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MONOTONIC SEQUENCES AND INEQUALITIES INVOLVING THE RATIO BETWEEN TWO ADJACENT NONZERO BERNOULLI NUMBERS

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Abstract. In the work, the authors concisely review monotonicity results of several sequences involving the ratio between two adjacent nonzero Bernoulli numbers, establish new monotonicity results of several sequences involving the ratio between two adjacent Bernoulli numbers, derive several inequalities for the ratio of two adjacent nonzero Bernoulli numbers, recover the first double inequality for the ratio of two adjacent nonzero Bernoulli numbers, and discover that the function $\left(2^{t-1}-1\right)\zeta(t)$ is logarithmically concave in $t\in(0,\infty)$, where $\zeta(t)$ denoted the Riemann zeta function.

1. A concise review

The Bernoulli numbers B_j are generated by the Maclaurin power series expansion

$$\frac{x}{\mathrm{e}^x-1}=\sum_{j=0}^{\infty}B_j\frac{x^j}{j!}=1-\frac{x}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!},\quad 0<|x|<2\pi;$$

see [43, p. 3] and [55], for example.

We now chronologically review the double inequalities for the Bernoulli number B_{2j} and their ratio $\frac{B_{2j+2}}{B_{2j}}$ and concisely survey some recent monotonicity results for several sequences involving the ratio $\frac{B_{2j+2}}{B_{2j}}$.

There have been many inequalities for bounding the Bernoulli number B_{2j} in the literature. For detailed information, please refer to the short reviews in [3, Section 3], [5, Section 1], [36, Section 1], and [41, Section 1].

In 1994 and 2000, D'Aniello [19] and Alzer [3] established the following double and sharp inequalities.

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THEOREM 1. ([3, 19]) *The double inequality*

$$\frac{2(2j)!}{(2\pi)^{2j}}\frac{1}{1-2^{\alpha-2j}} \leqslant |B_{2j}| \leqslant \frac{2(2j)!}{(2\pi)^{2j}}\frac{1}{1-2^{\beta-2j}}, \quad j\in\mathbb{N} \tag{1}$$

is valid, where $\alpha=0$ *and*

$$\beta = 2+\log_2\left(1-\frac{6}{\pi^2}\right) = 0.649\ldots$$

are the best possible in the sense that they can not be replaced respectively by any bigger and smaller constants in the double inequality (1).

In 2023, Bagul [5] refined the double inequality (1) as follows.

THEOREM 2. ([5, Theorem 2.9]) *The best possible constants* α *and* β *satisfying the double inequality*

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\alpha)} < |B_{2k}| < \frac{2(2k)!}{\pi^{2k}(2^{2k}-1)}\frac{3^{2k}}{(3^{2k}-\beta)}, \quad k\in\mathbb{N}$$

are 1 *and* $9\left(1-\frac{8}{\pi^2}\right)=1.704875\ldots$*, respectively.*

To the best of our knowledge, the first double inequality for bounding the ratio $\frac{B_{2j+2}}{B_{2j}}$ was discovered by Qi [36] in 2019.

THEOREM 3. ([36, Theorem 1.1]) *The double inequality*

$$\frac{2(2j+1)(j+1)}{\pi^2}\frac{2^{2j-1}-1}{2^{2j+1}-1} < \left|\frac{B_{2j+2}}{B_{2j}}\right| < \frac{2(2j+1)(j+1)}{\pi^2}\frac{2^{2j}-1}{2^{2j+2}-1} \tag{2}$$

is sound for $j\in\mathbb{N}$.

Hereafter, the double inequality (2) was rapidly discussed, generalized, improved, sharpened, applied, and recovered in the papers [37, 48, 49, 63] and a lot of works such as [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 34, 38, 39, 40, 42, 44, 46, 47, 50, 51, 56, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76]. Especially, Yang–Tian [49] and Zhu [63] found the following refined and sharp form of (2).

