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ESTIMATES FOR GENERALIZED FRACTIONAL INTEGRALS ASSOCIATED WITH OPERATORS ON MORREY–CAMPANATO SPACES

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Dedicated to the memory of Li Xue

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Abstract. Let $\mathscr{L}$ be the infinitesimal generator of an analytic semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ satisfying the Gaussian upper bounds. For given $0<\alpha<n$, let $\mathscr{L}^{-\alpha/2}$ be the generalized fractional integral associated with $\mathscr{L}$, which is defined as

$$\mathscr{L}^{-\alpha/2}(f)(x) := \frac{1}{\Gamma(\alpha/2)}\int_0^{+\infty} e^{-t\mathscr{L}}(f)(x)t^{\alpha/2-1}\,dt,$$

where $\Gamma(\cdot)$ is the usual gamma function. For a locally integrable function $b(x)$ defined on $\mathbb{R}^n$, the related commutator operator $\big[b,\mathscr{L}^{-\alpha/2}\big]$ generated by b and $\mathscr{L}^{-\alpha/2}$ is defined by

$$\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x) := b(x)\cdot\mathscr{L}^{-\alpha/2}(f)(x) - \mathscr{L}^{-\alpha/2}(bf)(x).$$

A new class of Morrey–Campanato spaces associated with $\mathscr{L}$ is introduced in this paper. The authors establish some new estimates for the commutators $\big[b,\mathscr{L}^{-\alpha/2}\big]$ on Morrey–Campanato spaces. The corresponding results for higher-order commutators $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ ($m\in\mathbb{N}$) are also discussed.

1. Introduction and preliminaries

Let $\mathscr{L}$ be the infinitesimal generator of an analytic semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ on $L^2(\mathbb{R}^n)$ with Gaussian upper bounds on its heat kernel, and suppose that $\mathscr{L}$ has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space endowed with the Lebesgue measure dx and the Euclidean norm $|\cdot|$. Then $\mathscr{L}$ is the linear operator which generates an analytic semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ with kernel $\mathscr{P}_t(x,y)$ satisfying

$$e^{-t\mathscr{L}}(f)(x) := \int_{\mathbb{R}^n}\mathscr{P}_t(x,y)f(y)\,dy, \quad t>0,$$

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and there exist two positive constants C and A such that for all $x,y\in\mathbb{R}^n$ and all $t>0$, we have

$$\big|\mathscr{P}_t(x,y)\big| \leqslant \frac{C}{t^{n/2}}\cdot e^{-A\frac{|x-y|^2}{t}}. \tag{1.1}$$

For any $0<\alpha<n$, the generalized fractional integral $\mathscr{L}^{-\alpha/2}$ associated with the operator $\mathscr{L}$ is defined by

$$\mathscr{L}^{-\alpha/2}(f)(x) := \frac{1}{\Gamma(\alpha/2)}\int_0^{+\infty} e^{-t\mathscr{L}}(f)(x)t^{\alpha/2-1}dt. \tag{1.2}$$

Let Δ be the Laplacian operator on $\mathbb{R}^n$, that is,

$$\Delta := \frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}.$$

Note that if $\mathscr{L}=-\Delta$ is the Laplacian operator on $\mathbb{R}^n$, then $\mathscr{L}^{-\alpha/2}$ is exactly the classical fractional integral operator I_α of order $\alpha\,(0<\alpha<n)$, which is given by

$$I_\alpha(f)(x) := \frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad x\in\mathbb{R}^n,$$

where $\gamma(\alpha) := \frac{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}$ and $\Gamma(\cdot)$ being the usual gamma function. The problem of fractional-order derivatives was an early motivation to study fractional integrals. Let $(-\Delta)^{\alpha/2}$ (under $0<\alpha<n$) denote the $\alpha/2$-th order Laplacian operator. Then $u=I_\alpha f$ is viewed as a solution of the $\alpha/2$-th order Laplace equation

$$(-\Delta)^{\alpha/2}u=f$$

in the sense of the Fourier transform, i.e., $(-\Delta)^{\alpha/2}$ exists as the inverse of I_α. It is well known that the classical fractional integral operator I_α of order α plays an important role in harmonic analysis, potential theory and PDEs, particularly in the study of smoothness properties of functions. Let $0<\alpha<n$ and $1<p<q<\infty$. The classical Hardy–Littlewood–Sobolev theorem states that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $1/q=1/p-\alpha/n$. Since the semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ has a kernel $\mathscr{P}_t(x,y)$ which satisfies the Gaussian upper bound (1.1), it is easy to check that for all $x\in\mathbb{R}^n$,

$$\big|\mathscr{L}^{-\alpha/2}(f)(x)\big| \leqslant C\cdot I_\alpha(|f|)(x). \tag{1.3}$$

In fact, if we denote the kernel of $\mathscr{L}^{-\alpha/2}$ by $\mathscr{K}_\alpha(x,y)$, then it follows immediately from (1.2) and Fubini's theorem that (see [10] and [19])

$$\mathscr{K}_\alpha(x,y) = \frac{1}{\Gamma(\alpha/2)}\int_0^{+\infty}\mathscr{P}_t(x,y)t^{\alpha/2-1}dt, \tag{1.4}$$

where $\mathscr{P}_t(x,y)$ is the kernel of $e^{-t\mathscr{L}}$. Thus, by using the Gaussian upper bound (1.1) and the expression (1.4), we can deduce that

$$\begin{aligned}\big|\mathscr{K}_\alpha(x,y)\big| &\leqslant \frac{1}{\Gamma(\alpha/2)}\int_0^{+\infty}\big|\mathscr{P}_t(x,y)\big|t^{\alpha/2-1}dt\\ &\leqslant C\cdot\int_0^{+\infty}e^{-A\frac{|x-y|^2}{t}}\cdot t^{\alpha/2-n/2-1}dt\\ &\leqslant C\cdot\frac{1}{|x-y|^{n-\alpha}}\int_0^{+\infty}e^{-v}\cdot v^{n/2-\alpha/2-1}dv\\ &\leqslant C\cdot\frac{1}{|x-y|^{n-\alpha}}.\end{aligned} \tag{1.5}$$

This proves (1.3) with $C>0$ independent of f (see [10] and [19]). For $x_0\in\mathbb{R}^n$ and $r>0$, let $B(x_0,r)=\{x\in\mathbb{R}^n:|x-x_0|<r\}$ denote the open ball centered at x_0 of radius r, $B(x_0,r)^{\complement}$ denote its complement and $m(B(x_0,r))$ be the Lebesgue measure of the ball $B(x_0,r)$. Recall that, for any given $1\leqslant p<\infty$, the space $L^p(\mathbb{R}^n)$ is defined as the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p}:=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p}<+\infty.$$

Let $L^\infty(\mathbb{R}^n)$ denote the Banach space of all essentially bounded measurable functions f on $\mathbb{R}^n$. The norm of $f\in L^\infty(\mathbb{R}^n)$ is given by

$$\|f\|_{L^\infty}:=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}|f(x)|<+\infty.$$

A locally integrable function f is said to belong to the space $\mathrm{BMO}(\mathbb{R}^n)$ (bounded mean oscillation space, see [15]), if

$$\|f\|_{\mathrm{BMO}}:=\sup_{\mathscr{B}\subset\mathbb{R}^n}\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}|f(x)-f_{\mathscr{B}}|\,dx<+\infty,$$

where $f_{\mathscr{B}}$ denotes the mean value of f on the ball $\mathscr{B}$, i.e.,

$$f_{\mathscr{B}}:=\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}f(y)\,dy$$

and the supremum is taken over all balls $\mathscr{B}$ in $\mathbb{R}^n$. Modulo constants, the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to the BMO norm $\|\cdot\|_{\mathrm{BMO}}$.

