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q–DEFORMED HILBERT TRANSFORM AND ITS RELATED PROPERTIES AND INEQUALITIES

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Abstract. We present a new formulation of the Hilbert transform constructed via the q-deformation of convolution, which is called the q-deformed Hilbert transform. We also find the q-deformed Hilbert transform of some basic functions and examine its connection with the q-Fourier transform. In particular, a number of new related inequalities and embeddings are proved such as a q-analuge of the Chebyshev inequality and a Hardy-type inequality. In additionally, we present a direct application of Hardy-type inequality to study some inequalities for the q-deformed Hilbert transform on $L^p(\mathbb{R}_q)$ and $L^{p,r}(\mathbb{R}_q)$. Finally, we prove a weak (1.1) inequality for the q-deformed Hilbert transform.

1. Introduction

The most common language of quantum calculus is based on the q-calculus (or q-deformation), though this type of calculus was already introduced by L. Euler [17] in the 18th century. The study of q-deformation began in 1748 when he considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} \frac{1}{1-q^{k+1}}$, $|q| < 1$, as a generating function for $p(n)$, where $p(n)$ denotes the partition function, i.e., the number of ways to express n as a sum of positive integers. In the early 20th century, F. H. Jackson introduced the concept of the q-derivative and the definite q-integral [25, 26], marking the foundation of what is now known as q-deformation. During the last two decades, the study of q-deformation has garnered significant attention from researchers. For examples, the book by V. Kac and P. Cheung [11] explores many fundamental aspects of q-deformation. In the book [15] of T. Ernst (see also [14]), it has gained renewed interest due to its relevance in mathematical models for quantum computing. Moreover, in [6], N. Bettaibi and R. H. Bettaieb introduced a new q-deformated Dunkl operator and examined the corresponding Fourier transform in [19, 20] (see, [31]). The q-deformated Dunkl operator is defined by Rubin's q-differential operator ∂_q, as presented in [39, 40]. For a more detailed overview of the development and recent advances in q-calculus, we refer the reader to the monographs [1, 3, 5, 14, 15, 16, 21] and the references therein. The first notable results on q-deformed integral inequalities appeared with the work of H. Gauchman

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[22] in 2004. In 2014, L. Maligranda, R. Oinarov and L.-E. Persson [33] derived a q-analogue of the classical Hardy inequality, originally introduced by Hardy in 1925 (see [23, 24]). The development of Hardy-type inequalities has since become a vast and active area of research, with many significant results and applications discussed in the monograph [35]. Consequently, exploring which of these classical Hardy-type inequalities admit meaningful q-analogues remains a promising direction for further research.

Recently, in this paper [34], the authors defined a q-deformated of the Hilbert transform $\mathscr{H}$ of a real-valued function $f(t)$ is defined as:

$$\mathscr{H}_{q,\alpha}(f)(x) = a_{q,\alpha} p.v. \int_{-\infty}^{\infty} \frac{T_{q,\alpha}^{x}(f)(-\lambda)}{\lambda} d_q\lambda = a_{q,\alpha} \lim_{\varepsilon\to 0} \int_{|\lambda|>\varepsilon} \frac{T_{q,\alpha}^{x}(f)(-\lambda)}{\lambda} d_q\lambda, \quad (1)$$

where the generalized q-deformated Dunkl translation operator $T_{q,\alpha}^{x}$ (see, [34, formula (32)]) and the constant $a_{q,\alpha}$ is given by

$$a_{q,\alpha} = \frac{(1+q)\Gamma_{q^2}\left(\alpha+\frac{3}{2}\right)}{2q\Gamma_{q^2}\left(\frac{1}{2}\right)\Gamma_{q^2}(\alpha+1)},$$

with the q^2-Gamma function $\Gamma_{q^2}(\cdot)$ is defined by fomula (5). Historically, the Hilbert transform emerged from D. Hilbert's work [37] on integral equations and boundary value problems in 1905 (see [29]). Moreover, they investigated its fundamental properties through the harmonic analysis framework related to the q-deformated Dunkl operator in [6].

One main purpose of this work is to introduce and investigate another q-deformation of the Hilbert transform (23) constructed via the q-deformated convolution (16). In the particular case when $\alpha = \frac{1}{2}$, the defination (1) is equivalent to our formulation of the q-deformed Hilbert transform (23). However, the Definition (23) provides an efficient tool for the direct derivation of the main results concerning the q-deformed Hilbert transform in our further analysis. Moreover, we study the q-deformed Hardy-type inequalities (38) and (37) and apply them to derive the Riesz inequality (45), the inequality (42) and the inequality (49) related to the q-deformed Hilbert transform (23). Note that $\lim_{q\to 1} f(x \ominus_q y) = f(x-y)$, as a consequence of the approximation property of the q-Transform in the limit $q \to 1$ (see, [40, Section 3]). Therefore, we get $\lim_{q\to 1}(\mathscr{H}_q f)(x) = (\mathscr{H} f)(x)$ which is the classical Hilbert transform $\mathscr{H}$ of a real-valued function f is defined as:

$$(\mathscr{H} f)(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau, \quad (2)$$

where p.v. denotes the Cauchy principal value (see, [28]). The historical development of the Hilbert transform began with David Hilbert's study of integral equations and boundary value problems in 1905 (see, [29]). If $f \in L^p(\mathbb{R})$ with $1 < p < \infty$, then there exists a positive constant A independent of f such that the following inequality holds:

$$(\mathscr{H} f)(t) \leqslant A\left(\frac{1}{t}\int_0^t f(s)ds + \int_t^\infty \frac{f(s)}{s}ds\right) \quad t > 0. \tag{3}$$

The inequality (3) demonstrates that the Hilbert transform maps functions from $L^p(\mathbb{R})$ spaces to other functional spaces (e.g., Lorentz spaces) while maintaining boundedness. The proof can be found in Bennett & Sharpley [10, p. 134–138.]. Specifically, a weak-type (1,1) inequality found in [10, Chapter 3, Section 4] (see also [43] and [45]). Similar constructions for discrete Hilbert transform were investigated in [2] (see also [44])

Finally, we obtain the inequality (53) which is a q-analogue of the inequlaity (3). As a applcation of the inequality (53) to investigate a weak (1.1) inequality (58) for the q-deformed Hilbert transform (2).

The outline of this paper is as follows: We will recall the necessary notions and definitions in Section 2. We introduce the definition of the q-deformed Hilbert transform and proceed to compute its action on some basic functions. Additionally, we will explore the relationship between the Hilbert transform and the Fourier transform in Section 3. We also prove several important and useful theorems related to q-decreasing rearrangement, which will be crucial for the subsequent developments of further investigation in Section 4. In Section 5, we study a special form of some q-deformed Hardy-type inequality. Moreover, In Section 6, we present a direct application of q-deformed Hardy-type inequalities to establish the q-deformed Hilbert transform L^p-boundedness and to obtain its is bounded in Lorentz $L^{p,r}(\mathbb{R}_q)$. Finally, in Section 7, We will investigate a weak (1.1) inequality (50) in Section.

2. Preliminaries

In this Section, we give notation that we will also use throughout this paper and assume that $0 < q < 1$. Let $\alpha \in \mathrm{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q := \frac{1-q^\alpha}{1-q},$$

where $\lim\limits_{q\to 1} \frac{1-q^\alpha}{1-q} = \alpha$.

We introduce for any $x, a \in \mathbb{R}$

$$(x,a)_q^0 = 1, \quad (x,a)_n = \prod_{k=0}^{n-1}\left(x - q^k a\right), \quad (x,a)_\infty = \lim_{n\to\infty}(x,a)_q^n.$$

The q-analog of the binomial coefficients are defined by

$$[n]_q! := \begin{cases} 1, & \text{if n} = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if n} \in \mathrm{N}, \end{cases} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

Let us define q-addition by

$$(a \oplus_q b)^0 = 1, \quad (a \oplus_q b)^1 = a - b, \quad (a \oplus_q b)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}, \quad a \neq b. \tag{4}$$

The gamma function Γ_q is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{5}$$

The q-analogue differential operator is defined as (see [25])

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}$$

Note that if f is differentiable at x, then $\lim_{q\to 1} D_q f(x) = f'(x)$.

Let $\mathbb{R}_q^+ = \{q^k, k \in \mathbb{Z}\}$ and $\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\}$.

