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WEIGHTED MAXIMAL POLYA–KNOPP INEQUALITIES

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Abstract. We characterize the pairs of weights (u,v) for the maximal operator G_0, defined for nonnegative functions on $(0,\infty)$ by

$$G_0 f(x) = \sup_{b>x} \exp\left(\frac{1}{b}\int_0^b \log f\right),$$

to be bounded from $L^p(v)$ to $L^q(u)$, $p \leqslant q$, or from $L^p(v)$ to $L^{q,\infty}(u)$.

1. Introduction and results

If f is a positive measurable function defined on $(0,\infty)$, the inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f\right) dx \leqslant e\int_0^\infty f(x)dx \tag{1}$$

is known as Polya-Knopp inequality. It was proved by Polya and, independently, by Knopp [8], who extended the discrete result due to Carleman [1]. For more information on this inequality and its connection with Hardy's one, see the monograph [9].

It is simple to get the next stronger result:

$$\int_0^\infty \left(\sup_{b>x} \exp\left(\frac{1}{b}\int_0^b \log f\right)\right) dx \leqslant e\int_0^\infty f(x)dx, \tag{2}$$

which we call maximal Polya-Knopp inequality.

Inequality (2) is nothing but a strong-type inequality for the maximal geometric mean operator G_0 defined for nonnegative functions on $(0,\infty)$ and $x \in (0,\infty)$ by

$$G_0 f(x) = \sup_{b>x} \exp\left(\frac{1}{b}\int_0^b \log f\right),$$

where we mean that $\exp\left(\frac{1}{b}\int_0^b \log f\right) = 0$ if $f = 0$ on $A \subset (0,b)$ with $|A| > 0$.

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The weighted versions of Polya-Knopp inequality (1) have been widely studied (see, for instance, [7], [10], [11] and [13]). The weighted inequalities for the maximal geometric operator

$$Gf(x) = \sup_I \exp\left(\frac{1}{|I|}\int_I \log|f|\right),$$

where the supremum is taken over all open bounded intervals I containing x, have also been characterized (see [2], [12] and [14]). However, as far as we know, it seems that not much attention has been paid to weighted maximal Polya-Knopp inequalities.

The condition on the weight w that characterizes the one-weight strong-type inequality for G is that $w \in A_\infty$. The classical A_∞ condition can be expressed in several equivalent ways (see, for instance, [6], chapter IV, Corollary 2.13 and Theorem 2.15). In this sense, it is well known that $w \in A_\infty$ if and only if $w \in \cup_{p>1} A_p$, which turns to be equivalent to condition $A_{\exp}$, i. e.,

$$\sup_I \left(\frac{1}{|I|}\int_I w\right) \exp\left(\frac{1}{|I|}\int_I \log(w^{-1})\right) < \infty.$$

However, as Duoandikoetxea, Martín-Reyes and Ombrosi have pointed out in [3] and [4], these equivalences do not hold for other bases. Specifically, this is the case for the basis involved in G_0. It is therefore interesting to determine which is the A_∞-type condition that characterizes the boundedness of G_0 in weighted L^p spaces.

In this paper, we tackle the problem of characterizing the pairs of weights (u,v) for which the inequality

$$\left(\int_0^\infty G_0 f(x)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}} \tag{3}$$

holds for all nonnegative functions f, with a constant $C > 0$ independent of f, in the case $1 < p \leqslant q < \infty$. We will also deal with the weighted weak-type inequality

$$\sup_{\lambda>0} \lambda \left(\int_{\{x\in(0,\infty):G_0 f(x)>\lambda\}} u(x)dx\right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty f(x)^p v(x)dx\right)^{\frac{1}{p}}, \tag{4}$$

as well as the relationship between the weighted weak and strong-type inequalities in the case $p = q$ and $u = v$.

Our results are the following ones. The first one characterizes inequality (3) in the case $1 < p \leqslant q < \infty$.