Let $b(x)$ be a locally integrable function on $\mathbb{R}^n$ and $0<\alpha<n$. Then the commutator operator generated by b and $\mathscr{L}^{-\alpha/2}$ is defined by

$$\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x):=b(x)\cdot\mathscr{L}^{-\alpha/2}(f)(x)-\mathscr{L}^{-\alpha/2}(bf)(x). \tag{1.6}$$

The function b is called *the symbol function* of $\big[b,\mathscr{L}^{-\alpha/2}\big]$. The commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]$ was first introduced and studied by Duong and Yan in [10]. When $\mathscr{L}=-\Delta$ is the

Laplacian operator on $\mathbb{R}^n$, the commutator $[b,\mathscr{L}^{-\alpha/2}]=[b,I_\alpha]$ generated by b and I_α was first defined by Chanillo in [6]. Then for all $0<\alpha<n$,

$$[b,I_\alpha](f)(x)=\frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}}f(y)\,dy,$$

and

$$[b,\mathscr{L}^{-\alpha/2}](f)(x)=\int_{\mathbb{R}^n}[b(x)-b(y)]\mathscr{K}_\alpha(x,y)f(y)\,dy,$$

where $\mathscr{K}_\alpha(x,y)$ denotes the kernel of $\mathscr{L}^{-\alpha/2}$.

In the present paper, we will study the boundedness of generalized fractional integral operators and commutators.

REMARK 1.1. The property (1.1) is satisfied by a large class of differential operators, such as (magnetic) Schrödinger operators and second-order elliptic operators of divergence form, see [10, 11, 12] for more details.

For the classical fractional integral I_α and the commutator $[b,I_\alpha]$ acting on Lebesgue spaces, we have

THEOREM 1.1. *The following statements are true:*

1. *Let* $0<\alpha<n$, $1<p<n/\alpha$ *and* $1/q=1/p-\alpha/n$. *Then the classical fractional integral operator* I_α *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
2. *If* $b\in\mathrm{BMO}(\mathbb{R}^n)$, *then the commutator operator* $[b,I_\alpha]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

For the proof of this result, see, for example, Stein [28, Chapter V], Grafakos [13, Chapter 6] and [6, Theorem 1]. For the weighted version of this result, see also Muckenhoupt–Wheeden [21, 22], Segovia–Torrea [25, Theorem 2.3] and Lu–Ding–Yan [17, Chapter 3].

The result in Theorem 1.1 can be extended from $-\Delta$ to the more general operator $\mathscr{L}$ with Gaussian upper bound (1.1). In [10], by using a new sharp maximal function, $M^{\#}_{\mathscr{L}}$, adapted to the semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ (see Definition 2.1 below), Duong and Yan studied unweighted estimates for commutators of generalized fractional integrals, and proved that for all $0<\alpha<n$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$, both $\mathscr{L}^{-\alpha/2}$ and $[b,\mathscr{L}^{-\alpha/2}]$ are all bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. By the pointwise inequality (1.5) and the kernel estimate for the difference operator $\mathscr{L}^{-\alpha/2}-e^{-t\mathscr{L}}\mathscr{L}^{-\alpha/2}$ in [9, 10] (with $0<\alpha<n$), we have

THEOREM 1.2. *The following statements are true:*

1. *Let* $0<\alpha<n$, $1<p<n/\alpha$ *and* $1/q=1/p-\alpha/n$. *Then the generalized fractional integral operator* $\mathscr{L}^{-\alpha/2}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.
2. *If* $b\in\mathrm{BMO}(\mathbb{R}^n)$, *then the commutator operator* $[b,\mathscr{L}^{-\alpha/2}]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

For the proof of this result, see Duong–Yan [10, Theorem 1.1 and Lemma 2.2]. Another proof was given by Cruz-Uribe–Martell–Pérez in [7, Proposition 3.2], which used a variant of the A_p extrapolation theorem and fractional Orlicz maximal operators. For the weighted version of this result, see Auscher–Martell [3, Theorems 1.3 and 1.4]. See also [4, Section 5.4] for another proof of such weighted estimates.

For $0<\beta\leqslant 1$, we say that a real-valued function f on $\mathbb{R}^n$ is a Lipschitz function of order β, if

$$\|f\|_{\mathrm{Lip}_\beta} := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < +\infty.$$

Let $\mathrm{Lip}_\beta(\mathbb{R}^n)$ be the space of Lipschitz functions of order β, that is,

$$\mathrm{Lip}_\beta(\mathbb{R}^n) := \big\{f : \|f\|_{\mathrm{Lip}_\beta} < +\infty\big\}.$$

If $b\in\mathrm{Lip}_\beta(\mathbb{R}^n)$ with $0<\beta\leqslant 1$ and $0<\alpha+\beta<n$, then by the pointwise inequality (1.5) and the definition of $\mathrm{Lip}_\beta(\mathbb{R}^n)$, we can deduce that for any $x\in\mathbb{R}^n$,

$$\begin{aligned}\Big|\big[b,I_\alpha\big](f)(x)\Big| &\leqslant \big\|b\big\|_{\mathrm{Lip}_\beta}\frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha-\beta}}dy \\ &= \big\|b\big\|_{\mathrm{Lip}_\beta} I_{\alpha+\beta}(|f|)(x),\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}\Big|\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x)\Big| &\leqslant \int_{\mathbb{R}^n}\big|b(x)-b(y)\big|\cdot|\mathscr{K}_\alpha(x,y)||f(y)|\,dy \\ &\leqslant C\big\|b\big\|_{\mathrm{Lip}_\beta}\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha-\beta}}dy \\ &\leqslant C\big\|b\big\|_{\mathrm{Lip}_\beta} I_{\alpha+\beta}(|f|)(x).\end{aligned} \tag{1.8}$$

Thus, by Theorem 1.1, we can prove that the commutators $\big[b,I_\alpha\big]$ and $\big[b,\mathscr{L}^{-\alpha/2}\big]$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, whenever $1<p<n/(\alpha+\beta)$ and $1/q = 1/p-(\alpha+\beta)/n$. This result was obtained by Paluszyński in [24] and Mo–Lu in [19]. On the other hand, the classical Morrey spaces $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ were originally introduced by Morrey in [20] to study the local regularity of solutions to second order elliptic partial differential equations. Nowadays these spaces have been studied extensively in the literature, and found a wide range of applications in harmonic analysis, potential theory and PDEs. Let us now recall the definition of the classical Morrey space. Let $1\leqslant p<\infty$ and $-n/p\leqslant\beta\leqslant 0$. We denote by $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ the Morrey space of all p-locally integrable functions f on $\mathbb{R}^n$ such that

$$\begin{aligned}\big\|f\big\|_{\mathscr{M}^{p,\beta}} &:= \sup_{\mathscr{B}\subset\mathbb{R}^n}\frac{1}{m(\mathscr{B})^{\beta/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}|f(x)|^p\,dx\right)^{1/p} \\ &= \sup_{\mathscr{B}\subset\mathbb{R}^n}\frac{1}{m(\mathscr{B})^{\beta/n+1/p}}\big\|f\cdot\chi_{\mathscr{B}}\big\|_{L^p} < +\infty.\end{aligned}$$

Here and in what follows, we denote by $\chi_{\mathscr{B}}$ the characteristic function of the ball $\mathscr{B}$. It is obvious that $\mathscr{M}^{p,-n/p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, the classical Lebesgue space. Also note that $\mathscr{M}^{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ by the Lebesgue differentiation theorem. If $\beta < -n/p$ or $\beta > 0$, then $\mathscr{M}^{p,\beta}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

For boundedness properties of the classical fractional integral I_α and the commutator $[b, I_\alpha]$ on Morrey spaces, we have

THEOREM 1.3. *The following estimates hold:*

1. *Let* $0 < \alpha < n$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$. *Suppose that* $-n/p \leqslant \beta < (-\alpha)$, *then the classical fractional integral operator* I_α *is bounded from* $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ *to* $\mathscr{M}^{q,\alpha+\beta}(\mathbb{R}^n)$.

2. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then the commutator operator* $[b, I_\alpha]$ *is bounded from* $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ *to* $\mathscr{M}^{q,\alpha+\beta}(\mathbb{R}^n)$.

For the proof of this result, see, for example, Adams [1, 2] and Peetre [23]. See also [16, 29] for the weighted case. Motivated by Theorem 1.2, it is natural to ask whether the result in Theorem 1.3 also holds for the generalized fractional integrals and related commutators. We give a positive answer to this question.