The q-integral (or Jackson integral) is defined as (see [26])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m); \quad \int_{\mathbb{R}_q^+} f(x) d_q x = (1-q) \sum_{m=-\infty}^{\infty} q^m f(q^m) \tag{6}$$

and

$$\int_{\mathbb{R}_q} f(x) d_q x = (1-q) \sum_{m=-\infty}^{\infty} q^m \left(f(q^m) + f(-q^m)\right), \tag{7}$$

provided the sums converge absolutely.

We denote

$$L_q^p(\mathbb{R}_q) = \{f : \int_{\mathbb{R}_q} |f(x)|^p d_q x < \infty, 0 < p < \infty\}; \quad L_q^\infty\left(\mathbb{R}_q^+\right) = \{f : \sup_{x\in\mathbb{R}_q} |f(x)| < \infty\}.$$

The normalized q-Bessel function is defined by (see, [12] and [31])

$$j_\alpha(x,q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2},q^2)_n (q^2,q^2)_n} x^{2n}.$$

Note that we have

$$j_\alpha(x,q^2) = (1-q^2)^\alpha \Gamma_q^2(\alpha+1)\left((1-q)x\right)^{-\alpha} J_\alpha\left((1-q)x;q^2\right),$$

where

$$J_\alpha(x;q^2) = \frac{x^\alpha \left(q^{2\alpha+2},q^2\right)_\infty}{(q^2,q^2)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} \left(q^2x^2\right)^k}{(q^{2\alpha+2},q^2)_k \, (q^2,q^2)_\infty}.$$

Moreover,

$$\frac{(1+q)^t \Gamma_{q^2}\left((\alpha+1+t)/2\right)}{\Gamma_{q^2}\left((\alpha+1-t)/2\right)} = \int_{\mathbb{R}_q^+} x^t J_\alpha\left((1-q)x;q^2\right) d_q x, \tag{8}$$

for $\log(1-q)/\log(q) \in \mathbb{Z}$. The normalized q-Bessel function $j_\alpha(.,q^2)$ satisfies the orthogonality relation (see e.g. [12])

$$K_{\alpha,q}^2 \int_{\mathbb{R}_q^+} j_\alpha(\lambda_1 x,q^2) j_\alpha(\lambda_2 x,q^2) x^{2\alpha+1} d_q x = \delta_q(\lambda_1,\lambda_2), \quad \forall \lambda_1,\lambda_2 \in \mathbb{R}_q \tag{9}$$

where $K_{\alpha,q} := \frac{(q^{2\alpha+2},q^2)_\infty}{(1-q)(q^2,q^2)_\infty}$ and

$$\delta_q(\lambda_1,\lambda_2) = \begin{cases} 0, & \lambda_1 \neq \lambda_2; \\ \frac{1}{(1-q)\lambda_1^{2(\alpha+1)}}, & \lambda_1 = \lambda_2. \end{cases}$$

Let f be a function defined on R_q^+ then

$$\int_{\mathbb{R}_q^+} f(\lambda_2)\delta_q(\lambda_1,\lambda_2)\lambda_2^{2\alpha+1} d_q\lambda_2 = f(\lambda_1). \tag{10}$$

The q^2-exponentials (see, [31] and [40])

$$\exp(x;q^2) = \cos(-ix;q^2) + i\sin(-ix;q^2), \tag{11}$$

where

$$\cos(x;q^2) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}((1-q)x)^{\frac{1}{2}} J_{-\frac{1}{2}}((1-q)x;q^2),$$

$$\sin(x;q^2) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}((1-q)x)^{\frac{1}{2}} J_{\frac{1}{2}}((1-q)x;q^2). \tag{12}$$

The q^2-Fourier transform $\widehat{f}$ is defined as follows (see [18] and [39])

$$\widehat{f}(x;q^2) = K_q \int_{\mathbb{R}_q} f(t)\exp(-ixt;q^2) d_q t \tag{13}$$

and its inverse

$$f(t) = K_q \int_{\mathbb{R}_q} \exp(ixt;q^2) \widehat{f}(x;q^2) d_q x, \tag{14}$$

where $K_q = \frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}\left(\frac{1}{2}\right)}$. For $x,y \in \mathbb{R}_q$, the Fourier multiplier operator corresponding to translation by y is

$$f(x \ominus_q y) = K_q \int_{\mathbb{R}_q} \exp(-iyt;q^2)\hat{f}(t;q^2)\exp(ixt;q^2) d_q t, \tag{15}$$

For $f \in L_q^2(\mathbb{R}_q)$, $g \in L_q^1(\mathbb{R}_q)$, define the multiplier corresponding to Fourier convolution of f with g to be

$$(f *_q g)(x) = \int_{\mathbb{R}_q} f(x \ominus_q y) g(y) d_q y. \tag{16}$$

If $f, g \in L_q^1(\mathbb{R}_q) \bigcap L_q^2(\mathbb{R}_q)$. Then

$$\widehat{f * g} = \widehat{f}\widehat{g}. \tag{17}$$

PROPOSITION 1. (q-Dirichlet Integral) *Let $x \in \mathbb{R}_q$. Then*

$$\int_{\mathbb{R}_q^+} \frac{\sin(xt;q^2)}{t} d_q t = \frac{\pi_q}{(1+q)^{\frac{1}{2}}} \operatorname{sgn}(x), \tag{18}$$

where $\pi_q := \Gamma_{q^2}^2\left(\frac{1}{2}\right)$ and $\operatorname{sgn}(\cdot)$ *is the signum function:*

$$\operatorname{sgn}(x) = \begin{cases} -1 & \textit{if } x < 0, \\ 0 & \textit{if } x = 0, \\ 1 & \textit{if } x > 0. \end{cases}$$

Proof. Let $x \in \mathbb{R}_q$ and $x < 0$. Then $x = -|x|$ $\sin(t;q^2)$ is even function and using (12) we find that

$$-\sin(|x|t;q^2) \overset{(12)}{=} -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}((1-q)|x|t)^{\frac{1}{2}} J_{\frac{1}{2}}((1-q)|x|t;q^2). \tag{19}$$

Now $|x|t$ replaced by z and (5), (8) and (19) we have that

$$\begin{aligned} \int_{\mathbb{R}_q^+} \frac{\sin(xt;q^2)}{t} d_q t &\overset{(19)}{=} -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(1-q)^{\frac{1}{2}} \int_{\mathbb{R}_q^+} \left[\frac{|x|}{t}\right]^{\frac{1}{2}} J_{\frac{1}{2}}((1-q)|x|t;q^2) d_q t \\ &= -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(1-q)^{\frac{1}{2}} \int_{\mathbb{R}_q^+} z^{-\frac{1}{2}} J_{\frac{1}{2}}((1-q)z;q^2) d_q z \\ &\overset{(8)}{=} -\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(1-q)^{\frac{1}{2}}(1+q)^{-\frac{1}{2}} \frac{\Gamma_{q^2}\left(\frac{1}{2}\right)}{\Gamma_{q^2}(1)} \\ &\overset{(5)}{=} -\frac{\Gamma_{q^2}^2\left(\frac{1}{2}\right)}{(1+q)^{\frac{1}{2}}}. \end{aligned} \tag{20}$$

Next, assume that $x > 0$, applying the same argument as in the calculation of integral (20), we obtain

$$\int_{\mathbb{R}_q^+} \frac{\sin(xt;q^2)}{t} d_q t = \frac{\Gamma_{q^2}^2\left(\frac{1}{2}\right)}{(1+q)^{\frac{1}{2}}}. \tag{21}$$

Therefore, combining (20) and (21) we conclude that (18) holds for all $x \in \mathbb{R}_q$. This completes the proof. $\square$

REMARK 1. Since $\lim_{q\to 1}\Gamma_{q^2}\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\lim_{q\to 1}\sin(t;q^2) = \sin(t)$ we get that

$$\lim_{q\to 1}\int_{\mathbb{R}_q^+}\frac{\sin(xt;q^2)}{t}d_qt = \operatorname{sgn}(x)\lim_{q\to 1}\frac{\pi_q}{(1+q)^{\frac{1}{2}}} = \operatorname{sgn}(x)\frac{\pi}{2}.$$

Hence, when $q \to 1$ we find that the Dirichlet Integral:

$$\lim_{q\to 1}\int_{\mathbb{R}_q^+}\frac{\sin(xt;q^2)}{t}d_qt = \int_{\mathbb{R}}\frac{\sin(xt)}{t}dt.$$

PROPOSITION 2. *Let* $\varphi(t) = \frac{1}{t}$ *and* $x \in \mathbb{R}_q$. *Then*

$$\widehat{\varphi}(x;q^2) = -i\Gamma_{q^2}\left(\frac{1}{2}\right)\operatorname{sgn}(x). \tag{22}$$