THEOREM 4. ([49, Corollary 2] and [63, Theorem 1]) *For* $j,k\in\mathbb{N}$*, the double inequality*

$$\frac{1}{\pi^{2k}}\frac{2^{2j+\beta}-1}{2^{2(j+k)+\beta}-1}\frac{(2j+2k)!}{(2j)!} \leqslant \frac{|B_{2j+2k}|}{|B_{2j}|} < \frac{1}{\pi^{2k}}\frac{2^{2j+\alpha}-1}{2^{2(j+k)+\alpha}-1}\frac{(2j+2k)!}{(2j)!}$$

holds if and only if $\alpha\geqslant 0$ *and*

$$-2<\beta\leqslant\log_2\left(\frac{1}{4}\frac{(2k+2)!-12\pi^{2k}|B_{2k+2}|}{(2k+2)!-12(2\pi)^{2k}|B_{2k+2}|}\right).$$

In particular, the double inequality

$$\frac{2(j+1)(2j+1)}{\pi^2}\frac{2^{2j+\lambda}-1}{2^{2j+2+\lambda}-1} \leqslant \left|\frac{B_{2j+2}}{B_{2j}}\right| < \frac{2(j+1)(2j+1)}{\pi^2}\frac{2^{2j+\mu}-1}{2^{2j+2+\mu}-1} \tag{3}$$

is valid for $j \in \mathbb{N}$ *if only if* $\mu \geqslant 0$ *and*

$$\lambda \leqslant \log_2\left(\frac{1}{16}\frac{60-\pi^2}{15-\pi^2}\right) = -0.711\dots. \tag{4}$$

In 2021, the increasing property of two sequences involving the ratio $\frac{B_{2j+2}}{B_{2j}}$ was established by Qi and his coauthor as follows.

THEOREM 5. ([41, Theorems 1.1 and 1.2]) *The sequences* $\left|\frac{B_{2j+2}}{B_{2j}}\right|$ *for* $j \in \mathbb{N}_0$ *and*

$$\frac{(2j+\ell+2)(2j+\ell+1)}{(j+1)(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|, \quad j,\ell \in \mathbb{N} \tag{5}$$

are both increasing in j.

REMARK 1. The sequence (5) in Theorem 5 can be rewritten as

$$\begin{aligned}\frac{(2j+\ell+2)(2j+\ell+1)}{(j+1)(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| &= \frac{1}{(2j+1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| \\ &\quad\times \frac{(2j+\ell+2)(2j+\ell+1)}{(j+1)(2j+1)}\frac{(2j+1)(j+1)(2^{2j}-1)}{2^{2j+2}-1} \\ &\to \begin{cases}\infty, & j\to\infty \\ \dfrac{(\ell+4)(\ell+3)}{30}, & j=1\end{cases}\end{aligned}$$

for $j,\ell \in \mathbb{N}$. Accordingly, we obtain

$$\begin{aligned}\left|\frac{B_{2j+2}}{B_{2j}}\right| &\geqslant \frac{(\ell+4)(\ell+3)}{30}\frac{(j+1)(2j+1)}{(2j+\ell+2)(2j+\ell+1)} \\ &\to \begin{cases}\dfrac{2j+1}{3(2j+3)}, & \ell=1 \\ \dfrac{(j+1)(2j+1)}{30}, & \ell\to\infty\end{cases}\end{aligned}$$

for $j,\ell \in \mathbb{N}$. As a result, we deduce a simple inequality

$$\left|\frac{B_{2j+2}}{B_{2j}}\right| \geqslant \frac{(j+1)(2j+1)}{30}, \quad j \in \mathbb{N}.$$

The lower bound in this inequality is smaller than the corresponding one in the double inequality (2) for all $j \in \mathbb{N}$ and is smaller than the corresponding one in the one-sided inequality (8) for $j \geqslant 3$.

Since 2023, when studying some properties of normalized remainders of the Maclaurin power series expansions for the tangent function and its square [33, 54], for a function with relation to normalized remainder of the cosine function [35], and for a trigonometric function originating from an integral representation of the reciprocal of the gamma function [53], the authors presented the following monotonicity results.

THEOREM 6. ([54, Lemma 1]) *Let* $\zeta(t)$ *stands for the Riemann zeta function. The function* $(2^t-1)\zeta(t)$ *is logarithmically convex in* $t\in(1,\infty)$. *Consequently, the sequence*

$$\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{6}$$

is increasing in $j\in\mathbb{N}$ *and tends to* $\frac{2}{\pi^2}$ *as* $j\to\infty$.