THEOREM 1.4. *The following estimates hold:*

1. *Let* $0 < \alpha < n$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$. *Suppose that* $-n/p \leqslant \beta < (-\alpha)$, *then the generalized fractional integral operator* $\mathscr{L}^{-\alpha/2}$ *is bounded from* $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ *to* $\mathscr{M}^{q,\alpha+\beta}(\mathbb{R}^n)$.

2. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then the commutator operator* $[b, \mathscr{L}^{-\alpha/2}]$ *is bounded from* $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ *to* $\mathscr{M}^{q,\alpha+\beta}(\mathbb{R}^n)$.

The proof is based on the pointwise inequality (1.5) and an estimate on the kernel of the difference operator $\mathscr{L}^{-\alpha/2} - e^{-t\mathscr{L}}\mathscr{L}^{-\alpha/2}$ in [9, 10] (with $0 < \alpha < n$). A weighted version of this result has been obtained by the author in [30, Theorem 1.4] (see also [31, Corollaries 18 and 24]).

REMARK 1.2. When $-n/p \leqslant \beta < (-\alpha)$ and $1/q = 1/p - \alpha/n$, one can see that

$$-n/q \leqslant \alpha + \beta < 0.$$

If $b \in \mathrm{Lip}_{\beta_1}(\mathbb{R}^n)$ with $0 < \beta_1 \leqslant 1$ and $0 < \alpha + \beta_1 < n$, then by estimates (1.7), (1.8) and Theorem 1.3, the commutators $[b, I_\alpha]$ and $[b, \mathscr{L}^{-\alpha/2}]$ are bounded operators from $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ to $\mathscr{M}^{q,\gamma}(\mathbb{R}^n)$, where

$$-\frac{n}{p} \leqslant \beta < -(\alpha + \beta_1), \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta_1}{n}, \quad \text{and} \quad \gamma = \alpha + \beta_1 + \beta.$$

In the present situation, we see that

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta_1}{n} < \frac{1}{p} \Longrightarrow q > p,$$

and

$$\gamma = \beta + (\alpha + \beta_1) \Longrightarrow \gamma > \beta.$$

This result was established by the author in [30, Theorems 1.5 and 1.6].

It is well known that the classical Morrey–Campanato spaces play an important role in the study of partial differential equations and harmonic analysis, see [8, 14, 27] for more details.

DEFINITION 1.1. Let $1 \leqslant p < \infty$ and $-n/p \leqslant \beta \leqslant 1$. A locally integrable function f is said to belong to the Morrey–Campanato space $\mathscr{C}^{p,\beta}(\mathbb{R}^n)$, if

$$\|f\|_{\mathscr{C}^{p,\beta}} := \sup_{\mathscr{B} \subset \mathbb{R}^n} \frac{1}{m(\mathscr{B})^{\beta/n}} \left(\frac{1}{m(\mathscr{B})} \int_{\mathscr{B}} |f(x) - f_{\mathscr{B}}|^p \, dx \right)^{1/p} < +\infty, \tag{1.9}$$

where $f_{\mathscr{B}}$ denotes the mean value of f and $m(\mathscr{B})$ is the Lebesgue measure of the ball $\mathscr{B}$. The quantity $\|f\|_{\mathscr{C}^{p,\beta}}$ is called the Morrey–Campanato norm of f.

The space $\mathscr{C}^{p,\beta}(\mathbb{R}^n)$ was first introduced by Campanato in [5], and was studied extensively in the literature. Both the space $\mathrm{BMO}(\mathbb{R}^n)$ and (homogenous) Lipschitz function space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ are special cases of Morrey–Campanato spaces.

- It is well known that when $\beta = 0$ and $1 \leqslant p < \infty$, then

  $$\mathscr{C}^{p,\beta}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$$

  with equivalent norms, see [9, 13, 17] for example.

- If $1 \leqslant p < \infty$ and $0 < \beta \leqslant 1$, then

  $$\mathscr{C}^{p,\beta}(\mathbb{R}^n) = \mathrm{Lip}_\beta(\mathbb{R}^n) \tag{1.10}$$

  with equivalent norms, see [14] and [24] for example.

- We remark that the equation (1.10) allows us to give an integral characterization of the (homogeneous) Lipschitz function space $\mathrm{Lip}_\beta(\mathbb{R}^n)$. This fact further leads to a generalization of the classical Sobolev embedding theorem. It is also well known that $\mathscr{C}^{1,1/p-1}(\mathbb{R}^n)$ is the dual space of Hardy space $H^p(\mathbb{R}^n)$ when $0 < p < 1$, and $\mathscr{C}^{1,0}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$ is the dual space of Hardy space $H^1(\mathbb{R}^n)$, see [13] and [14] for further details.

- If $-n/p \leqslant \beta < 0$ and $1 \leqslant p < \infty$, then in this case, it can be shown that

  $$\mathscr{C}^{p,\beta}(\mathbb{R}^n) \supseteq \mathscr{M}^{p,\beta}(\mathbb{R}^n).$$

  This inclusion relation can be found, for instance, in [26, 27].

As mentioned above, many authors are interested in the study of commutators for which the symbol functions belong to BMO spaces and Lipschitz spaces. It is interesting to consider the boundedness of commutators with symbol functions belonging to $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$, when $-n/p_1 \leqslant \beta_1 < 0$ and $1 \leqslant p_1 < \infty$. It was proved by Shi and Lu that the commutator $\left[b, I_\alpha\right]$ is a bounded operator from $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ to $\mathscr{C}^{q,\gamma}(\mathbb{R}^n)$ for appropriate indices, under the assumption that $b \in \mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ with $-n/p_1 \leqslant \beta_1 < 0$ and $1 \leqslant p_1 < \infty$. Moreover, some new characterizations of Morrey–Campanato spaces via the boundedness of $\left[b, I_\alpha\right]$ were also given, by new methods instead of the sharp maximal function theorem. See [27, Theorem 1.1]. For the corresponding results of commutators associated with Calderón–Zygmund singular integral operators, see [26, Theorems 1.2 and 1.3].

The main purpose of this paper is to prove that the commutator $\left[b, \mathscr{L}^{-\alpha/2}\right]$ is bounded from $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ to $\mathscr{C}^{q,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ for suitable indices, which is an extension of the result of Shi–Lu. Here the Morrey–Campanato space $\mathscr{C}^{q,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ is defined for all functions f with suitable bounds on growth. See Definition 2.3 below.

2. New Morrey–Campanato spaces associated with $\mathscr{L}$

To study the operators $\mathscr{L}^{-\alpha/2}$ and $\left[b, \mathscr{L}^{-\alpha/2}\right]$, we will introduce the Morrey–Campanato space $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ associated with the operator $\mathscr{L}$. Let us first introduce some notations and definitions related to the new Morrey–Campanato space.

A family of operators $\{\mathbf{A}_t\}_{t>0}$ is said to be "generalized approximation to the identity", if for every $t > 0$, $\mathbf{A}_t$ is represented by the kernel $p_t(x,y)$ that satisfies an upper bound

$$\left|p_t(x,y)\right| \leqslant h_t(x,y) := t^{-n/2} g\Big(\frac{|x-y|}{\sqrt{t}}\Big),$$

for all $x,y \in \mathbb{R}^n$ and $t > 0$. Here g is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r) = 0 \tag{2.1}$$

for some $\varepsilon > 0$.