Proof. Using (11) we find

$$\exp(-ixt;q^2) = \cos(xt;q^2) + i\sin(xt;q^2).$$

Therefore, by (13) and (18) we get that

$$\begin{aligned}
\widehat{\varphi}(x;q^2) &\overset{(13)}{=} K_q\int_{\mathbb{R}_q}\frac{\exp(-ixt;q^2)}{t}d_qt \\
&= K_q\int_{\mathbb{R}_q}\left[\frac{\cos(-xt;q^2)}{t} + i\frac{\sin(-xt;q^2)}{t}\right]d_qt \\
&\overset{(18)}{=} -2iK_q\int_{\mathbb{R}_q^+}\frac{\sin(xt;q^2)}{t}d_qt \\
&= -i\Gamma_{q^2}\left(\frac{1}{2}\right)\operatorname{sgn}(x).
\end{aligned}$$

Here we use the fact that $\frac{\cos(xt;q^2)}{t}$ is an odd function and $\frac{\sin(xt;q^2)}{t}$ is an even function. This completes the proof. $\square$

In the following, we denote by

- $\mathscr{S}_q(\mathbb{R}^q)$, the space of functions f defined on $\mathbb{R}^q$ satisfying

$$\forall n,m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x\in\mathbb{R}^q}\left|x^m\partial_q^n f(x)\right| < \infty.$$

3. The q-deformed Hilbert transform

3.1. Hilbert transform on $\mathbb{R}_q$

Let us start with the following main definition which is a new modification of (2) on $\mathbb{R}_q$.

DEFINITION 1. Let $f \in L_q^p(\mathbb{R}_q)$ be a function for $1 \leqslant p < \infty$. Then $\mathscr{H}_q$ is the Hilbert transform of f given by

$$(\mathscr{H}_q f)(x) := \frac{1}{\pi_q} f(t) *_q \frac{1}{t} = p.v. \frac{1}{\pi_q} \int\limits_{\mathbb{R}_q} \frac{f(y)}{x \ominus_q y} d_q y. \tag{23}$$

We now proceed to verify that (23) is well-defined. For this purpose, let us consider a function f from the Schwartz class $\mathscr{S}_q(\mathbb{R})$. Then,

$$\begin{aligned}
(\mathscr{H}_q f)(x) &= \frac{1}{\pi_q} \lim_{\varepsilon \to 0^+} \int\limits_{|y-x|>\varepsilon} \frac{f(y)}{x-y} d_q y \\
&= \frac{1-q}{\pi_q} \lim_{\varepsilon \to 0^+} \sum_{|q^n - x|>\varepsilon} q^n \frac{f(q^n)}{x-q^n} \\
&= \frac{1-q}{\pi_q} \lim_{\varepsilon \to 0^+} \left(\sum_{\varepsilon<|x-q^n|<1} q^n \frac{f(q^n)-f(x)}{x-q^n} + \sum_{|x-q^n|>1} q^n \frac{f(q^n)}{x-q^n} \right) \\
&= \frac{1-q}{\pi_q} \lim_{\varepsilon \to 0^+} \left(\sum_{\varepsilon<|x-q^n|<1} q^n \frac{f(q^n)-f(x)}{x-q^n} + \sum_{\varepsilon<|x-q^n|<1} q^n \frac{f(x)}{x-q^n} \right. \\
&\quad \left. + \sum_{|x-q^n|>1} q^n \frac{f(q^n)}{x-q^n} \right) \\
&= \lim_{\varepsilon \to 0^+} \frac{1-q}{\pi_q} \sum_{\varepsilon<|x-q^n|<1} q^n \frac{f(q^n)-f(x)}{x-q^n} - \frac{1-q}{\pi_q} \sum_{|x-q^n|>1} q^n \frac{f(q^n)}{x-q^n}.
\end{aligned}$$

Here we used the fact that

$$\sum_{\varepsilon<|x-q^n|<1} \frac{q^n}{x-q^n} = \sum_{-1<-|x-q^n|<-\varepsilon} \frac{q^n}{x-q^n} + \sum_{\varepsilon<|x-q^n|<1} \frac{q^n}{x-q^n} = 0.$$

Therefore,

$$|(\mathscr{H}_q f)(x)| \leqslant \lim_{\varepsilon \to 0^+} \frac{1-q}{\pi_q} \sum_{\varepsilon<|x-q^n|<1} q^n \left| \frac{f(q^n)-f(x)}{q^n - x} \right| + \frac{1-q}{\pi_q} \sum_{|x-q^n|>1} \frac{|q^n f(q^n)|}{|x-q^n|}.$$

By applying the mean value theorem (see, [38, Theorem 3.2]), we get $|f(q^n) - f(x)| = |q^n - x| D_q f(\xi_t)$, where ξ_t lies between q^n and x. Since $D_q f(x)$ is bounded, we can conclude that:

$$\left| \frac{f(q^n)-f(x)}{q^n - x} \right| = |D_q f(\xi_t)| \leqslant \|D_q f\|_{L^\infty(\mathbb{R}_q)}$$

and $q^n = \frac{q^n - x + x - q^{n+2}}{(1-q^2)} \leqslant \frac{2}{1-q^2}|x-q^n|$. Thus,

$$\begin{aligned} |(\mathscr{H}_q f)(x)| &\leqslant \frac{\|D_q f\|_{L^\infty(\mathbb{R}_q)}}{\pi_q} \frac{2}{1+q} \sum_{|x-q^n|<1} |x-q^n| \\ &\quad + \frac{1-q}{\pi_q} \sum_{|x-q^n|>1} \frac{1}{|x-q^n|} P_{1,0,q}(f) \\ &\leqslant \frac{4}{(1+q)\pi_q} \left(\|D_q f\|_{L^\infty(\mathbb{R}_q)} + P_{1,0,q}(f) \right) < \infty, \end{aligned}$$

which means that $\mathscr{H}_q$ is well-defined.

Let us calculate the Hilbert transform of some basic functions.

EXAMPLE 1. The q-Hilbert transform for a constant function $f(t) = c$ is easy to calculate. Using the Definition 1 we obtain that

$$(\mathscr{H}_q f)(x) = \frac{c}{\pi_q} \int_{\mathbb{R}_q} \frac{d_q s}{t \ominus_q s} = 0.$$

The last equality is due to the integrand $\frac{1}{t \ominus_q s}$ being an odd function over $\mathbb{R}_q$ (see, (15)). Hence, $H(c) = 0$ for any constant c.

EXAMPLE 2. Let $f(y) = \exp(i\alpha y; q^2)$ and $\alpha \in \mathbb{R}$. Then, since

$$\widehat{\delta}_t(y : q^2) = K_q \exp(-iyt; q^2)$$

(see, [40, Propery 2, p. 780]), using (10), (14), (15), Proposition 2 and (23) we obtain that

$$\begin{aligned} (\mathscr{H}_q f)(x) &\overset{(23)}{=} \frac{1}{\pi_q} \int_{\mathbb{R}_q} \frac{\exp(i\alpha y; q^2)}{x \ominus_q y} d_q y \\ &\overset{(15)}{=} \frac{K_q}{\pi_q} \int_{\mathbb{R}_q} \exp(i\alpha y; q^2) \int_{\mathbb{R}_q} \exp(-iyt; q^2) \widehat{\varphi}(t; q^2) \exp(ixt; q^2) d_q t \, d_q y \\ &= \frac{1}{\pi_q} \int_{\mathbb{R}_q} \widehat{\varphi}(t; q^2) \exp(ixt; q^2) \int_{\mathbb{R}_q} K_q \exp(-iyt; q^2) \exp(i\alpha y; q^2) d_q y \, d_q t \\ &= \frac{1}{\pi_q} \int_{\mathbb{R}_q} \widehat{\varphi}(t; q^2) \exp(ixt; q^2) \int_{\mathbb{R}_q} \widehat{\delta}_t(y; q^2) \exp(i\alpha y; q^2) d_q y \, d_q t \\ &\overset{(14)}{=} \frac{1}{\pi_q} \int_{\mathbb{R}_q} \widehat{\varphi}(t; q^2) \exp(ixt; q^2) \delta_t(\alpha) d_q t \\ &\overset{(10)}{=} \frac{1}{\pi_q} \widehat{\varphi}(\alpha; q^2) \exp(i\alpha x; q^2) \\ &\overset{(22)}{=} -\frac{i}{\Gamma_{q^2}\left(\frac{1}{2}\right)} \exp(i\alpha x; q^2) sng(\alpha). \end{aligned}$$