THEOREM 1. *Let $1 < p \leqslant q < \infty$ and let u, v be positive measurable functions on $(0,\infty)$. Then there exists a positive constant C such that inequality (3) holds for all nonnegative functions f on $(0,\infty)$ if and only if*

$$C_1 \equiv \sup_{b>0} \frac{1}{b^{\frac{q}{p}}} \int_0^b \left(G_0(\chi_{(0,b)} v^{-1})(x)\right)^{\frac{q}{p}} u(x)dx < \infty.$$

Moreover, the best constant C in (3) verifies $C_1^{\frac{1}{q}} \leqslant C \leqslant 4\left(\frac{1}{p}\right)^{\frac{q}{p}} (p')^{\frac{1}{q}} C_1^{\frac{1}{q}}$, where p' is the conjugate exponent of p.

Our second result characterizes the weak-type inequality (4).

THEOREM 2. *Let $0 < p,q < \infty$ and let u, v be positive measurable functions on $(0,\infty)$. Then there exists a positive constant C such that inequality (4) holds for all nonnegative functions f on $(0,\infty)$ if and only if the pair (u,v) verifies condition $A_{0,p,q,exp}$, which means that*

$$[u,v]_{A_{0,p,q,exp}} \equiv \sup_{b>0} \frac{1}{b}\left(\int_0^b u\right)^{\frac{p}{q}} \exp\left(\frac{1}{b}\int_0^b \log v^{-1}\right) < \infty.$$

Moreover, the best constant in inequality (4) is $[u,v]_{A_{0,p,q,exp}}^{\frac{1}{p}}$.

The next result shows that when $p = q$ and $u = v$, the weighted weak and strong-type maximal Polya-Knopp inequalities are equivalent. The result is the next one.

THEOREM 3. *Let q be a real number with $0 < q < \infty$ and let w be a positive measurable function on $(0,\infty)$. Then the following statements are equivalent:*

(i) There exists a positive constant K_q such that inequality

$$\left(\int_0^\infty G_0 f(x)^q w(x)dx\right)^{\frac{1}{q}} \leqslant K_q \left(\int_0^\infty f(x)^q w(x)dx\right)^{\frac{1}{q}}$$

holds for all nonnegative functions f on $(0,\infty)$.

(ii) There exists a positive constant C_q such that inequality

$$\left(\int_{\{x\in(0,\infty):G_0 f(x)>\lambda\}} w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{C_q}{\lambda}\left(\int_0^\infty f(x)^q w(x)dx\right)^{\frac{1}{q}}$$

holds for all nonnegative functions f on $(0,\infty)$ and all $\lambda > 0$.

(iii) There exists a positive constant K_1 such that inequality

$$\int_0^\infty G_0 f(x) w(x)dx \leqslant K_1 \int_0^\infty f(x) w(x)dx$$

holds for all nonnegative functions f on $(0,\infty)$.

(iv) There exists a positive constant C_1 such that inequality

$$\int_{\{x\in(0,\infty):G_0 f(x)>\lambda\}} w(x)dx \leqslant \frac{C_1}{\lambda}\int_0^\infty f(x) w(x)dx$$

holds for all nonnegative functions f on $(0,\infty)$ and all $\lambda > 0$.

(v) The weight w verifies condition $A_{0,\exp}$, which means that

$$[w]_{A_{0,exp}} \equiv \sup_{b>0}\left(\frac{1}{b}\int_0^b w\right)\exp\left(\frac{1}{b}\int_0^b \log w^{-1}\right) < \infty.$$

Furthermore, the best constants K_q, C_q, K_1 *and* C_1 *in (i), (ii), (iii) and (iv) verify*

$$1 = C_1 = C_q^q \leqslant K_1 = K_q^q = e$$

if $[w]_{A_{0,exp}} = 1$, *and*

$$[w]_{A_{0,exp}} = C_1 = C_q^q \leqslant K_1 = K_q^q \leqslant [w]_{A_{0,exp}}^{p_0} \left(\frac{p_0}{p_0 - 1}\right)^{p_0}$$

if $[w]_{A_{0,exp}} > 1$, *where* p_0 *is the only real number greater than* 1 *which is solution of the equation*