Let ε be as in (2.1) and let $0 < \rho < \varepsilon$. For any $1 \leqslant p < \infty$, a locally integrable function f is said to be a function of (p,ρ)-type, if it satisfies

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^p}{(1+|x|)^{n+\rho}} dx\right)^{1/p} \leqslant C < +\infty, \tag{2.2}$$

and we denote by $M_{(p,\rho)}$ the collection of all functions of (p,ρ)-type. The smallest bound C satisfying the condition (2.2) is then taken to be the norm of f, and is denoted by $\|f\|_{M_{(p,\rho)}}$. It is easy to see that $M_{(p,\rho)}$ is a Banach function space under the norm $\|f\|_{M_{(p,\rho)}}$. We set

$$M_p := \bigcup_{\rho:0<\rho<\varepsilon} M_{(p,\rho)}.$$

For any $f \in L^p(\mathbb{R}^n)$ with $p \geqslant 1$, the sharp maximal function $M^{\#}_{\mathbf{A}}(f)$ associated with "generalized approximation to the identity" is defined as follows:

$$M^{\#}_{\mathbf{A}}(f)(x) := \sup_{x\in\mathscr{B}} \frac{1}{m(\mathscr{B})} \int_{\mathscr{B}} \big|f(y) - \mathbf{A}_{t_{\mathscr{B}}} f(y)\big|\, dy,$$

where $t_{\mathscr{B}} = r^2_{\mathscr{B}}$, $r_{\mathscr{B}}$ is the radius of the ball $\mathscr{B}$,

$$\mathbf{A}_{t_{\mathscr{B}}} f(y) := \int_{\mathbb{R}^n} p_{t_{\mathscr{B}}}(y,z) f(z)\, dz,$$

and the supremum is taken over all balls $\mathscr{B}$ containing the point x. The sharp maximal function $M^{\#}_{\mathbf{A}}$ was first introduced and studied by Martell in [18]. We remark that our analytic semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ is "generalized approximation to the identity". In particular, we give

DEFINITION 2.1. For every $f \in L^p(\mathbb{R}^n)$ with $p \geqslant 1$, the sharp maximal function $M^{\#}_{\mathscr{L}}(f)$ is defined by

$$M^{\#}_{\mathscr{L}}(f)(x) := \sup_{x\in\mathscr{B}} \frac{1}{m(\mathscr{B})} \int_{\mathscr{B}} \big|f(y) - e^{-t_{\mathscr{B}}\mathscr{L}} f(y)\big|\, dy,$$

where $t_{\mathscr{B}} = r^2_{\mathscr{B}}$, $r_{\mathscr{B}}$ is the radius of the ball $\mathscr{B}$ and the supremum is taken over all balls $\mathscr{B}$ in $\mathbb{R}^n$.

DEFINITION 2.2. Let $f \in L^p(\mathbb{R}^n)$ with $p \geqslant 1$ and $f \in M_p$. We say that a function f is in the space $\mathrm{BMO}_{\mathscr{L}}(\mathbb{R}^n)$ associated with the operator $\mathscr{L}$, if the sharp maximal function $M^{\#}_{\mathscr{L}}(f) \in L^{\infty}(\mathbb{R}^n)$, and we define

$$\|f\|_{\mathrm{BMO}_{\mathscr{L}}} = \big\|M^{\#}_{\mathscr{L}}(f)\big\|_{L^{\infty}}.$$

The new $\mathrm{BMO}_{\mathscr{L}}$ space associated with the operator $\mathscr{L}$ was introduced and studied by Duong and Yan in [11] (see also [12, 8, 9]). The idea is that the quantity $e^{-t\mathscr{L}}(f)$ can be viewed as an average version of f (at the scale t) and the quantity $e^{-t_{\mathscr{B}}\mathscr{L}} f(x)$ can be used to replace the mean value $f_{\mathscr{B}}$ in the definition of the classical BMO space. Here $t_{\mathscr{B}}$ is equal to the square of the radius of $\mathscr{B}$. Notice that $\mathrm{BMO}_{\mathscr{L}}$ is a semi-normed vector space, with the semi-norm vanishing on the kernel space $\mathrm{Ker}_{\mathscr{L}}$ defined by

$$\mathrm{Ker}_{\mathscr{L}} := \Big\{ f \in M_p : e^{-t\mathscr{L}}(f) = f, \ \text{ for all } \ t > 0 \Big\}.$$

The class of $\mathrm{BMO}_{\mathscr{L}}$ functions (modulo $\mathrm{Ker}_{\mathscr{L}}$) is a Banach function space.

- A natural question arising from Definition 2.2 is to compare the classical BMO space and the $\mathrm{BMO}_{\mathscr{L}}$ space associated with the operator $\mathscr{L}$. Denote by $e^{t\Delta}$ the heat semigroup on $\mathbb{R}^n$. It can be shown that the classical BMO space (modulo all constant functions) and the BMO_{Δ} space (modulo Ker_{Δ}) coincide, and their norms are equivalent, see [11, 12].

- Assume that for every $t > 0$, the equation

$$e^{-t\mathscr{L}}(\mathbf{1})(x) = \mathbf{1}$$

holds for almost everywhere $x \in \mathbb{R}^n$, that is,

$$\int_{\mathbb{R}^n} \mathscr{P}_t(x,y)\,dy = 1$$

for almost all $x \in \mathbb{R}^n$. Then we have $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_{\mathscr{L}}(\mathbb{R}^n)$, and there exists a positive constant $C > 0$ such that

$$\|f\|_{\mathrm{BMO}_{\mathscr{L}}} \leqslant C\|f\|_{\mathrm{BMO}}. \tag{2.3}$$

However, the converse inequality does not hold in general. See Martell [18, Proposition 3.1] and Deng–Duong–Sikora–Yan [9, Proposition 2.3]. As pointed out in [9], the condition $e^{-t\mathscr{L}}(\mathbf{1})(x) = \mathbf{1}$ is also necessary for (2.3). Indeed, it follows from (2.3) that $\|\mathbf{1}\|_{\mathrm{BMO}_{\mathscr{L}}} = 0$. This in turn implies that for all $t > 0$, $e^{-t\mathscr{L}}(\mathbf{1})(x) = \mathbf{1}$ holds for almost everywhere $x \in \mathbb{R}^n$.

For further details about the properties and applications of the $\mathrm{BMO}_{\mathscr{L}}(\mathbb{R}^n)$ space, we refer the reader to [8, 9, 11, 12] and the references therein. Motivated by Definitions 1.1 and 2.2, we now introduce our space $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ associated with $\mathscr{L}$, by using the function $e^{-t_{\mathscr{B}}\mathscr{L}}f(x)$ to replace the average $f_{\mathscr{B}}$ in the Morrey–Campanato norm (1.9), and $t_{\mathscr{B}}$ equals the square of the radius of $\mathscr{B}$.

DEFINITION 2.3. Assume that the semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ has a kernel $\mathscr{P}_t(x,y)$ satisfying the Gaussian upper bound (1.1). Let $1 \leqslant p < \infty$ and $-n/p \leqslant \gamma \leqslant 1$. A locally integrable function $f \in M_p$ is said to belong to the Morrey–Campanato space $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ associated with the operator $\mathscr{L}$, if

$$\|f\|_{\mathscr{C}^{p,\gamma}_{\mathscr{L}}} := \sup_{\mathscr{B}\subset\mathbb{R}^n} \frac{1}{m(\mathscr{B})^{\gamma/n}} \left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}} \left|f(x) - e^{-t_{\mathscr{B}}\mathscr{L}}f(x)\right|^p dx\right)^{1/p} < +\infty,$$

where $t_{\mathscr{B}} = r^2_{\mathscr{B}}$, $r_{\mathscr{B}}$ is the radius of the ball $\mathscr{B}$ and the supremum is taken over all balls $\mathscr{B}$ in $\mathbb{R}^n$.

- A natural question arising from Definition 2.3 is to compare the classical Morrey–Campanato space $\mathscr{C}^{p,\gamma}(\mathbb{R}^n)$ with the new function space $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$. Following the idea of [8, 11, 12], we can also show that when $\mathscr{L}$ is the Laplacian Δ in $\mathbb{R}^n$, then the corresponding space $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ coincides with the classical Morrey–Campanato space $\mathscr{C}^{p,\gamma}(\mathbb{R}^n)$, and their norms are equivalent.

- When $\gamma = 0$ and $1 \leqslant p < \infty$, then

$$\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n) = \mathrm{BMO}_{\mathscr{L}}(\mathbb{R}^n)$$

with equivalent norms, since a variant of the John–Nirenberg inequality holds for functions in $\mathrm{BMO}_{\mathscr{L}}(\mathbb{R}^n)$. See [8, 11, 12] for more details.

- Using a similar argument to that used in the proof of [18, Proposition 3.1] and [9, Proposition 2.3], we can also show that $\mathscr{C}^{p,\gamma}(\mathbb{R}^n)$ is a subspace of $\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$, under the assumption that $\mathscr{L}$ satisfies a conservation property of the semigroup $e^{-t\mathscr{L}}(\mathbf{1})=\mathbf{1}$ for every $t>0$.