EXAMPLE 3. Let $f(y)=\sin(y;q^2)$. Then, using (11), we find that

$$\sin(y;q^2)=\frac{\exp(iy;q^2)-\exp(-iy;q^2)}{2i};\quad \cos(x;q^2)=\frac{\exp(-ix;q^2)+\exp(ix;q^2)}{2}.$$

Thus, by Example 2 we have that

$$\begin{aligned}(\mathscr{H}_q f)(x) &= \frac{\left(\mathscr{H}_q \exp(i(\cdot);q^2)\right)(x)-\left(\mathscr{H}_q \exp(-i(\cdot);q^2)\right)(x)}{2i}\\ &\overset{(2)}{=} \frac{-\frac{i}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\exp(ix;q^2)-\frac{i}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\exp(-ix;q^2)}{2i}\\ &= \frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\frac{-\exp(ix;q^2)-\exp(-ix;q^2)}{2}\\ &= -\frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\cos(x;q^2).\end{aligned}$$

EXAMPLE 4. Let $f(y)=\cos(y;q^2)$. Then, it follows from (11) and Example 2 that

$$\mathscr{H}_q\left(\cos((\cdot);q^2)\right)(x)\overset{(2)}{=}\frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\sin(x;q^2).$$

3.2. The Fourier transform of the Hilbert transform

It is a well-known result that the Fourier transform of the Hilbert transform $(Hf)(x)$ satisfies the following (see, [28]) relation

$$\mathscr{F}(Hf)(\xi)=-i\frac{\operatorname{sgn}(\xi)}{\pi}\mathscr{F}(f)(\xi), \tag{24}$$

where $\operatorname{sgn}(\xi)$ is the signum function and the Fourier transform $\mathscr{F}(f)(\xi)$ of $f(x)$ is given by

$$\mathscr{F}(f)(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\xi x}dx.$$

We obtain a q-analogue of relation (24).

THEOREM 1. *Suppose $f\in L_q^p(\mathbb{R}_q)$, $1<p\leqslant 2$. Then the Fourier transform of $\mathscr{H}_q(x(t))$ is given by*

$$\widehat{(\mathscr{H}_q f)}(t;q^2)=-i\frac{\operatorname{sgn}(x)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}\widehat{f}(t;q^2).$$

for $x\in\mathbb{R}_q$.

Proof. Using (13), (17), (23) and Proposition 2 we have that

$$
\begin{aligned}
(\widehat{\mathscr{H}_q f})(t;q^2) &\overset{(13)}{=} \frac{1}{\pi_q}\widehat{f(s)*\varphi(s)}(t;q^2)\\
&\overset{(17)}{=} \frac{1}{\pi_q}\widehat{f}(t;q^2)\widehat{\varphi}(t;q^2)\\
&\overset{(22)}{=} -i\frac{\operatorname{sgn}(x)}{\pi_q}\Gamma_{q^2}(\frac{1}{2})\widehat{f}(t;q^2)\\
&= -i\frac{\operatorname{sgn}(x)}{\Gamma_{q^2}(\frac{1}{2})}\widehat{f}(t;q^2).
\end{aligned}
$$

The proof is complete. □

4. Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces

This section is devoted to the Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces. In the first subsection contains the decreasing rearrangement which is, in fact, a distribution function with respect to the q-measure. Moreover, we provide proofs for some of its fundamental properties. These investigations will be very important when we discuss the Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces in second subsection.

4.1. A q-measure and q-distribution function

Let $\mathscr{B}$ denote the Borel σ-algebra of $\mathbb{R}$. Let $\Omega \subset \mathbb{R}$ be a measurable space, and let $\alpha_0 \in \mathbb{R}$. The *unit point measure* (or Dirac measure) at α_0, denoted by δ_{α_0}, is as follows

$$
\delta_{\alpha_0}(\Omega) = \begin{cases} 1 & \text{if } \alpha_0 \in \Omega,\\ 0 & \text{if } \alpha_0 \notin \Omega. \end{cases}
$$

More generally, if $\{\alpha_n : n \in \mathbb{Z}\}$ is a countable set of points in $\mathbb{R}$ and $\{\beta_n \geqslant 0 : n \in \mathbb{Z}\}$, we define a point measure μ by

$$
\mu = \sum_{n\in\mathbb{Z}} \beta_n \delta_{\alpha_n}, \quad \mu(\Omega) = \sum_{\alpha_n\in\Omega} \beta_n.
$$

This measure is σ-finite, and finite if $\sum \beta_n < \infty$ (see, [41, Chapter 4] and [8, Chapter 1]). Let $0 < q < 1$. Assume that $\alpha_n = q^n$ and $\beta_n = (1-q)q^n$ for $n \in \mathbb{Z}$. Then we define a *q-measure* μ_q is defined as

$$
\mu_q = \sum_{n\in\mathbb{Z}} q^n \delta_{q^n}, \quad \mu_q(\Omega) = (1-q)\sum_{q^n\in\Omega} q^n. \tag{25}
$$

EXAMPLES.

1. Let $\Omega_1 = \{q^n : n \in \mathbb{N}\}$. Then the q-measure of Ω_1 is

$$
\mu_q(\Omega_1) = (1-q)\sum_{q^n\in\Omega_1} q^n = (1-q)\sum_{k=1}^{\infty} q^k = \int_0^q d_q x = q.
$$

2. Let $\Omega_2 := [0, q^m] \subset \mathbb{R}_q^+$ for some $m \in \mathbb{Z}$. Then, by (7) and (25) we find

$$\begin{aligned} \mu_q(\Omega_2) &= (1-q) \sum_{q^n \in \Omega_2} q^n \\ &= (1-q) \sum_{q^n \leqslant q^m} q^n \\ &= (1-q) \sum_{n=-\infty}^{\infty} q^n \chi_{[0,q^m]}(q^n) \\ &= \int_{\mathbb{R}_q^+} \chi_{(0,q^m]}(x) \mathrm{d}_q x. \end{aligned} \tag{26}$$

3. Let $\Omega_2 := [-q^k, q^m] \subset \mathbb{R}_q^+$ for some $m, k \in \mathbb{Z}$. Then, by (7), (25) and (26) we have

$$\begin{aligned} \mu_q(\Omega_2) &= (1-q) \sum_{q^n \in \Omega_3} q^n \\ &= (1-q) \sum_{0 \leqslant q^n \leqslant q^m} q^n + (1-q) \sum_{-q^m \leqslant -q^n \leqslant 0} q^n \\ &= (1-q) \sum_{n=-\infty}^{\infty} q^n \chi_{[0,q^m]}(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n \chi_{[0,q^m]}(-q^n) \\ &\overset{(26)}{=} \int_{\mathbb{R}_q} \chi_{\Omega_2}(x) \mathrm{d}_q x. \end{aligned} \tag{27}$$

DEFINITION 2. The q-distribution function $d_f(\lambda; q)$ of $f : \mathbb{R}_q \to \mathbb{R}$ is a real-valued function, which expressed as

$$d_f(\lambda; q) = \mu_q\{x \in \mathbb{R}_q : |f(x)| > \lambda\}, \quad \lambda > 0. \tag{28}$$

Moreover, we observe that

$$d_{f+g}(2\lambda; q) \leqslant d_f(\lambda; q) + d_g(\lambda; q).$$

Let $\Omega_3 = \{x \in \mathbb{R}_q : |f(x)| > \lambda\}$. Then, it follows from (27) and (28) that

$$d_f(\lambda; q) = (1-q) \sum_{q^n \in \Omega_3} q^n = \int_{\mathbb{R}_q} \chi_{\Omega_3}(x) d_q x. \tag{29}$$

DEFINITION 3. The non-increasing rearrangement f^* of f is defined by

$$f^* = \inf\{\lambda \geqslant 0 : d_f(\lambda; q) \leqslant t\}, \quad t > 0. \tag{30}$$

where we use the convention that $\inf \varnothing = \infty$.

It is important to note that the function f^* exhibits the following properties

1) *Subadditive:*
$$(f+g)^* \leqslant f^* + g^*; \tag{31}$$

2) *Submultiplicative:*
$$(fg)^* \leqslant f^* g^*. \tag{32}$$

REMARK 2. The non-increasing rearrangement f^* can be understood as a sequence $\{f^*(q^n)\}$ by defining

$$f^*(q^n) = f^*(\lambda) = \inf\{\lambda > 0 : d_f(\lambda;q) \leqslant q^n\}, \quad q^n \leqslant \lambda < q^{n-1}, \tag{33}$$

for $n \in \mathbb{Z}$.