$$1 + \log[w]_{A_{0,exp}} + \log\left(\frac{p}{p-1}\right) = \frac{p}{p-1}.$$

Observe that, by Jensen's inequality,

$$\left(\frac{1}{b}\int_0^b w\right)\exp\left(\frac{1}{b}\int_0^b \log w^{-1}\right) \geqslant \exp\left(\frac{1}{b}\int_0^b \log w\right)\exp\left(\frac{1}{b}\int_0^b \log w^{-1}\right) = 1$$

for all $b > 0$, and then $[w]_{A_{0,\exp}} \geqslant 1$. Observe also that $[w]_{A_{0,\exp}} = 1$ if and only if w is a constant a.e. function. It is clear that if w is constant a.e., then $[w]_{A_{0,\exp}} = 1$. For the converse, if $[w]_{A_{0,\exp}} = 1$, then, as we have just seen,

$$\left(\frac{1}{b}\int_0^b w\right)\exp\left(\frac{1}{b}\int_0^b \log w^{-1}\right) = 1$$

for all $b > 0$, i. e.,

$$\frac{1}{b}\int_0^b w = \exp\left(\frac{1}{b}\int_0^b \log w\right)$$

for all $b > 0$. This is a case of equality in Jensen's inequality and since the exponential function is strictly convex, necessarily w is constant a.e.

Now, we include a theorem for power weights which is a straightforward consequence of Theorems 1 and 2. It reads as follows.

THEOREM 4. *Let* $u, v : (0,\infty) \to \mathbb{R}$, $u(x) = x^{\alpha}$, $v(x) = x^{\beta}$.

(i) If $1 < p \leqslant q < \infty$, *then inequality* (3) *holds if and only if*

- *(a)* $\alpha + 1 > 0$ *and* $\alpha + 1 = \frac{q}{p}(\beta + 1)$ *when* $\beta < 0$;
- *(b)* $\alpha + 1 = \frac{q}{p}(\beta + 1)$ *when* $\beta \geqslant 0$.

(ii) If $0 < p \leqslant q < \infty$, *then inequality* (4) *holds if and only if* $\alpha + 1 > 0$ *and* $\alpha + 1 = \frac{q}{p}(\beta + 1)$.

Finally, we will apply the results on G_0 in order to characterize the weighted inequality

$$\int_{\mathbb{R}^n} \mathscr{G} f(x) w(x) dx \leqslant C \int_{\mathbb{R}^n} f(x) w(x) dx,$$

where $\mathscr{G}$ is the operator defined for nonnegative functions on $\mathbb{R}^n$ by

$$\mathscr{G}f(x) = \sup_{b \geqslant 1} \exp\left(\frac{1}{b}\int_0^b \log(f(xt))dt\right).$$

The result for $\mathscr{G}$ is the next one.

THEOREM 5. *Let w be a positive weight on $\mathbb{R}^n$. Then, the following statements are equivalent:*

(i) There exists a positive constant K_1 such that inequality

$$\int_{\mathbb{R}^n} \mathscr{G}f(x)w(x)dx \leqslant K_1 \int_{\mathbb{R}^n} f(x)w(x)dx$$

holds for all nonnegative functions f on $\mathbb{R}^n$.

(ii) There exists a positive constant C_1 such that inequality

$$\int_{\{x\in\mathbb{R}^n : \mathscr{G}f(x)>\lambda\}} w(x)dx \leqslant \frac{C_1}{\lambda}\int_{\mathbb{R}^n} f(x)w(x)dx$$

holds for all nonnegative functions f on $\mathbb{R}^n$ and all $\lambda > 0$.

(iii) The weight w verifies condition $\tilde{A}_{0,\exp}$, which means that

$$[w]_{\tilde{A}_{0,exp}} \equiv ess \sup_{x\in\mathbb{R}^n} \left(\int_0^1 w(tx)t^{n-1}dt\right) \exp\left(\int_0^1 \log(w^{-1}(tx)t^{1-n})dt\right) < \infty.$$

Furthermore, the best constants K_1 and C_1 verify

$$1 \leqslant C_1 \leqslant K_1 \leqslant e$$

if $[w]_{\tilde{A}_{0,exp}} = 1$, and

$$[w]_{\tilde{A}_{0,exp}} \leqslant C_1 \leqslant K_1 \leqslant [w]_{\tilde{A}_{0,exp}}^{p_0}\left(\frac{p_0}{p_0-1}\right)^{p_0}$$

if $[w]_{\tilde{A}_{0,exp}} > 1$, where p_0 is the only real number greater than 1 which is solution of the equation

$$1 + \log[w]_{\tilde{A}_{0,exp}} + \log\left(\frac{p}{p-1}\right) = \frac{p}{p-1}.$$

We will prove Theorems 1, 2, 3 and 5 in the next sections.