We have the following lemma.

LEMMA 2.1. *Let $1\leqslant p<\infty$ and $-n/p\leqslant\gamma\leqslant 1$ with $\gamma\neq 0$. Assume that the semigroup $\{e^{-t\mathscr{L}}\}_{t>0}$ has a kernel $\mathscr{P}_t(x,y)$ satisfying the Gaussian upper bound (1.1) and for every $t>0$, the equation*

$$e^{-t\mathscr{L}}(\mathbf{1})(x)=\mathbf{1}$$

holds for almost all $x\in\mathbb{R}^n$. Then we have $\mathscr{C}^{p,\gamma}(\mathbb{R}^n)\subset\mathscr{C}^{p,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$, and there exists a positive constant $C>0$ such that

$$\|f\|_{\mathscr{C}^{p,\gamma}_{\mathscr{L}}}\leqslant C\|f\|_{\mathscr{C}^{p,\gamma}}. \tag{2.4}$$

We also remark that the condition $e^{-t\mathscr{L}}(\mathbf{1})=\mathbf{1}$ is necessary for (2.4).

In this paper, the symbols $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of all real numbers and natural numbers, respectively. $C>0$ denotes a universal constant which is independent of the main parameters involved and may change from line to line. The notation $\mathbf{X}\approx\mathbf{Y}$ means that $C_1\mathbf{Y}\leqslant\mathbf{X}\leqslant C_2\mathbf{Y}$ with some positive constants C_1 and C_2. $B(x_0,r_B)$ denotes the ball centered at x_0 and with radius r_B. For given $B=B(x_0,r_B)$ and $\lambda>0$, we write λB for the λ-dilate ball, which is the ball with the same center x_0 and with radius λr_B. For a measurable set E in $\mathbb{R}^n$, $m(E)$ denotes the Lebesgue measure of the set E and χ_E denotes the characteristic function of the set E.

3. Some lemmas

In this section, we will give some preliminary lemmas, which will be used in the proofs of our main theorems. First, we establish several direct estimates related to the Morrey and Campanato spaces.

LEMMA 3.1. *Let $b\in\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ with $1\leqslant p_1<\infty$ and $-n/p_1\leqslant\beta_1<0$. Then the following properties hold.*

(1) *For any ball $\mathscr{B}$ in $\mathbb{R}^n$, we get*

$$\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}|b(x)-b_{\mathscr{B}}|\,dx\leqslant m(\mathscr{B})^{\beta_1/n}\cdot\|b\|_{\mathscr{C}^{p_1,\beta_1}}.$$

(2) *For every $k\in\mathbb{N}$, we get*

$$\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}|b(x)-b_{\mathscr{B}}|\,dx\leqslant(2^n+1)k\Big[m(2\mathscr{B})^{\beta_1/n}\cdot\|b\|_{\mathscr{C}^{p_1,\beta_1}}\Big].$$

(3) *Let* $f \in \mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ *with* $1 \leqslant p_2 < \infty$ *and* $-n/p_2 \leqslant \beta_2 < 0$. *Then for every* $k \in \mathbb{N}$, *we get*

$$\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}} \big|b(x) - b_{\mathscr{B}}\big| \cdot \big|f(x)\big|\,dx \leqslant (2^n+1)k\Big[m\big(2\mathscr{B}\big)^{(\beta_1+\beta_2)/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}} \big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\Big].$$

Here we give the proof for the sake of completeness.

Proof. We begin with the proof of (1). From Hölder's inequality and the definition of $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$, it then follows that

$$\begin{aligned}\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}} \big|b(x) - b_{\mathscr{B}}\big|\,dx &\leqslant \left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}} \big|b(x) - b_{\mathscr{B}}\big|^{p_1}\,dx\right)^{1/p_1}\\ &\leqslant m(\mathscr{B})^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}.\end{aligned}$$

Next, we give the proof of (2). Observe that for any ball $\mathscr{B}$ in $\mathbb{R}^n$, we have

$$\begin{aligned}\big|b_{2\mathscr{B}} - b_{\mathscr{B}}\big| &\leqslant \frac{1}{m(\mathscr{B})}\int_{\mathscr{B}} \big|b(x) - b_{2\mathscr{B}}\big|\,dx\\ &\leqslant \frac{2^n}{m(2\mathscr{B})}\int_{2\mathscr{B}} \big|b(x) - b_{2\mathscr{B}}\big|\,dx \leqslant 2^n m(2\mathscr{B})^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}},\end{aligned}$$

by part (1). Similarly, for each $1 \leqslant i \leqslant k$,

$$\big|b_{2^i\mathscr{B}} - b_{2^{i-1}\mathscr{B}}\big| \leqslant 2^n m\big(2^i\mathscr{B}\big)^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}.$$

Since $\beta_1 < 0$, in this case, it is clear that

$$m\big(2^k\mathscr{B}\big)^{\beta_1/n} \leqslant m\big(2^i\mathscr{B}\big)^{\beta_1/n} \leqslant m(2\mathscr{B})^{\beta_1/n}, \tag{3.1}$$

for $i = 1,2,\dots,k$. Hence, for any $k \in \mathbb{N}$, we can deduce that

$$\begin{aligned}\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}} \big|b(x) - b_{\mathscr{B}}\big|\,dx &\leqslant \frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}} \big|b(x) - b_{2^k\mathscr{B}}\big|\,dx + \big|b_{2^k\mathscr{B}} - b_{\mathscr{B}}\big|\\ &\leqslant m\big(2^k\mathscr{B}\big)^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}} + \sum_{i=1}^{k} \big|b_{2^i\mathscr{B}} - b_{2^{i-1}\mathscr{B}}\big|\\ &\leqslant m\big(2^k\mathscr{B}\big)^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}} + \sum_{i=1}^{k} 2^n m\big(2^i\mathscr{B}\big)^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\\ &\leqslant (2^n+1)k\Big[m\big(2\mathscr{B}\big)^{\beta_1/n} \cdot \big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\Big].\end{aligned}$$

Let us now prove (3). We take $1 \leqslant p < \infty$ such that

$$1/p = 1/p_1 + 1/p_2.$$

By using Hölder's inequality and the definitions of $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ and $\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, we obtain that for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|b(x)-b_{2^k\mathscr{B}}\big|\cdot\big|f(x)\big|\,dx\\
&\leqslant\left(\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|b(x)-b_{2^k\mathscr{B}}\big|^p\cdot\big|f(x)\big|^p\,dx\right)^{1/p}\\
&\leqslant\frac{1}{m(2^k\mathscr{B})^{1/p}}\left(\int_{2^k\mathscr{B}}\big|b(x)-b_{2^k\mathscr{B}}\big|^{p_1}dx\right)^{1/p_1}\times\left(\int_{2^k\mathscr{B}}\big|f(x)\big|^{p_2}dx\right)^{1/p_2}\\
&\leqslant\frac{1}{m(2^k\mathscr{B})^{1/p}}\cdot m\big(2^k\mathscr{B}\big)^{\beta_1/n+1/p_1}\cdot m\big(2^k\mathscr{B}\big)^{\beta_2/n+1/p_2}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&=m\big(2^k\mathscr{B}\big)^{(\beta_1+\beta_2)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Hence, by part (2), we have

$$\begin{aligned}
&\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|b(x)-b_{\mathscr{B}}\big|\cdot\big|f(x)\big|\,dx\\
&\leqslant\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|b(x)-b_{2^k\mathscr{B}}\big|\cdot\big|f(x)\big|\,dx+\big|b_{2^k\mathscr{B}}-b_{\mathscr{B}}\big|\cdot\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|f(x)\big|\,dx\\
&\leqslant m\big(2^k\mathscr{B}\big)^{(\beta_1+\beta_2)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&\quad+\sum_{i=1}^{k}2^n m\big(2^i\mathscr{B}\big)^{\beta_1/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\cdot\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|f(x)\big|\,dx.
\end{aligned}$$

Moreover, by Hölder's inequality again,

$$\begin{aligned}
\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|f(x)\big|\,dx&\leqslant\left(\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|f(x)\big|^{p_2}dx\right)^{1/p_2}\\
&\leqslant m\big(2^k\mathscr{B}\big)^{\beta_2/n}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Note that $\beta_1 < 0$ and $\beta_2 < 0$(which implies $\beta_1+\beta_2 < 0$). Therefore, by using (3.1), we can deduce that for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m(2^k\mathscr{B})}\int_{2^k\mathscr{B}}\big|b(x)-b_{\mathscr{B}}\big|\cdot\big|f(x)\big|\,dx\\
&\leqslant m\big(2\mathscr{B}\big)^{(\beta_1+\beta_2)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}+\sum_{i=1}^{k}2^n m\big(2\mathscr{B}\big)^{(\beta_1+\beta_2)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&\leqslant(2^n+1)k\Big[m\big(2\mathscr{B}\big)^{(\beta_1+\beta_2)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\Big].
\end{aligned}$$

We are done. □

We also need the following key lemma, which gives the kernel estimate of the difference operator $\mathscr{L}^{-\alpha/2}-e^{-t\mathscr{L}}\mathscr{L}^{-\alpha/2}$ for any $t > 0$ and $0 < \alpha < n$.