PROPOSITION 3. *Let $1 \leqslant p \leqslant \infty$. Then, the following properties hold:*

(i) $f^*(t) > \lambda$ *if and only if* $d_f(\lambda;q) > t$.

(ii) f *and* f^* *are q-equimeasurable, that is,*

$$\mu_q\{x \in \mathbb{R}_q : |f(x)| > \lambda\} = \mu_q\{t > 0 : f^*(t) > \lambda\}, \quad \textit{for all} \quad \lambda > 0.$$

Proof.

(i) Assume that $d_f(\lambda;q) > t$. Since $d_f(\lambda;q)$ is a decreasing function we have

$$\lambda < \inf\{\eta : d_f(\eta;q) \leqslant t\}.$$

Therefore, by Definition 3 we get $f^*(t) > \lambda$. For the reverse direction, suppose $f^*(t) > \lambda$. Then, by Definition 3 we find that

$$\inf\{\eta : d_f(\eta;q) \leqslant t\} > \lambda.$$

Hence, it follows from the fact that d_f is a decreasing function that $d_f(\lambda;q) > t$.

(ii) By (i) and (30) we have

$$\begin{aligned} d_{f^*}(\lambda;q) &= \mu_q\{t > 0 : f^*(t) > \lambda\} = \mu_q\{t > 0 : d_f(\lambda;q) > t\} \\ &= \mu_q\{[0, d_f(\lambda;q))\} = d_f(\lambda;q). \end{aligned}$$

(iii) Since $0 < \lambda < \infty$, using (30) we obtain

$$\begin{aligned} (|f|^p)^*(t) &= \inf\{\lambda > 0 : \mu_q\{x \in \Omega : |f(x)|^p > \lambda\} \leqslant t\} \\ &= \inf\{\lambda^p > 0 : \mu_q\{x \in \Omega : |f(x)| > \lambda\} \leqslant t\} = f^*(t)^p. \end{aligned}$$

This completes the proof. □

In terms of the distribution function, we state and prove the following important description of the $L_q^p(\mathbb{R}_q)$ norm.

PROPOSITION 4. *Let* $0 < p < \infty$ *and* $f \in L_q^p(\mathbb{R}_q)$. *Then*

$$\|f\|^p_{L_q^p(\mathbb{R}_q)} = [p]_q \int\limits_{\mathbb{R}_q^+} \lambda^{p-1} d_f(\lambda;q) d_q\lambda = \int\limits_{\mathbb{R}_q^+} f^*(t)^p d_q t. \tag{34}$$

Proof. Using (7) and (29) we get

$$\begin{aligned}[p]_q \int\limits_{\mathbb{R}_q^+} \lambda^{p-1} d_f(\lambda;q) d_q\lambda &\overset{(29)}{=} [p]_q \int\limits_{\mathbb{R}_q^+} \lambda^{p-1} \int\limits_{\mathbb{R}_q} \chi_{\{x\in\mathbb{R}_q : |f(x)|>\lambda\}}(x) d_q x d_q \\ &\overset{(7)}{=} [p]_q (1-q)^2 \sum_{n=-\infty}^{\infty} q^{np} \sum_{|f(q^m)|>q^n} q^m. \end{aligned} \tag{35}$$

Since $[p]_q(1-q)^2 = (1-q)(1-q^p)$, we have

$$\begin{aligned}[p]_q (1-q)^2 \sum_{n=-\infty}^{\infty} q^{np} \sum_{|f(q^m)|>q^n} q^m &= (1-q)(1-q^p) \sum_{m=-\infty}^{\infty} q^m \sum_{|f(q^m)|\leqslant q^n} q^{np} \\ &= (1-q) \sum_{m=-\infty}^{\infty} q^m \sum_{|f(q^m)|\leqslant q^n} (q^{np} - q^{(n+1)p}) \\ &= (1-q) \sum_{m=-\infty}^{\infty} q^m |f(q^m)|^p \qquad (36) \\ &\overset{(7)}{=} \int\limits_{\mathbb{R}_q} |f(x)|^p d_q x. \end{aligned}$$

Finally, according to (35) and (36), we derive the first equality in (34). The second equality in (34) follows directly from the q-equimeasurability of $|f|^p$ and $(f^*)^p$ in Proposition 3. The proof is complete. □

LEMMA 1. *Let* $f \in L_q^p(\mathbb{R}_q)$ *for* $0 < p < \infty$. *Then*

i) We assume that $\Omega_3 = \{x \in \mathbb{R}_q : |f(x)| > \lambda\}$

$$d_f(\lambda;q) \leqslant \frac{1}{\lambda} \int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x) |f(x)| d_q x \leqslant \frac{1}{\lambda} \int\limits_{\mathbb{R}_q} |f(x)| d_q x;$$

ii) The following (The q*-Chebyshev inequality) holds*

$$d_f(\lambda;q) \leqslant \frac{1}{\lambda^p} \int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x) |f(x)|^p d_q x,$$

Proof. i) Let $x \in \Omega_3$. The, the relation $0 \leqslant \lambda\chi_{\Omega_3}(x) \leqslant |f(x)|\chi_{\Omega_3}(x) \leqslant |f(x)|$ implies that

$$\int\limits_{\mathbb{R}_q} \lambda\chi_{\Omega_3}(x)d_qx \leqslant \int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x)|f(x)|d_qx \leqslant \int\limits_{\mathbb{R}_q} |f(x)|d_qx.$$

Since $\int\limits_0^\infty \lambda\chi_{\Omega_3}(x)d_qx = \lambda d_f(\lambda;q)$, the inequality in i) is proved.

ii) Using the fact that $\frac{|f(x)|}{\lambda} > 1$ for $x \in \Omega_3$ and (29), we obtain that

$$\begin{aligned}
\frac{1}{\lambda^p}\int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x)\,|f(x)|^p\,d_qx &= \int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x)\left(\frac{|f(x)|}{\lambda}\right)^p d_qx \\
&\geqslant \int\limits_{\mathbb{R}_q} \chi_{\Omega_3}(x)d_qx \\
&\overset{(29)}{=} d_f(\lambda;q),
\end{aligned}$$

for $0 < p < \infty$. So also the inequality in ii) is proved. □

4.2. Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces

After covering the fundamental properties of decreasing rearrangements of functions (see, Subsection 4.1), now we introduce the definition of Lorentz spaces on $\mathbb{R}_q$.

DEFINITION 4. Let f a function on $\mathbb{R}_q$ and $0 < p,r \leqslant \infty$. The Lorentz space $L^{p,r}(\mathbb{R}_q)$ is defined as the set of real-value functions f for which the following quasi-norm is finite;

$$\|f\|_{L^{p,r}(\mathbb{R}_q)} = \left(\int\limits_0^\infty \left(t^{\frac{1}{p}}f^*(t)\right)^r \frac{d_qt}{t}\right)^{\frac{1}{r}}.$$

If $q = \infty$, then the quasi-norm is defined as

$$\|f\|_{L^{p,\infty}(\mathbb{R}_q)} = \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}}f^*(t).$$

Note that, $L^{p,r}(\mathbb{R}_q)$ is a linear space for all $0 < p \leqslant \infty$ and $0 < r < \infty$. From our proof above, it follows that the functional $\|\cdot\|_{L^{p,r}(\mathbb{R}_q)}$ is a quasi-norm. Moreover, the functional $\|\cdot\|_{L^{p,r}(\mathbb{R}_q)}$ is a norm if and only if either $1 \leqslant r \leqslant p$ or $p = r = 1$.

4.3. The maximal function on $\mathbb{R}_q$

The maximal function plays a crucial role in the study of Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces, with its applications becoming evident in further analysis. The formal definition is as follows:

DEFINITION 5. The function $f^{**}:(0,\infty)\to[0,\infty]$ is defined as

$$f^{**}(t)=\frac{1}{t}\int_0^t f^*(s)d_qs.$$

Note that, as $q\to 1$, we obtain $\frac{1}{t}\int_0^t f^*(s)d_qs$ which is sometimes referred to as the *maximal function* of f. In classical analysis, it represents the supremum of all average values of f^*.

4.4. The normed space $\mathcal{N}^{p,r}(\mathbb{R}_q)$

In this subsection, we introduce the space $\mathcal{N}^{p,r}(\mathbb{R}_q)$ and prove its equivalence to spaces $L^{p,r}(\mathbb{R}_q)$.