2. Proof of Theorem 1

Assume that (3) holds. It is equivalent to

$$\int_0^\infty \left(G_0(fv^{-1})(x)\right)^{\frac{q}{p}} u(x)dx \leqslant C^q \left(\int_0^\infty f(x)dx\right)^{\frac{q}{p}}. \tag{5}$$

Let $b > 0$ and $f = \chi_{(0,b)}$. Then (5) implies

$$\int_0^b \left(G_0(\chi_{(0,b)}v^{-1})(x)\right)^{\frac{q}{p}} u(x)dx \leqslant C^q b^{\frac{q}{p}},$$

and since this inequality holds for all $b > 0$, we get $C_1 \leqslant C^q$.

Assume now that $C_1 < \infty$. Let f be a nonnegative function on $(0,\infty)$. We can suppose that $f = 0$ outside an interval $(0,c)$. Since the function $G_0 f$ is nonincreasing, then for every $k \in \mathbb{Z}$ the set $O_k = \{x \in (0,\infty) : G_0 f(x) > 2^k\}$ is an interval $(0,b_k)$, where b_k verifies $2^k = \exp\left(\frac{1}{b_k}\int_0^{b_k} \log f\right)$. Thus, by Jensen's inequality

$$\begin{aligned}
\int_0^\infty G_0 f(x)^q u(x)dx &= \sum_{k\in\mathbb{Z}} \int_{\{x\in(0,\infty):2^k<G_0f(x)\leqslant 2^{k+1}\}} G_0 f(x)^q u(x)dx \\
&= \sum_{k\in\mathbb{Z}} \int_{b_{k+1}}^{b_k} G_0 f(x)^q u(x)dx \leqslant 2^q \sum_{k\in\mathbb{Z}} \int_{b_{k+1}}^{b_k} 2^{kq} u(x)dx \\
&= 2^q \sum_{k\in\mathbb{Z}} \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log f\right)\right)^q \int_{b_{k+1}}^{b_k} u \\
&= 2^q \sum_{k\in\mathbb{Z}} \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log(fv^{\frac{1}{p}})\right)\right)^q \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log v^{-1}\right)\right)^{\frac{q}{p}} \int_{b_{k+1}}^{b_k} u \\
&\leqslant 2^q \sum_{k\in\mathbb{Z}} \left(\frac{1}{b_k}\int_0^{b_k} fv^{\frac{1}{p}}\right)^q \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log v^{-1}\right)\right)^{\frac{q}{p}} \int_{b_{k+1}}^{b_k} u \\
&= 2^q \sum_{k\in\mathbb{Z}} \left(T(fv^{\frac{1}{p}})(k)\right)^q \gamma_k,
\end{aligned} \tag{6}$$

where, for a nonnegative function h on $(0,\infty)$,

$$Th(k) = \frac{1}{b_k}\int_0^{b_k} h \qquad \text{and} \qquad \gamma_k = \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log v^{-1}\right)\right)^{\frac{q}{p}} \int_{b_{k+1}}^{b_k} u.$$

If we prove that the operator T is bounded from $L^\infty(0,\infty)$ to $l^\infty(\{\gamma_k\})$ and that T is also bounded from $L^1(0,\infty)$ to $l^{\frac{q}{p},\infty}(\{\gamma_k\})$, then by Marcinkiewicz's interpolation theorem the operator T will be bounded from $L^p(0,\infty)$ to $l^q(\{\gamma_k\})$.

