufficiently small. Define

$$\psi_\varepsilon(x) = \begin{cases} c + c^{-\frac{1}{n-1}}\left(-\frac{n-1}{\lambda_n}\log(1+\kappa_n^{\frac{1}{n-1}}(\frac{F^o(x)}{\varepsilon})^{\frac{n}{n-1}}) + b\right) & F^o(x) \leqslant L\varepsilon, \\ \frac{G(F^o(x))}{c^{\frac{1}{n-1}}} & F^o(x) > L\varepsilon, \end{cases}$$

where L, b and c are functions of ε to be determined later which satisfy

(i) $L\to\infty$, $c\to\infty$ and $L\varepsilon \to 0$ as $\varepsilon\to 0$;

(ii) $c + \frac{-\frac{n-1}{\lambda_n}\log(1+\kappa_n^{\frac{1}{n-1}}L^{\frac{n}{n-1}})+b}{c^{\frac{1}{n-1}}} = \frac{G(L\varepsilon)}{c^{\frac{1}{n-1}}}$;

(iii) $\frac{\log L}{c^{\frac{n^2}{(n-1)^2}}} \to 0$ as $\varepsilon \to 0$.

From (ii), we obtain

$$c^{\frac{n}{n-1}} = \frac{1}{\lambda_n}\log\kappa_n - b - \frac{n}{\lambda_n}\log\varepsilon + C_G + O(L^{-\frac{n}{n-1}}) + O(L\varepsilon). \tag{48}$$

Next suppose $\|\psi_\varepsilon\|_{F,n,\gamma,p} = 1$, we shall verify the relation

$$\begin{aligned}\int_{\mathscr{W}^c_{L\varepsilon}} (F^n(\nabla\psi_\varepsilon) + \psi_\varepsilon^n)dx &= \frac{1}{c^{\frac{n}{n-1}}}\int_{\mathscr{W}^c_{L\varepsilon}} (F^n(\nabla G) + |G|^n)dx \\ &= \frac{\gamma\|G\|_p^n + G(L\varepsilon) + O(\log^p(L\varepsilon)(L\varepsilon)^n) + O(\log(L\varepsilon)^n(L\varepsilon)^n)}{c^{\frac{n}{n-1}}}.\end{aligned}$$

and

$$\int_{\mathscr{W}^c_{L\varepsilon}} |\psi_\varepsilon|^p dx = \frac{\|G\|_p^p + O(\log^p(L\varepsilon)(L\varepsilon)^n)}{c^{\frac{p}{n-1}}}.$$

On the other hand

$$\int_{\mathscr{W}_{L\varepsilon}} F^n(\nabla\psi_\varepsilon)dx = \frac{n-1}{\lambda_n}\left(\frac{\log(1+\kappa_n^{\frac{1}{n-1}}L^{\frac{n}{n-1}}) - \sum_{k=1}^{n-1}\frac{1}{k} + O(L^{-\frac{n}{n-1}})}{c^{\frac{n}{n-1}}}\right).$$

In addition

$$\int_{\mathscr{W}_{L\varepsilon}} |\psi_\varepsilon|^n dx = O((\log\varepsilon)^{n-1}(L\varepsilon)^n)$$

and

$$\int_{\mathscr{W}_{L\varepsilon}} |\psi_\varepsilon|^p dx = O((\log\varepsilon)^{\frac{n-1}{n}p}(L\varepsilon)^n).$$

Combining the previous estimates, we conclude

$$\|\psi_\varepsilon\|^n_{F,n,\gamma,p} = \frac{1}{c^{\frac{n}{n-1}}}\left(G(L\varepsilon) + \frac{n-1}{\lambda_n}\log\left(1+\kappa_n^{\frac{1}{n-1}}L^{\frac{n}{n-1}}\right) - \frac{n-1}{\lambda_n}\sum_{k=1}^{n-1}\frac{1}{k} + O(\psi_1)\right),$$

where $\psi_1 = \log^p(L\varepsilon)(L\varepsilon)^n + \log(L\varepsilon)^n(L\varepsilon)^n + \log^n(L\varepsilon)(L\varepsilon)^{\frac{n^2}{p}} + (\log\varepsilon)^{n-1}(L\varepsilon)^n + (\log\varepsilon)^{n-1}(L\varepsilon)^{\frac{n^2}{p}} + L^{-\frac{n}{n-1}}$. Therefore

$$\begin{aligned} c^{\frac{n}{n-1}} &= G(L\varepsilon) + \frac{n-1}{\lambda_n}\log\left(1+\kappa_n^{\frac{1}{n-1}}L^{\frac{n}{n-1}}\right) - \frac{n-1}{\lambda_n}\sum_{k=1}^{n-1}\frac{1}{k} + O(\psi_1) \\ &= -\frac{n}{\lambda_n}\log\varepsilon + C_G + \frac{1}{\lambda_n}\log\kappa_n - \frac{n-1}{\lambda_n}\sum_{k=1}^{n-1}\frac{1}{k} + O(\psi_1). \end{aligned}$$