DEFINITION 6. For any $f\in L^{p,r}(\mathbb{R}_q)$, the functional $\|\cdot\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)}$ is defined by

$$\|f\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)}=\left(\int_0^\infty\left(t^{\frac{1}{p}}f^{**}(t)\right)^r\frac{d_qt}{t}\right)^{\frac{1}{r}},$$

for $0<p<\infty,\ 0<r<\infty$, and

$$\|f\|_{\mathcal{N}^{p,\infty}(\mathbb{R}_q)}=\sup_{t\in\mathbb{R}_q^+}t^{\frac{1}{p}}f^{**}(t),$$

for $0<p<\infty,\ r=\infty$.

5. Some q-deformed Hardy-type inequality

The Hardy-type inequality, renowned for its rich historical significance and numerous variants, stands as one of the most widely utilized tools in classical analysis, alongside Sobolev inequalities (see [35]). In recent years, substantial progress has been made in developing the q-deformed Hardy inequality. This includes early generalizations incorporating weighted frameworks, as well as various modifications and extensions (see [4, 33, 36, 42]). In the following, we present a special form of the q-deformed Hardy-type inequality.

THEOREM 2. *Let* $1\leqslant p<\infty$ *and* $\alpha>0$. *Assume that* g *be a positive function on* $\mathbb{R}_q$. *Then we have the following inequalities*

$$\left(\int_{\mathbb{R}_q^+}t^{-\alpha-1}\left(\int_0^t g(s)\mathrm{d}_qs\right)^p d_qt\right)^{\frac{1}{p}}\leqslant\left[\frac{\alpha}{p}\right]_q^{-1}\left(\int_{\mathbb{R}_q^+}g^p(t)t^{-\alpha+p-1}d_qt\right)^{\frac{1}{p}},\tag{37}$$

and

$$\left(\int\limits_{\mathbb{R}_q^+} t^{\alpha-1}\left(\int\limits_t^{\infty} g(s)\mathrm{d}_q s\right)^p d_q t\right)^{\frac{1}{p}} \leqslant \left[\frac{\alpha}{p}\right]_q^{-1}\left(\int\limits_{\mathbb{R}_q^+} g^p(t)t^{\alpha+p-1}d_q t\right)^{\frac{1}{p}}. \tag{38}$$

Proof. First we prove the inequality (37). Using (6) and (26) we get

$$\begin{aligned} I_1^p(g) &:= \int\limits_{\mathbb{R}_q^+} t^{-\alpha-1}\left(\int\limits_0^t g(s)\mathrm{d}_q s\right)^p d_q t \\ &= (1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\left(\sum_{m=n}^{\infty} q^m g(q^m)\right)^p. \end{aligned} \tag{39}$$

Next, we demonstrate the necessary facts. It follows from Jensen inequality (see, [32])

$$\sum_i a_i\phi(x_i) \geqslant \left(\sum_i a_i\right)\phi\left(\frac{\sum_i a_i x_i}{\sum_i a_i}\right)$$

(where $a_i > 0$ are positive weights, x_i are points, and $\phi(t) := t^p$ is a convex function with $p > 1$) that

$$\left(\frac{\sum\limits_{m=n}^{\infty} q^m g(q^m) q^{m\frac{\alpha}{p}} q^{-m\frac{\alpha}{p}}}{\sum\limits_{m=n}^{\infty} q^{m\frac{\alpha}{p}}}\right)^r \sum_{m=n}^{\infty} q^{m\frac{\alpha}{p}} \leqslant \sum_{m=n}^{\infty}\left(q^m g(q^m) q^{-m\frac{\alpha}{p}}\right)^p q^{m\frac{\alpha}{p}} \tag{40}$$

and we have that

$$\begin{aligned} \left[\frac{\alpha}{p}\right]_q \sum_{k=\pm k_0}^{\infty} q^{k\frac{\alpha}{p}} &= \frac{(1-q^{\frac{\alpha}{p}})\sum\limits_{k=\pm k_0}^{\infty} q^{k\frac{\alpha}{p}}}{1-q} \\ &= \frac{(1-q^{\frac{\alpha}{p}})\sum\limits_{k=\pm k_0}^{\infty}\left[q^{k\frac{\alpha}{p}} - q^{(k+1)\frac{\alpha}{p}}\right]}{1-q} \\ &= \frac{q^{\pm k_0\frac{\alpha}{p}}}{1-q}, \end{aligned} \tag{41}$$

where $k_0 \in \mathbb{Z}$. It follows from (6), (39), (40) and (41) that

$$\begin{aligned} I_1^p(g) &\overset{(39)}{=} (1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\left(\sum_{m=n}^{\infty} q^m g(q^m)\right)^p \\ &= (1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\left(\frac{\sum\limits_{m=n}^{\infty} q^m g(q^m) q^{m\frac{\alpha}{p}} q^{-m\frac{\alpha}{p}}}{\sum\limits_{m=n}^{\infty} q^{m\frac{\alpha}{p}}}\right)^p\left(\sum_{m=n}^{\infty} q^{m\frac{\alpha}{p}}\right)^p \end{aligned}$$

$$\overset{(40)}{\leqslant} (1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\sum_{m=n}^{\infty}\left(q^m g(q^m) q^{-m\frac{\alpha}{p}}\right)^p q^{m\frac{\alpha}{p}}\left(\sum_{m=n}^{\infty} q^{m\frac{\alpha}{p}}\right)^{p-1}$$

$$= \left[\frac{\alpha}{p}\right]_q^{1-p}(1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\sum_{m=n}^{\infty}(q^m g(q^m))^p q^{m(\frac{\alpha}{p}-\alpha)}\left(\left[\frac{\alpha}{p}\right]_q\sum_{m=n}^{\infty} q^{m\frac{\alpha}{p}}\right)^{p-1}$$

$$\overset{(41)}{=} \left[\frac{\alpha}{p}\right]_q^{1-p}(1-q)^2\sum_{n=-\infty}^{\infty} q^{-n\frac{\alpha}{p}}\sum_{m=n}^{\infty} g^p(q^m) q^{m(\frac{\alpha}{p}-\alpha+p)}$$

$$= \left[\frac{\alpha}{p}\right]_q^{1-p}(1-q)^2\sum_{m=-\infty}^{\infty} g^p(q^m) q^{m(\frac{\alpha}{p}-\alpha+p)}\sum_{n=-\infty}^{m} q^{-n\frac{\alpha}{p}}$$

$$= \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)^2\sum_{m=-\infty}^{\infty} g^p(q^m) q^{m(\frac{\alpha}{p}-\alpha+p)}\left(\left[\frac{\alpha}{p}\right]_q\sum_{n=-m}^{\infty} q^{n\frac{\alpha}{p}}\right)$$

$$\overset{(41)}{=} \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\sum_{m=-\infty}^{\infty} g^p(q^m) q^{m(\frac{\alpha}{p}-\alpha+p)} q^{-m\frac{\alpha}{p}}$$

$$= \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\sum_{m=-\infty}^{\infty} q^m g^p(q^m) q^{-m(\alpha-1+p)}$$

$$\overset{(6)}{=} \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\int_{\mathbb{R}_q} g^p(t) t^{-\alpha-1+p} d_q t.$$

Hence, we have the following inequality:

$$I_1^p(g) \leqslant \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\int_{\mathbb{R}_q} g^p(t) t^{-\alpha-1+p} d_q t,$$

which gives the inequality (37). Next, by (6) we find that

$$I_2(g) := \left(\int_{\mathbb{R}_q^+} t^{-\alpha-1}\left(\int_t^{\infty} g(s)\mathrm{d}_q s\right)^p d_q t\right)^{\frac{1}{p}}$$

$$= \left((1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{-n\alpha}\left(\sum_{m=-\infty}^{n} q^m g(q^m)\right)^p\right)^{\frac{1}{p}}$$

$$= \left((1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{n\alpha}\left(\sum_{m=n}^{\infty} q^{-m} g(q^{-m})\right)^p\right)^{\frac{1}{p}}.$$

Moreover, by applying the same argument as in the calculation of $I_1(g)$ we have that

$$\begin{aligned} I_2(g) &= \left((1-q)^{1+p}\sum_{n=-\infty}^{\infty} q^{n\alpha}\left(\sum_{m=n}^{\infty} q^{-m}g(q^{-m})\right)^p\right)^{\frac{1}{p}} \\ &\leqslant \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\sum_{m=-\infty}^{\infty} q^{-m}g^p(q^{-m})q^{m(\alpha-1+p)} \\ &\overset{(6)}{=} \left[\frac{\alpha}{p}\right]_q^{-p}(1-q)\int_{\mathbb{R}_q} g^p(t)t^{\alpha-1+p}d_qt. \end{aligned}$$

which shows that (38) holds. So the proof is complete. □

THEOREM 3. *Let $1<p<\infty$, $1\leqslant r\leqslant\infty$ or $p=r=\infty$, then $\|\cdot\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)}$ is a norm on $L^{p,r}(\mathbb{R}_q)$ and hence $(L^{p,r}(\mathbb{R}_q),\|\cdot\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)})$ is a normed space. More precisely,*

$$\|f\|_{L^{p,r}(\mathbb{R}_q)} \leqslant \|f\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)} \leqslant \frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^{p,r}(\mathbb{R}_q)}. \tag{42}$$

That is, the quasi-norms $\|\cdot\|_{L^{p,r}(\mathbb{R}_q)}$ and $\|\cdot\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)}$ are equivalent.