The operator T is bounded from $L^\infty(0,\infty)$ to $l^\infty(\{\gamma_k\})$ with constant equal to 1. Indeed, observe that we use the weight $\{\gamma_k\}$ as a measure, not as a multiplier, and then, since $l^\infty \subset l^\infty(\{\gamma_k\})$ with $\|\{x_k\}\|_{l^\infty(\{\gamma_k\})} \leqslant \|\{x_k\}\|_{l^\infty}$ for all $\{x_k\} \in l^\infty$, we have

$$\|\{Th(k)\}\|_{l^\infty(\{\gamma_k\})} \leqslant \|\{Th(k)\}\|_{l^\infty} \leqslant \|h\|_{L^\infty(0,\infty)}.$$

Let us prove now that T is of weak-type $(1, \frac{q}{p})$. Let $\lambda > 0$ and $O_\lambda = \{k \in \mathbb{Z} : Th(k) > \lambda\}$. Then,

$$\sum_{k\in O_\lambda} \gamma_k = \lim_{j\to-\infty} \sum_{\{k\geqslant j: Th(k)>\lambda\}} \gamma_k.$$

We define $F_j = \{k \geqslant j : Th(k) > \lambda\}$. These sets verify $F_j \subset F_{j-1}$ and also $\{F_j\} \nearrow O_\lambda$ when $j \to -\infty$. Let us fix j and the corresponding F_j. Let $j_0 = \min F_j$. If $k \in F_j$, then $k \geqslant j_0$, which implies that $b_k \leqslant b_{j_0}$. Now, if $x \in (b_{k+1}, b_k)$ and $k \in F_j$, we have

$$\exp\left(\frac{1}{b_k}\int_0^{b_k} \log v^{-1}\right) \leqslant G_0(\chi_{(0,b_{j_0})}v^{-1})(x).$$

Then, by definition of C_1,

$$\begin{aligned}
\sum_{k\in F_j} \gamma_k &= \sum_{k\in F_j} \int_{b_{k+1}}^{b_k} \left(\exp\left(\frac{1}{b_k}\int_0^{b_k} \log v^{-1}\right)\right)^{\frac{q}{p}} u(x)dx \\
&\leqslant \sum_{k\in F_j} \int_{b_{k+1}}^{b_k} \left(G_0(\chi_{(0,b_{j_0})}v^{-1})\right)^{\frac{q}{p}} u(x)dx \\
&\leqslant \int_0^{b_{j_0}} \left(G_0(\chi_{(0,b_{j_0})}v^{-1})\right)^{\frac{q}{p}} u(x)dx \leqslant C_1 b_{j_0}^{\frac{q}{p}} \\
&\leqslant C_1 \left(\frac{1}{\lambda}\int_0^{b_{j_0}} h\right)^{\frac{q}{p}} \leqslant \frac{C_1}{\lambda^{\frac{q}{p}}} \left(\int_0^\infty h\right)^{\frac{q}{p}}.
\end{aligned}$$

This proves that T is of weak-type $(1, \frac{q}{p})$ with constant $C_1^{\frac{p}{q}}$, which implies the boundedness of T from $L^p(0,\infty)$ to $l^q(\{\gamma_k\})$ with constant $2\left(\frac{1}{p}\right)^{\frac{q}{p}}(p')^{\frac{1}{q}}C_1^{\frac{1}{q}}$ (see [5], Theorem (6.28), for the behaviour of constants in Marcinkiewicz interpolation theorem). Now, applying this fact in (6), we get

$$\begin{aligned}
\left(\int_0^\infty G_0 f(x)^q u(x)dx\right)^{\frac{1}{q}} &\leqslant 2\left(\sum_{k\in\mathbb{Z}} \left(T(fv^{\frac{1}{p}})(k)\right)^q \gamma_k\right)^{\frac{1}{q}} \\
&\leqslant 4\left(\frac{1}{p}\right)^{\frac{q}{p}} (p')^{\frac{1}{q}} C_1^{\frac{1}{q}} \left(\int_0^\infty f^p v\right)^{\frac{1}{p}},
\end{aligned}$$

which finishes the proof.