REMARK 2. The sequence in (6) increases in $j\in\mathbb{N}$ very slowly, slower than the increasing speeds of two sequences in Theorem 5. From Theorem 6, we can derive the inequality on the right-hand side in the double inequality (2). The limit in Theorem 6 implies that the upper bound in the double inequality (2) is sharp in the sense that the constant $\frac{2}{\pi^2}$ can not be replaced by any smaller number. This recovers the corresponding sharpness Theorem 4.

REMARK 3. Since the sequence (6) can be rearranged as

$$\frac{2^{2j+2}-1}{2^{2j}-1}\frac{1}{(2j+\ell+2)(2j+\ell+1)}\frac{(2j+\ell+2)(2j+\ell+1)}{(j+1)(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{2^{2j+2}-1}{2^{2j}-1}\frac{1}{(2j+\ell+2)(2j+\ell+1)}$ is decreasing in $j\in\mathbb{N}$ for given $\ell\in\mathbb{N}$, we see that Theorem 6 implies Theorem 5 and that the sequence in (5) increases more rapidly than the sequence in (6) does.

THEOREM 7. ([33, Lemma 1]) *The function* $(t-1)(2^t-1)\zeta(t)$ *is logarithmically concave in* $t\in(0,\infty)$. *Consequently, the sequence*

$$\frac{1}{(2j-1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{7}$$

is decreasing in $j\in\mathbb{N}$.

REMARK 4. Since the sequence in (7) can be rearranged as

$$\frac{2j+1}{2j-1}\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|,$$

employing the limit in Theorem 6, we obtain that, as $j\to\infty$, the sequence in (7) tends to $\frac{2}{\pi^2}$. Accordingly, Theorem 7 implies the inequality

$$\left|\frac{B_{2j+2}}{B_{2j}}\right|>\frac{2(2j-1)(j+1)}{\pi^2}\frac{2^{2j}-1}{2^{2j+2}-1},\quad j\in\mathbb{N}. \tag{8}$$

It is not difficult to verify that the lower bound in the inequality (8) is smaller than the left-hand side in the double inequality (2) in Theorem 3.

THEOREM 8. ([35, Lemma 2]) *The sequence*

$$\frac{j}{(j+1)^2(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{9}$$

is increasing in $j\in\mathbb{N}_0$.

THEOREM 9. ([53, Lemma 3]) *The sequence*

$$\frac{j(2j+3)}{(j+1)^2(2j+1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{10}$$

is increasing in $j\in\mathbb{N}_0$.

REMARK 5. Because the sequence in (10) can be reformulated as

$$\frac{2j+3}{2j+1}\frac{j}{(j+1)^2(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{2j+3}{2j+1}$ is decreasing in $j\in\mathbb{N}$, we see that the sequence in (10) increases in $j\in\mathbb{N}$ slower than the sequence in (9) does.

REMARK 6. Since the sequence (9) can be rewritten as

$$\frac{j}{j+1}\frac{2^{2j}-1}{2^{2j+2}-1}\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|\to\frac{1}{2\pi^2},\quad j\to\infty, \tag{11}$$

where we used the limit in Theorem 6, from Theorem 8, we acquire

$$\left|\frac{B_{2j+2}}{B_{2j}}\right|<\frac{1}{2\pi^2}\frac{(j+1)^2(2j+1)}{j},\quad j\in\mathbb{N}. \tag{12}$$

REMARK 7. Since the sequence (10) can be reformulated as

$$\frac{j(2j+3)}{(2j+1)(j+1)}\frac{2^{2j}-1}{2^{2j+2}-1}\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2}-1}{2^{2j}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|\to\frac{1}{2\pi^2} \tag{13}$$

as $j\to\infty$, where we used the limit in Theorem 6, considering Theorem 9, we gain

$$\left|\frac{B_{2j+2}}{B_{2j}}\right|<\frac{1}{2\pi^2}\frac{(j+1)^2(2j+1)^2}{j(2j+3)},\quad j\in\mathbb{N}. \tag{14}$$

It is easy to see that the upper bound in (14) is smaller than the corresponding one in (12), but it is bigger than the upper bound of the double inequality (2).

REMARK 8. Since the sequences

$$\frac{j}{j+1}\frac{2^{2j}-1}{2^{2j+2}-1} \quad \text{and} \quad \frac{j(2j+3)}{(2j+1)(j+1)}\frac{2^{2j}-1}{2^{2j+2}-1}$$

are both increasing in $j \in \mathbb{N}$, using the relations in (11) and (13), the sequence in (6) increases in $j \in \mathbb{N}$ slower than the sequences in (9) and (10) do.