Owing to (48), we deduce

$$b = \frac{n-1}{\lambda_n}\sum_{k=1}^{n-1}\frac{1}{k} + O(\psi_1).$$

We then compute

$$\begin{aligned} \psi_\varepsilon^{\frac{n}{n-1}} &\geqslant c^{\frac{n}{n-1}}\left(1 + \frac{n}{n-1}\,\frac{-\frac{n-1}{\lambda_n}\log\left(1+\kappa_n^{\frac{1}{n-1}}\left(\frac{F^o(x)}{\varepsilon}\right)^{\frac{n}{n-1}}\right) + b}{c^{\frac{n}{n-1}}}\right) \\ &= C_G + \frac{1}{\lambda_n}\left(\log\kappa_n + \sum_{k=1}^{n-1}\frac{1}{k}\right) - \frac{n}{\lambda_n}\left(\log\varepsilon + \log\left(1+\kappa_n^{\frac{1}{n-1}}\left(\frac{F^o(x)}{\varepsilon}\right)^{\frac{n}{n-1}}\right)\right) \\ &\quad + O(\psi_1) \end{aligned} \tag{49}$$

for any $x \in \mathscr{W}_{L\varepsilon}$, and hence

$$\begin{aligned} &\int_{\mathscr{W}_{L\varepsilon}} \Phi(\lambda_n\psi_\varepsilon^{\frac{n}{n-1}})dx \geqslant \int_{\mathscr{W}_{L\varepsilon}} e^{\lambda_n\psi_\varepsilon^{\frac{n}{n-1}}}dx + O(c^n(L\varepsilon)^n) \\ &\geqslant \kappa_n\varepsilon^{-n}e^{\lambda_n C_G + \sum_{k=1}^{n-1}\frac{1}{k} + O(\psi_1)}\int_{\mathscr{W}_{L\varepsilon}} \frac{1}{\left(1+\kappa_n^{\frac{1}{n-1}}\left(\frac{F^o(x)}{\varepsilon}\right)^{\frac{n}{n-1}}\right)^n}dx + O(c^n(L\varepsilon)^n) \\ &\geqslant \kappa_n e^{\lambda_n C_G + \sum_{k-1}^{n-1}\frac{1}{k}} + O(\psi_1) + O(c^n(L\varepsilon)^n) + O(L^{-\frac{n}{n-1}}). \end{aligned} \tag{50}$$

On the other hand, we have

$$\int_{\mathbb{R}^n\setminus \mathscr{W}_{L\varepsilon}} \Phi(\lambda_n \psi_\varepsilon^{\frac{n}{n-1}})dx \geqslant \frac{\lambda_n^n}{n!c^{\frac{n^2}{(n-1)^2}}}\left(\int_{\mathbb{R}^n} G^{\frac{n^2}{n-1}}dx + o_\varepsilon(1)\right).$$

Owing to (50), we deduce

$$\int_{\mathbb{R}^n} \Phi(\lambda_n \psi_\varepsilon^{\frac{n}{n-1}})dx \geqslant \kappa_n e^{\lambda_n C_G+\Sigma_{k=1}^{n-1}\frac{1}{k}} + \frac{\lambda_n^n}{n!c^{\frac{n^2}{(n-1)^2}}}\left(\int_{\mathbb{R}^n} G^{\frac{n^2}{n-1}}dx + o_\varepsilon(1)\right) + O(\psi_1) + O(c^n(L\varepsilon)^n) + O(L^{-\frac{n}{n-1}}).$$

We now set

$$L = (-\log\varepsilon)^2,$$

so that $L^{-\frac{n}{n-1}} = o(c^{-\frac{n^2}{(n-1)^2}})$, $c^n(L\varepsilon)^n = o(c^{-\frac{n^2}{(n-1)^2}})$ and $\psi_1 = o(c^{-\frac{n^2}{(n-1)^2}})$. We then obtain the inequality (47) and infer that c_ε must be bounded. The blow-up phenomenon in fact does not happen; whence the desired equality (46) holds, we finish the proof of point (5) of Thoerem 1 and Theorem 2.

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