LEMMA 3.2. *Assume that the semigroup* $\{e^{-t\mathscr{L}}\}_{t>0}$ *has a kernel* $\mathscr{P}_t(x,y)$ *satisfying the Gaussian upper bound* (1.1). *Then for any* $0<\alpha<n$, *the difference operator* $(I-e^{-t\mathscr{L}})\mathscr{L}^{-\alpha/2} := \mathscr{L}^{-\alpha/2}-e^{-t\mathscr{L}}\mathscr{L}^{-\alpha/2}$ *has an associated kernel* $\widetilde{K}_{\alpha,t}(x,y)$ *which satisfies*

$$\widetilde{K}_{\alpha,t}(x,y) \leqslant \frac{C}{|x-y|^{n-\alpha}}\cdot\frac{t}{|x-y|^2}.$$

Here the constant C is independent of $x,y\in\mathbb{R}^n$ and $t\in(0,+\infty)$. For the proof of this lemma, see Duong–Yan [10, Lemma 3.1] for $0<\alpha<1$ and Deng–Duong–Sikora–Yan [9, Lemma 5.3] for $0<\alpha<n$.

4. Main results

In this section, we will state and prove the main results of this paper. It is proved that for all $0<\alpha<n$, the commutator $[b,\mathscr{L}^{-\alpha/2}]$ is bounded from $\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ to $\mathscr{C}^{q,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$ for appropriate indices, when the symbol function $b(x)$ belongs to the space $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ with $1<p_1<\infty$ and $-n/p_1\leqslant\beta_1<0$. The corresponding results for the higher-order commutators are also obtained.

THEOREM 4.1. *Let* $0<\alpha<\min\{n,2\}$, $1<p_2<n/\alpha$ *and* $-n/p_2\leqslant\beta_2<(-\alpha)$. *Suppose that* $b\in\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $1<p_1<\infty$ *and* $-n/p_1\leqslant\beta_1<0$. *Then for any* $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, *there exists a positive constant* $C>0$ *independent of* b *and* f *such that*

$$\big\|[b,\mathscr{L}^{-\alpha/2}](f)\big\|_{\mathscr{C}^{q,\gamma}_{\mathscr{L}}} \leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}},$$

provided that

$$1/q=1/p_1+1/p_2-\alpha/n \quad \& \quad \gamma=\beta_1+\beta_2+\alpha.$$

Proof of Theorem 4.1. Let $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ with $1<p_2<n/\alpha$ and $-n/p_2\leqslant\beta_2<(-\alpha)$. By the definition of $\mathscr{C}^{q,\gamma}_{\mathscr{L}}(\mathbb{R}^n)$, it suffices to prove that for any given ball $\mathscr{B}$ in $\mathbb{R}^n$,

$$\begin{aligned}&\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|[b,\mathscr{L}^{-\alpha/2}](f)(x)-e^{-t_{\mathscr{B}}\mathscr{L}}\big([b,\mathscr{L}^{-\alpha/2}]f\big)(x)\right|^q dx\right)^{1/q}\\&\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}\tag{4.1}$$

Let $\mathscr{B}=B(x_0,r_{\mathscr{B}})$ be a fixed ball centered at x_0 and of radius $r_{\mathscr{B}}$, and let $t_{\mathscr{B}}=r_{\mathscr{B}}^2$. We decompose the function $f(x)$ as follows:

$$f(x)=f(x)\cdot\chi_{2\mathscr{B}}+f(x)\cdot\chi_{(2\mathscr{B})^{\complement}} := f_1(x)+f_2(x),$$

where $2\mathscr{B}=B(x_0,2r_{\mathscr{B}})$ and $(2\mathscr{B})^{\complement}=\mathbb{R}^n\setminus(2\mathscr{B})$. Observe that

$$\begin{aligned}&[b,\mathscr{L}^{-\alpha/2}](f)(x)\\&=[b(x)-b_{2\mathscr{B}}]\cdot\mathscr{L}^{-\alpha/2}(f)(x)-\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(x)-\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_2\big)(x),\end{aligned}$$

and

$$
\begin{aligned}
e^{-t_{\mathscr{B}}\mathscr{L}}\big([b,\mathscr{L}^{-\alpha/2}]f\big)(x) = {} & e^{-t_{\mathscr{B}}\mathscr{L}}\big([b-b_{2\mathscr{B}}]\cdot\mathscr{L}^{-\alpha/2}(f)\big)(x) \\
& - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(x) \\
& - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_2\big)(x).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|[b,\mathscr{L}^{-\alpha/2}](f)(x) - e^{-t_{\mathscr{B}}\mathscr{L}}\big([b,\mathscr{L}^{-\alpha/2}]f\big)(x)\Big|^q dx\right)^{1/q} \\
& \leqslant \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|[b(x)-b_{2\mathscr{B}}]\cdot\mathscr{L}^{-\alpha/2}(f)(x)\Big|^q dx\right)^{1/q} \\
& \quad + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(x)\Big|^q dx\right)^{1/q} \\
& \quad + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|e^{-t_{\mathscr{B}}\mathscr{L}}\big([b-b_{2\mathscr{B}}]\cdot\mathscr{L}^{-\alpha/2}(f)\big)(x)\Big|^q dx\right)^{1/q} \\
& \quad + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(x)\Big|^q dx\right)^{1/q} \\
& \quad + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_2\big)(x)\right. \\
& \quad \left. - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_2\big)(x)\Big|^q dx\right)^{1/q} \\
& := \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned}
$$

Let us now give the estimates of I, II, III, IV and V, respectively. For the first term I, we set

$$1/s = 1/p_2 - \alpha/n.$$

Then

$$1/q = 1/p_1 + 1/s. \tag{4.2}$$

We apply Hölder's inequality and (4.2) to obtain

$$
\begin{aligned}
\mathrm{I} & \leqslant \frac{1}{m(\mathscr{B})^{\gamma/n+1/q}}\left(\int_{\mathscr{B}}\big|b(x)-b_{2\mathscr{B}}\big|^{p_1}dx\right)^{1/p_1} \times \left(\int_{\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(x)\big|^s dx\right)^{1/s} \\
& \leqslant \|b\|_{\mathscr{C}^{p_1,\beta_1}} \times \frac{1}{m(\mathscr{B})^{\gamma/n+1/q-\beta_1/n-1/p_1}}\left(\int_{\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(x)\big|^s dx\right)^{1/s} \\
& = \|b\|_{\mathscr{C}^{p_1,\beta_1}} \times \frac{1}{m(\mathscr{B})^{(\gamma-\beta_1)/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(x)\big|^s dx\right)^{1/s}.
\end{aligned}
$$

Note that

$$\gamma - \beta_1 = \beta_2 + \alpha < 0. \tag{4.3}$$

According to Theorem 1.4, we know that $\mathscr{L}^{-\alpha/2}$ is bounded from $\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ to $\mathscr{M}^{s,\alpha+\beta_2}(\mathbb{R}^n)$. This fact, together with (4.3), gives

$$\mathrm{I} \leqslant \|b\|_{\mathscr{C}^{p_1,\beta_1}} \left\|\mathscr{L}^{-\alpha/2}(f)\right\|_{\mathscr{M}^{s,\alpha+\beta_2}} \leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