REMARK 9. Due to

$$\frac{j}{(j+1)^2(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{j}{(j+1)(2j+\ell+2)(2j+\ell+1)}\frac{(2j+\ell+2)(2j+\ell+1)}{(j+1)(2j+1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{j}{(j+1)(2j+\ell+2)(2j+\ell+1)}$ for given $\ell \in \mathbb{N}$ is decreasing in $j \in \mathbb{N}$, the sequence in (5) for given $\ell \in \mathbb{N}$ increases in $j \in \mathbb{N}$ quicker than the sequence in (9) does.

2. Several new decreasing sequences involving the ratio $\frac{B_{2j+2}}{B_{2j}}$

Influenced by the first double inequality (2) for bounding the ratio $\frac{B_{2j+2}}{B_{2j}}$ in Theorem 3, considering Theorem 7 and Remark 4, we pose a problem: Is the sequence

$$\frac{1}{(2j+1)(j+1)}\frac{2^{2j+1}-1}{2^{2j-1}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{15}$$

decreasing in $j \in \mathbb{N}$? The solution of this problem and Theorem 6 can be used to recover the double inequality (2).

Motivated by Theorems 6 to 8, we guess that the sequences

$$\frac{j}{(j+1)^2(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|, \quad \frac{j(2j+3)}{(j+1)^2(2j-1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right|, \tag{16}$$

and

$$\frac{j(2j+3)}{(j+1)^2(2j+1)(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{17}$$

are possibly decreasing in $j \in \mathbb{N}_0$. If this guess were true, due to

$$\frac{j(2j+3)}{(j+1)^2(2j+1)(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{2j-1}{2j+1}\frac{j(2j+3)}{(j+1)^2(2j-1)^2}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{2j-1}{2j+1}$ is increasing in $j \in \mathbb{N}$, the sequence in (17) would decrease slower in $j \in \mathbb{N}$ than the second sequence in (16) would. Hence, it suffices to show that the sequence in (17) is decreasing in $j \in \mathbb{N}$.

Furthermore, since

$$\frac{j}{(j+1)^2(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{2j+1}{2j+3}\frac{j(2j+3)}{(j+1)^2(2j+1)(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{2j+1}{2j+3}$ is increasing in $j \in \mathbb{N}$, in order to verify this guess, it is sufficient to prove that the first sequence in (16) is decreasing in $j \in \mathbb{N}$.

Because the sequence in (15) can be rearranged as

$$\frac{(j+1)(2j-1)}{j(2j+1)}\frac{2^{2j+1}-1}{2^{2j-1}-1}\frac{j}{(j+1)^2(2j-1)}\left|\frac{B_{2j+2}}{B_{2j}}\right|$$

and the sequence $\frac{(j+1)(2j-1)}{j(2j+1)}\frac{2^{2j+1}-1}{2^{2j-1}-1}$ is increasing in $j \geqslant 3$ and is decreasing in $j \in \{1,2,3\}$, in order to confirm that the sequence in (16) and (17) were decreasing in $j \in \mathbb{N}$, it is sufficient to show that the sequence in (15) is decreasing in $j \in \mathbb{N}$.

THEOREM 10. *The function* $\left(2^{t-1}-1\right)\zeta(t)$ *is logarithmically concave in* $t \in (0,\infty)$. *Consequently, the sequence in* (15) *is decreasing in* $j \in \mathbb{N}$.

Proof. In [43, p. 5, (1.14)], we find that

$$B_{2j} = (-1)^{j+1}\frac{2(2j)!}{(2\pi)^{2j}}\zeta(2j), \quad j \in \mathbb{N}. \tag{18}$$

By the relation (18), we gain

$$\frac{1}{(2j+1)(j+1)}\frac{2^{2j+1}-1}{2^{2j-1}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| = \frac{1}{2\pi^2}\frac{\left(2^{2j+1}-1\right)\zeta(2j+2)}{(2^{2j-1}-1)\zeta(2j)} = \frac{1}{2\pi^2}\frac{Z(2j+2)}{Z(2j)},$$

where

$$Z(t) = \left(2^{t-1}-1\right)\zeta(t) = 2^{t-1}\eta(t), \quad t > 0 \tag{19}$$

and $\eta(t)$ denotes the Dirichlet eta function; see the papers [29, 31, 52].