3. Proof of Theorem 2

Assume that (4) holds, which is clearly equivalent to

$$\sup_{\lambda>0} \lambda \left(\int_{\{x\in(0,\infty):G_0f(x)>\lambda\}} u(x)dx \right)^{\frac{p}{q}} \leqslant C^p \int_0^\infty f(x)v(x)dx. \tag{7}$$

Let $b > 0$, $0 < \alpha < 1$, $\lambda = \alpha \exp\left(\frac{1}{b}\int_0^b \log v^{-1}\right)$ and $f = v^{-1}\chi_{(0,b)}$. If $x \in (0,b)$, then $G_0f(x) > \lambda$, which shows that $(0,b) \subset \{x : G_0f(x) > \lambda\}$. Then, by (7), we have

$$\alpha \exp\left(\frac{1}{b}\int_0^b \log v^{-1}\right) \left(\int_0^b u\right)^{\frac{p}{q}} \leqslant C^p b.$$

Letting α tend to 1 and taking supremum in $b > 0$, we get $[u,v]_{A_{0,p,q,\exp}} \leqslant C^p$.

Assume now that $[u,v]_{A_{0,p,q,\exp}} < \infty$. Let $f \geqslant 0$ and $\lambda > 0$. We may assume that there exists $b_0 > 0$ such that $f(x) = 0$ for all $x > b_0$. This implies that $G_0f(x) = 0$ for all $x > b_0$. Since G_0f is a nonincreasing function, $O_\lambda = \{x \in (0,\infty) : G_0f(x) > \lambda\} = (0,b_\lambda)$, with $\exp\left(\frac{1}{b_\lambda}\int_0^{b_\lambda} \log f\right) = \lambda$. Then, by definition of $A_{0,p,q,\exp}$ and Jensen's inequality, we have

$$\begin{aligned}
\left(\int_{O_\lambda} u\right)^{\frac{p}{q}} &= \left(\int_0^{b_\lambda} u\right)^{\frac{p}{q}} \\
&= \frac{1}{\lambda}\left(\int_0^{b_\lambda} u\right)^{\frac{p}{q}} \exp\left(\frac{1}{b_\lambda}\int_0^{b_\lambda} \log f\right) \\
&= \frac{1}{\lambda}\left(\int_0^{b_\lambda} u\right)^{\frac{p}{q}} \exp\left(\frac{1}{b_\lambda}\int_0^{b_\lambda} \log (fv)\right) \exp\left(\frac{1}{b_\lambda}\int_0^{b_\lambda} \log v^{-1}\right) \\
&\leqslant \frac{[u,v]_{A_{0,p,q,\exp}}}{\lambda} b_\lambda \exp\left(\frac{1}{b_\lambda}\int_0^{b_\lambda} \log (fv)\right) \\
&\leqslant \frac{[u,v]_{A_{0,p,q,\exp}}}{\lambda} \int_0^{b_\lambda} fv \\
&\leqslant \frac{[u,v]_{A_{0,p,q,\exp}}}{\lambda} \int_0^{\infty} fv,
\end{aligned} \tag{8}$$

which proves (7) with constant $[u,v]_{A_{0,p,q,\exp}}$ or, equivalently, (4) with constant $[u,v]_{A_{0,p,q,\exp}}^{\frac{1}{p}}$.

4. Proof of Theorem 3

We only have to prove $(v) \Rightarrow (iii)$, because $(i) \Leftrightarrow (iii)$, $(ii) \Leftrightarrow (iv)$ and $(iii) \Rightarrow (iv)$ are clear and $(iv) \Rightarrow (v)$ has been proved in theorem 2.

Assume that (v) holds. Let $f \geqslant 0$, $x \in (0,\infty)$ and $b > x$. Then, by (v) and Jensen's inequality

$$\begin{aligned}
\exp\left(\frac{1}{b}\int_0^b \log f\right) &= \exp\left(\frac{1}{b}\int_0^b \log(fw)\right)\exp\left(\frac{1}{b}\int_0^b \log w^{-1}\right)\\
&\leqslant [w]_{A_{0,\exp}} \exp\left(\frac{1}{b}\int_0^b \log(fw)\right)\frac{b}{\int_0^b w} \leqslant [w]_{A_{0,\exp}}\frac{\int_0^b fw}{\int_0^b w} \qquad (9)\\
&\leqslant [w]_{A_{0,\exp}} N_w f(x),
\end{aligned}$$