For the second term II, we set

$$1/p = 1/p_1 + 1/p_2 \quad \& \quad p > 1 \quad (\text{which forces } p_1, p_2 > 1).$$

Then we have

$$1/q = 1/p - \alpha/n. \tag{4.4}$$

From Hölder's inequality and Theorem 1.2, it then follows that

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}} \left(\int_{2\mathscr{B}} \left|b(x) - b_{2\mathscr{B}}\right|^p \cdot \left|f(x)\right|^p dx \right)^{1/p} \\
&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}} \left(\int_{2\mathscr{B}} \left|b(x) - b_{2\mathscr{B}}\right|^{p_1} dx \right)^{1/p_1} \times \left(\int_{2\mathscr{B}} \left|f(x)\right|^{p_2} dx \right)^{1/p_2}.
\end{aligned}$$

Moreover, by using (4.4) and the definitions of $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ and $\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, one can see that

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}} \cdot m(2\mathscr{B})^{\beta_1/n+1/p_1} \cdot m(2\mathscr{B})^{\beta_2/n+1/p_2} \|b\|_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{\mathscr{M}^{p_2,\beta_2}} \\
&= \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}} \cdot m(2\mathscr{B})^{(\beta_1+\beta_2+\alpha)/n+1/q} \|b\|_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{\mathscr{M}^{p_2,\beta_2}} \\
&\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{\mathscr{M}^{p_2,\beta_2}},
\end{aligned}$$

where the last inequality follows from the fact that $\gamma = \beta_1 + \beta_2 + \alpha$. As for the estimate of the term III, we first note that for any $x \in \mathscr{B}$ and $y \in 2\mathscr{B}$, the following estimate

$$\left|\mathscr{P}_{t_{\mathscr{B}}}(x,y)\right| \leqslant \frac{C}{(t_{\mathscr{B}})^{n/2}} \leqslant \frac{C}{m(2\mathscr{B})} \tag{4.5}$$

holds by (1.1). Moreover, for any $x \in \mathscr{B}$ and $y \in 2^{k+1}\mathscr{B} \setminus 2^k\mathscr{B}$ with $k \in \mathbb{N}$, one has $|y-x| \approx |y-x_0|$, and hence

$$\left|\mathscr{P}_{t_{\mathscr{B}}}(x,y)\right| \leqslant C \cdot \frac{(t_{\mathscr{B}})^{n/2}}{|x-y|^{2n}} \leqslant C \cdot \frac{(t_{\mathscr{B}})^{n/2}}{|y-x_0|^{2n}} \tag{4.6}$$

by using (1.1) again. Therefore, it follows from (4.5) and (4.6) that for any $x \in \mathscr{B}$,

$$\begin{aligned}
&\Big|e^{-t_{\mathscr{B}}\mathscr{L}}\big(\big[b-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}(f)\big)(x)\Big| \\
&=\Big|\int_{\mathbb{R}^n}\mathscr{P}_{t_{\mathscr{B}}}(x,y)\cdot\big[b(y)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}(f)(y)\,dy\Big| \\
&\leqslant\int_{2\mathscr{B}}\big|\mathscr{P}_{t_{\mathscr{B}}}(x,y)\big|\cdot\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\quad+\sum_{k=1}^{\infty}\int_{2^{k+1}\mathscr{B}\backslash 2^k\mathscr{B}}\big|\mathscr{P}_{t_{\mathscr{B}}}(x,y)\big|\cdot\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\leqslant\frac{C}{m(2\mathscr{B})}\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\quad+C\sum_{k=1}^{\infty}\int_{2^{k+1}\mathscr{B}\backslash 2^k\mathscr{B}}\frac{(t_{\mathscr{B}})^{n/2}}{|y-x_0|^{2n}}\cdot\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\leqslant\frac{C}{m(2\mathscr{B})}\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\quad+C\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
\mathrm{III}&\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\bigg) \\
&\quad+\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\bigg) \\
&:=\mathrm{III}^{(1)}+\mathrm{III}^{(2)}.
\end{aligned}$$

As in the proof of the term I, we can also show that

$$\begin{aligned}
\mathrm{III}^{(1)}&\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^q\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|^q dy\bigg)^{1/q} \\
&\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

On the other hand, for any $k \in \mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\leqslant\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2^{k+1}\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy \\
&\quad+\big|b_{2^{k+1}\mathscr{B}}-b_{2\mathscr{B}}\big|\bigg\{\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\bigg\}.
\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}
&\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2^{k+1}\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\\
&\leqslant\left(\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2^{k+1}\mathscr{B}}\big|^{q}\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|^{q}dy\right)^{1/q}\\
&\leqslant Cm\big(2^{k+1}\mathscr{B}\big)^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

In addition, by Lemma 3.1, (4.3) and Hölder's inequality,

$$\begin{aligned}
&\big|b_{2^{k+1}\mathscr{B}}-b_{2\mathscr{B}}\big|\left\{\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\right\}\\
&\leqslant k\cdot 2^{n}m\big(2\mathscr{B}\big)^{\beta_1/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\times\left\{\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|^{s}dy\right\}^{1/s}\\
&=k\cdot 2^{n}\frac{m(2\mathscr{B})^{\beta_1/n}}{m(2^{k+1}\mathscr{B})^{\beta_1/n}}m\big(2^{k+1}\mathscr{B}\big)^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\\
&\quad\times\frac{1}{m(2^{k+1}\mathscr{B})^{(\gamma-\beta_1)/n}}\left\{\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|^{s}dy\right\}^{1/s}\\
&\leqslant k\cdot 2^{n}\frac{m(2\mathscr{B})^{\beta_1/n}}{m(2^{k+1}\mathscr{B})^{\beta_1/n}}m\big(2^{k+1}\mathscr{B}\big)^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|\mathscr{L}^{-\alpha/2}(f)\big\|_{\mathscr{M}^{s,\alpha+\beta_2}},
\end{aligned}$$

where the number s is the same as above. Furthermore, from Theorem 1.4, it then follows that

$$\begin{aligned}
&\big|b_{2^{k+1}\mathscr{B}}-b_{2\mathscr{B}}\big|\left\{\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\right\}\\
&\leqslant Ck\cdot\frac{m(2\mathscr{B})^{\beta_1/n}}{m(2^{k+1}\mathscr{B})^{\beta_1/n}}m\big(2^{k+1}\mathscr{B}\big)^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Observe that $\beta_1<0$. Summing up the above estimates, we conclude that for each $k\in\mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|\mathscr{L}^{-\alpha/2}(f)(y)\big|\,dy\\
&\leqslant Ck\cdot\frac{m(2\mathscr{B})^{\beta_1/n}}{m(2^{k+1}\mathscr{B})^{\beta_1/n}}m\big(2^{k+1}\mathscr{B}\big)^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathrm{III}^{(2)}&\leqslant C\sum_{k=1}^{\infty}\frac{k}{2^{kn}}\left[\frac{m(2\mathscr{B})}{m(2^{k+1}\mathscr{B})}\right]^{\beta_1/n}\cdot\left[\frac{m(2^{k+1}\mathscr{B})}{m(\mathscr{B})}\right]^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{k}{2^{k(n+\beta_1-\gamma)}}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}},
\end{aligned}$$

where in the last inequality we have used the fact that $n+\beta_1-\gamma=(n-\alpha)-\beta_2>0$. Consequently,

$$\mathrm{III}\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.$$

Let us now deal with the term IV. Using the estimates (4.5) and (4.6), we can deduce that for any $x\in\mathscr{B}$,

$$\begin{aligned}
&\Big|e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(x)\Big|=\Big|\int_{\mathbb{R}^n}\mathscr{P}_{t_{\mathscr{B}}}(x,y)\cdot\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\,dy\Big|\\
&\leqslant\frac{C}{m(2\mathscr{B})}\int_{2\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy\\
&\quad+C\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
\mathrm{IV}&\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy\bigg)\\
&\quad+\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy\bigg)\\
&:=\mathrm{IV}^{(1)}+\mathrm{IV}^{(2)}.
\end{aligned}$$