In 1998, Wang [45] obtained that the Dirichlet eta function $\eta(t)$ is logarithmically concave in $t \in (0,\infty)$. In 2015, Adell–Lekuona [1, Theorem 1.1] and Alzer–Kwong [4, Theorem 3.1] obtained a concavity of the Dirichlet eta function $\eta(t)$ in $t \in (0,\infty)$. This obviously implies that the function $Z(t)$ is logarithmically concave in $t \in (0,\infty)$. As a result, for $\alpha, t \in (0,\infty)$, we obtain

$$\left[\frac{Z(t+\alpha)}{Z(t)}\right]' = \frac{Z(t+\alpha)}{Z(t)}\left[\frac{Z'(t+\alpha)}{Z(t+\alpha)} - \frac{Z'(t)}{Z(t)}\right] < 0.$$

Accordingly, the ratio $\frac{Z(t+\alpha)}{Z(t)}$ for given $\alpha > 0$ is decreasing in $t \in (0,\infty)$. Therefore, the sequence $\frac{Z(2j+2)}{Z(2j)}$ is decreasing in $j \in \mathbb{N}$. Consequently, the sequence in (15) is decreasing in $j \in \mathbb{N}$. The proof of Theorem 10 is complete. □

3. More remarks

In this section, we give several more remarks on our main results and their proofs.

REMARK 10. Motivated by the lower bound in the double inequality (3), we pose a problem: Is the sequence

$$\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2+c}-1}{2^{2j+c}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right| \tag{20}$$

decreasing in $j\in\mathbb{N}$? where $c=\log_2\left(\frac{1}{16}\frac{60-\pi^2}{15-\pi^2}\right)=-0.711\dots$ is the negative scalar given by (4). A direct differentiation builds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2^{2t+2+\theta}-1}{2^{2t+\theta}-1}\frac{2^{2t-1}-1}{2^{2t+1}-1}\right)=\frac{3\left(2^{\theta+1}-1\right)\left(2^{\theta+4t+1}-1\right)4^t\ln 2}{(2^{2t+1}-1)^2(2^{\theta+2t}-1)^2}>0$$

if and only if $\theta>-1$ and $t\in\left[\frac{1}{2},\infty\right)$. From the relation

$$\begin{aligned}\frac{1}{(2j+1)(j+1)}&\frac{2^{2j+2+c}-1}{2^{2j+c}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|\\&=\frac{2^{2j+2+c}-1}{2^{2j+c}-1}\frac{2^{2j-1}-1}{2^{2j+1}-1}\frac{1}{(2j+1)(j+1)}\frac{2^{2j+1}-1}{2^{2j-1}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|\end{aligned}$$

and by virtue of the fact that the sequence $\frac{2^{2j+2+c}-1}{2^{2j+c}-1}\frac{2^{2j-1}-1}{2^{2j+1}-1}$ is increasing in $j\in\mathbb{N}$, we see that, in order to prove that the sequence in (15) is decreasing in $j\in\mathbb{N}$, it is enough to show that the sequence in (20) is decreasing in $j\in\mathbb{N}$.

In the proof of [41, Theorem 1.1], we find the relation

$$\left|\frac{B_{2n+2}}{B_{2n}}\right|=\frac{1}{2\pi^2}\left[(2n+1)(n+1)\frac{\zeta(2n+2)}{\zeta(2n)}\right],\quad n\in\mathbb{N}.$$

Hence, the sequence in (20) can be transformed as

$$\begin{aligned}\frac{1}{(2j+1)(j+1)}\frac{2^{2j+2+c}-1}{2^{2j+c}-1}\left|\frac{B_{2j+2}}{B_{2j}}\right|&=\frac{1}{2\pi^2}\frac{\left(2^{2j+2+c}-1\right)\zeta(2j+2)}{(2^{2j+c}-1)\zeta(2j)}\\&=\frac{1}{2\pi^2}\frac{Y(2j+2)}{Y(2j)},\quad j\in\mathbb{N},\end{aligned}$$

where

$$Y(t)=\left(2^{t+c}-1\right)\zeta(t),\quad t\geqslant 1.$$

As done in the proof of Theorem 10, it suffices to show that the function $Y(t)$ is logarithmically concave in $t\in[2,\infty)$.