Proof. It follows from f^* is a decreasing function that

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)d_qs \geqslant \frac{1}{t}\int_0^t f^*(t)d_qs = f^*(t).$$

Therefore, the first inequality follows immediately from the above inequality. To prove the second inequality, we will consider the following three cases:

1) If $1<p<\infty$ and $1<r<\infty$. Then, by Definition 5 and Definition 6 we get that

$$\begin{aligned} \|f\|_{\mathcal{N}^{p,r}(\mathbb{R}_q)} &= \left(\int_0^\infty \left(t^{\frac{1}{p}}f^{**}(t)\right)^r \frac{d_qt}{t}\right)^{\frac{1}{r}} \\ &= \left(\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{r}-1}\int_0^t f^*(s)d_qs\right)^r d_qt\right)^{\frac{1}{r}}. \end{aligned} \tag{43}$$

If we take $\alpha = r(1-\frac{1}{p})$ for Hardy inequality (37), then we obtain

$$\begin{aligned} \left(\int_0^\infty \left(t^{\frac{1}{p}-\frac{1}{r}-1}\int_0^t f^*(s)d_qs\right)^r d_qt\right)^{\frac{1}{r}} &\leqslant \frac{1}{\left[1-\frac{1}{p}\right]_q}\left(\int_0^\infty (f^*(t))^r t^{\frac{r}{p}-1}d_qt\right)^{\frac{1}{r}} \\ &= \frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^{p,r}(\mathbb{R}_q)}, \end{aligned} \tag{44}$$

Thus, according to (43) and (44) we have (42).

2) If $1 < p < \infty$ and $r = \infty$. Then

$$\begin{aligned}\|f\|_{\mathcal{N}^{p,\infty}(\mathbb{R}_q)} &= \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}} f^{**}(t) = \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}-1}\int_0^t f^*(s)d_qs \\ &= \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}-1}\int_0^t s^{-\frac{1}{p}} s^{\frac{1}{p}} f^*(s)d_qs \\ &\leqslant \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}-1}\int_0^t s^{-\frac{1}{p}} \sup_{u\in\mathbb{R}_q^+} u^{\frac{1}{p}} f^*(u)d_qs. \\ &= \|f\|_{L^{p,\infty}(\mathbb{R}_q)} \sup_{t\in\mathbb{R}_q^+} t^{\frac{1}{p}-1}\int_0^t s^{-\frac{1}{p}} d_qs \\ &= \frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^{p,\infty}(\mathbb{R}_q)}.\end{aligned}$$

3) If $p = r = \infty$. Then,

$$\|f\|_{\mathcal{N}^{p,\infty}(\mathbb{R}_q)} = \sup_{t\in\mathbb{R}_q} f^{**}(t) = \lim_{t\to 0+}\int_t^\infty f^*(s)d_qs = f^*(0) = \|f\|_{L^{\infty,\infty}(\mathbb{R}_q)}.$$

This completes the proof. □

6. Applications of the q-deformed Hardy-type inequality

6.1. The Riesz inequality for the q-deformed Hilbert transform

The Hilbert transform is a bounded linear operator on $L^p(\mathbb{R}_q)$, for $1 < p < \infty$. In the classical case next inequality was shown by M. Riesz and is known as the Riesz inequality [28].

THEOREM 4. *If $1 < p < \infty$, then there exists a positive constant $C_{p,q}$ such that*

$$\|\mathcal{H}_q f\|_{L^p(\mathbb{R}_q)} \leqslant C_{p,q}\|f\|_{L^p(\mathbb{R}_q)} \quad \text{for all } f \in L^p(\mathbb{R}_q), \tag{45}$$

where $C_{p,q}$ a positive constant independent of f.

Proof. It follows from (32), (34) and (53) that

$$\begin{aligned}\|\mathscr{H}_q f\|_{L^p(\mathbb{R}_q)} &= \|(\mathscr{H}_q f)^*\|_{L^p(\mathbb{R}_q)}\\ &\leqslant C_q^p\left\|\int_t^\infty f^*(s)\frac{d_qs}{s}+\frac{1}{x}\int_0^t f^*(s)d_qs\right\|_{L^p(\mathbb{R}_q)}\\ &\leqslant C_q^p\left\|\int_t^\infty f^*(s)\frac{d_qs}{s}\right\|_{L^p(\mathbb{R}_q)}+C_q^p\left\|\frac{1}{t}\int_0^t f^*(s)d_qs\right\|_{L^p(\mathbb{R}_q)}\\ &= C_q^p\left\|\int_t^\infty \frac{f^*(s)}{s}d_qs\right\|_{L^p(\mathbb{R}_q)}+C_q^p\|f\|_{\mathscr{N}^{p,p}(\mathbb{R}_q)}.\end{aligned}\tag{46}$$

If we take $\alpha=1$ for the Hardy's inequality (38), we get that

$$\begin{aligned}\left\|\int_t^\infty \frac{f^*(s)}{s}d_qs\right\|_{L^p(\mathbb{R}_q)} &= \left(\int_0^\infty\left(\int_t^\infty\frac{f^*(s)}{s}d_qs\right)^p d_qt\right)^{\frac{1}{p}}\\ &\leqslant \frac{1}{\left[\frac{1}{p}\right]_q}\left(\int_0^\infty\left(\frac{f^*(t)}{t}\right)^p t^p dt\right)^{\frac{1}{p}}\\ &\leqslant \frac{1}{\left[\frac{1}{p}\right]_q}\|f\|_{L^p(\mathbb{R}_q)}\end{aligned}\tag{47}$$

Using (42) we get

$$\|f\|_{\mathscr{N}^{p,p}(\mathbb{R}_q)}\leqslant\frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^{p,p}(\mathbb{R}_q)}=\frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^p(\mathbb{R}_q)}.\tag{48}$$

Hence, it follows from (46), (47) and (48) that

$$\begin{aligned}\|\mathscr{H}_q f\|_{L^p(\mathbb{R}_q)} &\leqslant \frac{C_q}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^p(\mathbb{R}_q)}+\frac{C_q}{\left[\frac{1}{p}\right]_q}\|f\|_{L^p(\mathbb{R}_q)}\\ &\leqslant\left(\frac{C_q}{\left[1-\frac{1}{p}\right]_q}+\frac{C_q}{\left[\frac{1}{p}\right]_q}\right)\|f\|_{L^p(\mathbb{R}_q)}\\ &= C_{p,q}\|f\|_{L^p(\mathbb{R}_q)},\end{aligned}$$

where $C_{p,q}:=\frac{C_q}{\left[1-\frac{1}{p}\right]_q}+\frac{C_q}{\left[\frac{1}{p}\right]_q}$. This completes the proof that the inequality (4) holds i.e that $\mathscr{H}_q$ is bounded on $L^p(\mathbb{R}_q)$ for $1<p<\infty$. This completes the proof. □

6.2. A inequality for the q-deformated Hilbert transform on Lorentz $L^{p,r}(\mathbb{R}_q)$

In this subsection, we study the boundedness of the q-deformated Hilbert transform on Lorentz $L^{p,r}(\mathbb{R}_q)$ spaces.

THEOREM 5. *Let $1<p<\infty$ and $1\leqslant r\leqslant\infty$. Then the following inequality holds*

$$\|\mathscr{H}_q f\|_{L^{p,r}(\mathbb{R}_q)} \leqslant B_{p,q}\|f\|_{L^{p,r}(\mathbb{R}_q)} \quad \text{for all} \quad f\in L^{p,r}(\mathbb{R}_q), \tag{49}$$

where $B_{p,q}$ a positive constant independent of f.