where N_w is the maximal operator defined by

$$N_w f(x) = \sup_{b>x}\frac{\int_0^b |f|w}{\int_0^b w}.$$

As an immediate consequence of (9), we get that $G_0 f(x) \leqslant [w]_{A_{0,\exp}} N_w f(x)$. Since the operator N_w is bounded in $L^p(w)$ for all $p > 1$ with norm $\frac{p}{p-1}$, we have

$$\begin{aligned}
\int_0^\infty G_0 f(x)^p w(x)dx &\leqslant [w]^p_{A_{0,\exp}} \int_0^\infty N_w f(x)^p w(x)dx\\
&\leqslant [w]^p_{A_{0,\exp}} \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p w(x)dx,
\end{aligned}$$

which is equivalent to

$$\int_0^\infty G_0 f(x) w(x)dx \leqslant [w]^p_{A_{0,\exp}} \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x) w(x)dx. \qquad (10)$$

If $[w]_{A_{0,\exp}} = 1$, then letting p tend to ∞, we get

$$\int_0^\infty G_0 f(x) w(x)dx \leqslant e\int_0^\infty f(x) w(x)dx.$$

If $[w]_{A_{0,\exp}} > 1$, then the function $\varphi(p) = [w]^p_{A_{0,\exp}}\left(\frac{p}{p-1}\right)^p$ has absolute minimum on $(1,\infty)$ and its minimum value is $\varphi(p_0)$, where p_0 is the only real number greater than 1 which is solution of the equation

$$1 + \log[w]_{A_{0,\exp}} + \log\left(\frac{p}{p-1}\right) = \frac{p}{p-1}.$$

Therefore, we have

$$\int_0^\infty G_0 f(x)w(x)dx \leqslant [w]_{A_{0,\exp}}^{p_0}\left(\frac{p_0}{p_0-1}\right)^{p_0}\int_0^\infty f(x)w(x)dx,$$

as we wished to prove.

5. Proof of Theorem 5

It is clear that (i) implies (ii). Let us see first that (iii) implies (i). We note that $[w]_{\tilde{A}_{0,\exp}} = \operatorname{ess\,sup}_{\alpha\in S^{n-1}}[w_\alpha(t)t^{n-1}]_{A_{0,\exp}}$, where $w_\alpha(t) = w(t\alpha)$ and, as we have seen in Theorem 3,

$$[w_\alpha(t)t^{n-1}]_{A_{0,\exp}} = \sup_{b>0}\left(\frac{1}{b}\int_0^b w_\alpha(t)t^{n-1}dt\right)\exp\left(\frac{1}{b}\int_0^b \log(w_\alpha^{-1}(t)t^{1-n})dt\right).$$

Working as in the proof of Theorem 3, we have that

$$\int_0^\infty G_0(f_\alpha)(t)w_\alpha(t)t^{n-1}dt \leqslant [w_\alpha(t)t^{n-1}]_{A_{0,\exp}}^p\left(\frac{p}{p-1}\right)^p\int_0^\infty f_\alpha(t)w_\alpha(t)t^{n-1}dt,$$

for almost every $\alpha\in S^{n-1}$ and all $p>1$ (see (10)). It implies that

$$\int_0^\infty G_0(f_\alpha)(t)w_\alpha(t)t^{n-1}dt \leqslant [w]_{\tilde{A}_{0,\exp}}^p\left(\frac{p}{p-1}\right)^p\int_0^\infty f_\alpha(t)w_\alpha(t)t^{n-1}dt$$

for almost every $\alpha\in S^{n-1}$ and all $p>1$. By integrating on S^{n-1}, we get

$$\int_{\mathbb{R}^n}\mathscr{G}f(x)w(x)dx \leqslant [w]_{\tilde{A}_{0,\exp}}^p\left(\frac{p}{p-1}\right)^p\int_{\mathbb{R}^n}f(x)w(x)dx$$

for all $p>1$, which implies

$$\int_{\mathbb{R}^n}\mathscr{G}f(x)w(x)dx \leqslant [w]_{\tilde{A}_{0,\exp}}^{p_0}\left(\frac{p_0}{p_0-1}\right)^{p_0}\int_{\mathbb{R}^n}f(x)w(x)dx,$$

where p_0 is the absolute minimum of the function $\varphi(p) = [w]_{\tilde{A}_{0,\exp}}^p\left(\frac{p}{p-1}\right)^p$ if $[w]_{\tilde{A}_{0,\exp}} > 1$, and

$$\int_{\mathbb{R}^n}\mathscr{G}f(x)w(x)dx \leqslant e\int_{\mathbb{R}^n}f(x)w(x)dx$$

if $[w]_{\tilde{A}_{0,\exp}} = 1$.