In fact, the function $Y(t)$ is possibly logarithmically concave in $t\in(t_0,\infty)$ for $t_0\in(3,4)$. Consequently, the sequence in (20) is possibly decreasing in $j\geqslant 2$. See also the question posed at the website `https://math.stackexchange.com/q/4981396/` (accessed on 8 October 2024).

REMARK 11. We guess that the function $Z(t)$ defined by (19) is convex in $t \in (0,\infty)$.

We guess that the function $Y(t)$ is concave in $t \in (0,1)$ and is convex in $t \in (1,\infty)$. See also the question posed at the website https://math.stackexchange.com/q/4981396/ (accessed on 8 October 2024).

What is the largest range for the constant ϑ such that the function

$$F_\vartheta(t) = \left(2^{t+\vartheta} - 1\right)\zeta(t)$$

is logarithmically concave in $t \in (0,\infty)$ or in $t \in (1,\infty)$? What is the largest range for the constant ϑ such that the function $F_\vartheta(t)$ is convex in $t \in (0,\infty)$ or in $t \in (1,\infty)$? What is the largest range for the constant ϑ such that the function $F_\vartheta(t)$ is concave in $t \in (0,1)$? For given ϑ, what is the smallest number t_ϑ such that the function $F_\vartheta(t)$ is logarithmically concave in $t \in (t_\vartheta,\infty)$?

REMARK 12. By the definition of the Riemann zeta function $\zeta(z)$ for $\Re(z) > 1$, we acquire

$$\begin{aligned}
Z(t) &= 2^{t-1}\sum_{k=1}^{\infty}\frac{1}{k^t} - \sum_{k=1}^{\infty}\frac{1}{k^t} \\
&= 2^{t-1}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^t} + 2^{t-1}\sum_{k=1}^{\infty}\frac{1}{(2k)^t} - \sum_{k=1}^{\infty}\frac{1}{k^t} \\
&= \frac{1}{2}\sum_{k=1}^{\infty}\left[\frac{1}{(k-1/2)^t} - \frac{1}{k^t}\right] \\
&= \frac{t}{2}\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)} u^{t-1}\,\mathrm{d}u \\
&= \frac{1}{2}\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)} \frac{\mathrm{d}u^t}{\mathrm{d}u}\,\mathrm{d}u,
\end{aligned}$$

$$[\ln Z(t)]' = \frac{\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u^t}{\mathrm{d}u}\right)\mathrm{d}u}{\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}u^t}{\mathrm{d}u}\,\mathrm{d}u},$$

and

$$[\ln Z(t)]'' = \frac{\left(\begin{gathered}\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}u^t}{\mathrm{d}u}\right)\mathrm{d}u\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}u^t}{\mathrm{d}u}\,\mathrm{d}u \\ -\left[\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u^t}{\mathrm{d}u}\right)\mathrm{d}u\right]^2\end{gathered}\right)}{\left[\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}u^t}{\mathrm{d}u}\,\mathrm{d}u\right]^2}.$$

Consequently, using the logarithmic concavity of the function $Z(t)$ in Theorem 10, we acquire

$$\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\biggl(\frac{\mathrm{d}u^t}{\mathrm{d}u}\biggr)\,\mathrm{d}u\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}u^t}{\mathrm{d}u}\,\mathrm{d}u - \Biggl[\sum_{k=1}^{\infty}\int_{1/k}^{1/(k-1/2)}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{\mathrm{d}u^t}{\mathrm{d}u}\biggr)\,\mathrm{d}u\Biggr]^2<0$$

for $t\in(1,\infty)$. This inequality can be reformulated as

$$\sum_{k=1}^{\infty}\biggl[\frac{\ln^2(k-1/2)}{(k-1/2)^t}-\frac{\ln^2k}{k^t}\biggr]\sum_{k=1}^{\infty}\biggl[\frac{1}{(k-1/2)^t}-\frac{1}{k^t}\biggr] - \Biggl[\sum_{k=1}^{\infty}\biggl(\frac{\ln(k-1/2)}{(k-1/2)^t}-\frac{\ln k}{k^t}\biggr)\Biggr]^2<0$$

for $t\in(1,\infty)$.

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