Proof. Using Definition 5 and the estimate (53) and the triangle inequality of the quasi-norm, we obtain that

$$\begin{aligned}\|\mathscr{H}_q f\|_{L^{p,r}(\mathbb{R}_q)} &\leqslant C_q\left\|f^{**}(x)+\int_t^\infty f^*(s)\frac{d_qs}{s}\right\|_{L^{p,r}(\mathbb{R}_q)}\\ &\leqslant C_q 2^{\frac{1}{p}+\frac{1}{r}+1}\left(\|f\|_{\mathscr{N}^{p,r}(\mathbb{R}_q)}+\left\|\int_t^\infty f^*(s)\frac{d_qs}{s}\right\|_{L^{p,r}(\mathbb{R}_q)}\right).\end{aligned} \tag{50}$$

If we take $\alpha=\frac{r}{p}$ for the Hardy's inequality (38) we get that

$$\begin{aligned}\left\|\int_t^\infty f^*(s)\frac{d_qs}{s}\right\|_{L^{p,r}(\mathbb{R}_q)} &\leqslant \left(\int_0^\infty t^{\frac{r}{p}-1}\left(\int_t^\infty f^*(s)d_qs\right)^r d_qt\right)^{\frac{1}{r}}\\ &\leqslant \frac{1}{\left[\frac{1}{p}\right]_q}\left(\int_0^\infty (f^*(s))^r t^{\frac{r}{p}-1}dt\right)^{\frac{1}{r}}\\ &\leqslant \frac{1}{\left[\frac{1}{p}\right]_q}\|f\|_{L^{p,r}(\mathbb{R}_q)}.\end{aligned} \tag{51}$$

Moreover, by using the inequality (42) we have that

$$\|f\|_{\mathscr{N}^{p,r}(\mathbb{R}_q)} \leqslant \frac{1}{\left[1-\frac{1}{p}\right]_q}\|f\|_{L^{p,r}(\mathbb{R}_q)}. \tag{52}$$

Consequently, from (50), (51) and (52) we deduce that

$$\|\mathscr{H}_q f\|_{L^{p,r}(\mathbb{R}_q)} \leqslant C_{p,q}\|f\|_{L^{p,r}(\mathbb{R}_q)},$$

where $B_{p,q}:=2^{\frac{1}{p}+\frac{1}{r}+1}\left(\frac{C_q}{\left[\frac{1}{p-1}\right]_q}+\frac{C_q}{\left[1-\frac{1}{p}\right]_q}\right)$. This completes the proof. □

7. Weak type $(1,1)$ inequality for the q-deformated Hilbert transform

The following inequality (53) involving the Hardy operator and its relationship to nonincreasing rearrangements is discussed extensively in the context of classical Lorentz spaces and integral inequalities. It can be found with detailed proofs in Bennett & Sharpley work on interpolation of operators, particularly around pages 134–138 in [10].

THEOREM 6. *If $f \in L^p(\mathbb{R}_q)$ for $1 < p < \infty$, then there exists a positive constant C_q independent of f such that*

$$(\mathscr{H}_q f)^*(t) \leqslant C_q \left(\frac{1}{t} \int_0^t f^*(s) d_q s + \int_t^\infty \frac{f^*(s)}{s} d_q s \right). \tag{53}$$

for all $t > 0$.

Proof. Using (7) and (1) we have

$$\begin{aligned} \left| (\mathscr{H}_q f)(q^m) \right| = & \left| \frac{1-q}{\pi_q} \sum_{n=-\infty}^{m-1} q^n \frac{f(q^n)}{q^m - q^n} + \frac{1-q}{\pi_q} \sum_{n=m+1}^{\infty} q^n \frac{f(q^n)}{q^m - q^n} \right. \\ & \left. + \frac{1-q}{\pi_q} \sum_{n=-\infty}^{m-1} q^n \frac{f(-q^n)}{q^m + q^n} + \frac{1-q}{\pi_q} \sum_{n=m+1}^{\infty} q^n \frac{f(-q^n)}{q^m + q^n} \right| \\ \leqslant & -\frac{1-q}{\pi_q} \sum_{n=-\infty}^{m-1} |f(q^n)| \frac{1}{|1-q^{m-n}|} + \frac{1-q}{\pi_q} q^m \sum_{n=m+1}^{\infty} q^n |f(q^n)| \frac{1}{|1-q^{n-m}|} \\ & + \frac{1-q}{\pi_q} \sum_{n=-\infty}^{m-1} |f(-q^n)| \frac{1}{1+q^{m-n}} + \frac{1-q}{\pi_q} q^m \sum_{n=m+1}^{\infty} q^n |f(-q^n)| \frac{1}{1+q^{n-m}}, \end{aligned}$$

If we set $m \geqslant 0$. Then, for $n \in \mathbb{Z}$, it is simple to confirm that

$$\begin{cases} m - n > 0 \text{ for } n < m; \\ n - m > 0 \text{ for } m < n. \end{cases} \tag{54}$$

If we assume $m < 0$. Then, for $n \in \mathbb{Z}$, we have

$$\begin{cases} m - n > 0 \text{ for } n < m; \\ n - m > 0 \text{ for } m < n. \end{cases} \tag{55}$$

Hence, (54) and (55) imply that

$$\begin{cases} \frac{1}{|1-q^{m-n}|} = \frac{1}{1-q^{m-n}} < 1 \text{ for } n < m; \\ \frac{1}{|1-q^{n-m}|} = \frac{1}{1-q^{n-m}} < 1 \text{ for } m < n. \end{cases} \tag{56}$$

Since $\frac{1}{1+q^{n-m}} < 1$ for all $n, m \in \mathbb{Z}$, and we apply (56) in order to derive the inequality

$$|(\mathscr{H}_q f)(q^m)| \leqslant \frac{1}{\pi_q} \sum_{n=-\infty}^{m-1} \{|f(q^n)| + |f(-q^n)|\} + \frac{1}{\pi_q} \frac{1}{q^m} \sum_{n=m+1}^{\infty} q^n \{|f(q^n)| + |f(-q^n)|\}.$$

Thus, applying (33), we obtain the following inequality:

$$\begin{aligned}(\mathscr{H}_q f)^*(q^m) &\overset{(33)}{\leqslant} \frac{2}{\pi_q}\sum_{n=-\infty}^{m-1} f^*(q^n) + \frac{2}{\pi_q}\frac{1}{q^m}\sum_{n=m}^{\infty} q^n f^*(q^n) \\ &= \frac{2}{\pi_q}\left((1-q)\sum_{n=-\infty}^{m-1} q^n \frac{f^*(q^n)}{q^n} + \frac{2}{\pi_q}\frac{1}{q^m}\sum_{n=m}^{\infty} q^n f^*(q^n)\right). \end{aligned} \tag{57}$$

Here we also used the well-known fact that $\sum_{n\in\mathbb{Z}} a(n)b(n) \leqslant \sum_{n\in\mathbb{Z}} a^*(n)b^*(n)$ holds when $a^*(n)$ and $b^*(n)$ are the decreasing rearrangements of $a(n)$ and $b(n)$, respectively. From (6), (7) and (57) we immediately obtain that

$$(\mathscr{H}_q f)^*(x) \leqslant C_q\left(\int_x^\infty f^*(t)\frac{d_q t}{t} + \frac{1}{x}\int_0^x f^*(t)d_q t\right).$$

This completes the proof. □

THEOREM 7. *Let $f \in L^1(\mathbb{R}_q)$. Then the following inequality holds:*

$$\|\mathscr{H}_q f\|_{L^{1,\infty}(\mathbb{R}_q)} \leqslant C_q \|f\|_{L^1(\mathbb{R}_q)}. \tag{58}$$

where C_q a positive constant independent of f.

Proof. Form Definition 5 and the estimate (53) that

$$\begin{aligned}\|\mathscr{H}_q f\|_{L^{1,\infty}(\mathbb{R}_q)} &= \|(\mathscr{H}_q f)^*\|_{L^{1,\infty}(\mathbb{R}_q)} \\ &= \sup_{t\in\mathbb{R}_q^+} t(\mathscr{H}_q f)^*(t) \\ &\leqslant C_q \sup_{t\in\mathbb{R}_q^+} t\left(\int_t^\infty f^*(s)\frac{d_q s}{s} + \frac{1}{t}\int_0^t f^*(s)d_q s\right) \\ &\leqslant C_q \sup_{t\in\mathbb{R}_q^+} t\left(\frac{1}{t}\int_t^\infty f^*(s)d_q s + \frac{1}{t}\int_0^t f^*(s)d_q s\right) \\ &\leqslant C_q \int_0^\infty f^*(s)d_q s \\ &= \|\mathscr{H}_q f\|_{L^1(\mathbb{R}_q)}\end{aligned}$$

and the inequality (50) holds, so the proof is complete. □

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