Finally, let us see that (ii) implies (iii). Then, assume that

$$\int_{\{x\in\mathbb{R}^n:\mathscr{G}f(x)>\lambda\}} w \leqslant \frac{C_1}{\lambda}\int_{\mathbb{R}^n} fw,$$

which is equivalent to the following inequality, by changing into polar coordinates:

$$\int_{S^{n-1}}\int_0^\infty \chi_{\{\alpha t\in\mathbb{R}^n:\mathscr{G}f(\alpha t)>\lambda\}}(\alpha t)w(\alpha t)t^{n-1}dtd\alpha \leqslant \frac{C_1}{\lambda}\int_{S^{n-1}}\int_0^\infty f(\alpha t)w(\alpha t)t^{n-1}dtd\alpha.$$

By a simple change of variables, it is easy to see that $\mathscr{G}f(\alpha t)=G_0(f_\alpha)(t)$. Then, the last inequality can be written as

$$\int_{S^{n-1}}\int_0^\infty \chi_{\{\alpha t\in\mathbb{R}^n:G_0(f_\alpha)(t)>\lambda\}}(\alpha t)w(\alpha t)t^{n-1}dtd\alpha \leqslant \frac{C_1}{\lambda}\int_{S^{n-1}}\int_0^\infty f_\alpha(t)w(\alpha t)t^{n-1}dtd\alpha.$$

For a fixed $\alpha\in S^{n-1}$, αt verifies $G_0(f_\alpha)(t)>\lambda$ if and only if t verifies $G_0(f_\alpha)(t) >\lambda$. Then, we get

$$\int_{S^{n-1}}\int_{\{t\in(0,\infty):G_0(f_\alpha)(t)>\lambda\}} w_\alpha(t)t^{n-1}dtd\alpha \leqslant \frac{C_1}{\lambda}\int_{S^{n-1}}\int_0^\infty f_\alpha(t)w_\alpha(t)t^{n-1}dtd\alpha. \tag{11}$$

Let $A\subset S^{n-1}$ with positive measure and let $f_\alpha(t)=\chi_A(\alpha)h(t)$. Thus, we have that

$$G_0(f_\alpha)(t)=\sup_{b\geqslant t}\exp\left(\frac{1}{b}\int_0^b \log(\chi_A(\alpha)h(s))ds\right)=G_0h(t)$$

for all $\alpha\in A$. Then, (11) implies that

$$\int_A\int_{\{t\in(0,\infty):G_0h(t)>\lambda\}} w_\alpha(t)t^{n-1}dtd\alpha \leqslant \frac{C_1}{\lambda}\int_A\int_0^\infty h(t)w_\alpha(t)t^{n-1}dtd\alpha.$$

By differentiation, the inequality above implies that

$$\int_{\{t\in(0,\infty):G_0h(t)>\lambda\}} w_\alpha(t)t^{n-1}dt \leqslant \frac{C_1}{\lambda}\int_0^\infty h(t)w_\alpha(t)t^{n-1}dt, \tag{12}$$

for almost every $\alpha\in S^{n-1}$, where the constant C_1 is independent of α and h. Applying Theorem 2, (12) implies that

$$[w_\alpha(t)t^{n-1}]_{A_{0,\exp}}\equiv\sup_{b>0}\left(\frac{1}{b}\int_0^b w_\alpha(t)t^{n-1}dt\right)\exp\left(\frac{1}{b}\int_0^b \log(w_\alpha^{-1}(t)t^{1-n})dt\right)\leqslant C_1,$$

for almost every $\alpha\in S^{n-1}$, and this gives (iii) and also $[w]_{\tilde{A}_{0,\exp}}\leqslant C_1$.

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