As in the proof of the term II, we can also prove that

$$\begin{aligned}
\mathrm{IV}^{(1)}&\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|^q dy\bigg)^{1/q}\\
&\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

On the other hand, Theorem 1.2 and Hölder's inequality imply that for any $k\in\mathbb{N}$,

$$\begin{aligned}
&\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy\\
&\leqslant\bigg(\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|^q dy\bigg)^{1/q}\\
&\leqslant\frac{1}{m(2^{k+1}\mathscr{B})^{1/q}}\bigg(\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^p\cdot\big|f(y)\big|^p dy\bigg)^{1/p},
\end{aligned}$$

where the number $p>1$ is the same as above. Since $1/p=1/p_1+1/p_2$, another application of the Hölder inequality yields

$$\begin{aligned}
&\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big([b-b_{2\mathscr{B}}]f_1\big)(y)\Big|\,dy\\
&\leqslant\frac{1}{m(2^{k+1}\mathscr{B})^{1/q}}\bigg(\int_{2\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^{p_1}dy\bigg)^{1/p_1}\times\bigg(\int_{2\mathscr{B}}\big|f(y)\big|^{p_2}dy\bigg)^{1/p_2}\\
&\leqslant\frac{1}{m(2^{k+1}\mathscr{B})^{1/q}}\cdot m(2\mathscr{B})^{(\beta_1+\beta_2)/n+1/p}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

This, together with (4.4), gives us that

$$\begin{aligned}\mathrm{IV}^{(2)} &\leqslant C\cdot\frac{m(2\mathscr{B})^{\gamma/n}}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\left[\frac{m(2\mathscr{B})}{m(2^{k+1}\mathscr{B})}\right]^{1/q}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}$$

Consequently,

$$\mathrm{IV}\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.$$

It remains to estimate the last term V. Note that if $x\in\mathscr{B}$ and $y\in 2^{k+1}\mathscr{B}\setminus 2^k\mathscr{B}$ with $k\in\mathbb{N}$, then $|y-x|\approx|y-x_0|$. This fact, together with Lemma 3.2, implies that for any $x\in\mathscr{B}$,

$$\begin{aligned}&\Big|\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x)-e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x)\Big|\\ &=\Big|\big(I-e^{-t_{\mathscr{B}}\mathscr{L}}\big)\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x)\Big|\\ &\leqslant\int_{(2\mathscr{B})^{\complement}}\big|\widetilde{K}_{\alpha,t_{\mathscr{B}}}(x,y)\big|\cdot\big|\big[b(y)-b_{2\mathscr{B}}\big]f(y)\big|\,dy\\ &\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}\mathscr{B}\setminus 2^k\mathscr{B}}\frac{1}{|x-y|^{n-\alpha}}\cdot\frac{r_{\mathscr{B}}^2}{|x-y|^2}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|f(y)\big|\,dy\\ &\leqslant C\sum_{k=1}^{\infty}\frac{1}{2^{2k}}\cdot\frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|f(y)\big|\,dy.\end{aligned}$$

Hence, by using Lemma 3.1, we obtain

$$\begin{aligned}\mathrm{V}&=\frac{1}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x)\\ &\quad-e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x)\Big|^q dx\bigg)^{1/q}\\ &\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{1}{2^{2k}}\cdot\frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|\cdot\big|f(y)\big|\,dy\\ &\leqslant\frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{k}{2^{2k}}\cdot m\big(2\mathscr{B}\big)^{(\beta_1+\beta_2)/n}m\big(2^{k+1}\mathscr{B}\big)^{\alpha/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}$$

Since $\gamma=\beta_1+\beta_2+\alpha$, it then follows that

$$\begin{aligned}\mathrm{V}&\leqslant C\cdot\frac{m(2\mathscr{B})^{\gamma/n}}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{k}{2^{2k}}\cdot\left[\frac{m(2^{k+1}\mathscr{B})}{m(2\mathscr{B})}\right]^{\alpha/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\sum_{k=1}^{\infty}\frac{k}{2^{(2-\alpha)k}}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}},\end{aligned}$$

where in the last inequality we have used the fact that $\alpha < 2$. Combining the above estimates I, II, III, IV with V, we obtain the desired result (4.1), and hence the proof of Theorem 4.1 is complete. □

In particular, if $\beta_2 = -n/p_2$, then we can get the corresponding result for $\big[b,\mathscr{L}^{-\alpha/2}\big]$.

COROLLARY 4.2. *Let* $0<\alpha<\min\{n,2\}$. *Suppose that* $b\in\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $1<p_1<\infty$ *and* $-n/p_1\leqslant\beta_1<0$. *Then for any* $f\in L^{p_2}(\mathbb{R}^n)$ *with* $1<p_2<n/\alpha$, *there exists a positive constant* $C>0$ *independent of* b *and* f *such that*

$$\big\|\big[b,\mathscr{L}^{-\alpha/2}\big](f)\big\|_{\mathscr{C}^{q,\gamma}_{\mathscr{L}}}\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{L^{p_2}},$$

provided that

$$1/q=1/p_1+1/p_2-\alpha/n \quad \& \quad \gamma=\beta_1+\alpha-n/p_2.$$

In the present situation, we have

$$1/q=1/p_1+(\beta_1-\gamma)/n \quad \& \quad \gamma=\beta_1+n(1/p_1-1/q).$$

Let us now study the corresponding estimates for the higher-order commutators. Let $0<\alpha<n$, $2\leqslant m\in\mathbb{N}$ and $b(x)$ be a locally integrable function on $\mathbb{R}^n$. The higher-order commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ generated by $\mathscr{L}^{-\alpha/2}$ and b is defined by

$$\big[b,\mathscr{L}^{-\alpha/2}\big]^m(f)(x):=\big[b,\dots\big[b,\big[b,\mathscr{L}^{-\alpha/2}\big]\big]\big](f)(x).$$

Since for $0<\alpha<n$, the kernel of $\mathscr{L}^{-\alpha/2}$ is $\mathscr{K}_\alpha(x,y)$, we then have

$$\big[b,\mathscr{L}^{-\alpha/2}\big]^m(f)(x)=\int_{\mathbb{R}^n}\big[b(x)-b(y)\big]^m\mathscr{K}_\alpha(x,y)f(y)\,dy. \tag{4.7}$$

- When $m\geqslant 2$ and $b\in\mathrm{BMO}(\mathbb{R}^n)$, it was first proved by Mo and Lu that for any $0<\alpha<1$, the higher-order commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$, see [19, Theorem 1.7]. Actually, for all $0<\alpha<n$, the conclusion of Theorem 1.7 in [19] is also true, by using Lemma 3.2 with the range of $0<\alpha<n$. See also [3, Theorem 1.4] for the weighted case. Furthermore, for all $0<\alpha<n$, it can be shown that the higher-order commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ is bounded from $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ to $\mathscr{M}^{q,\alpha+\beta}(\mathbb{R}^n)$ by induction on m, where $-n/p\leqslant\beta<(-\alpha)$. Here we omit the details for brevity.

- Let us see what happens if the symbol function $b(x)$ belongs to the space $\mathrm{Lip}_{\beta_1}(\mathbb{R}^n)$. When $m\geqslant 2$ and $b\in\mathrm{Lip}_{\beta_1}(\mathbb{R}^n)$ with $0<\beta_1\leqslant 1$ and $0<\alpha+m\beta_1<n$, then by (4.7), the pointwise inequality (1.5) and the definition of $\mathrm{Lip}_{\beta_1}(\mathbb{R}^n)$, we can deduce that for any $x\in\mathbb{R}^n$,

$$\begin{aligned}\Big|\big[b,\mathscr{L}^{-\alpha/2}\big]^m(f)(x)\Big|&\leqslant\int_{\mathbb{R}^n}\big|b(x)-b(y)\big|^m\cdot|\mathscr{K}_\alpha(x,y)||f(y)|\,dy\\&\leqslant C\|b\|^m_{\mathrm{Lip}_{\beta_1}}\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n-\alpha-m\beta_1}}dy\\&\leqslant C\|b\|^m_{\mathrm{Lip}_{\beta_1}}I_{\alpha+m\beta_1}(|f|)(x).\end{aligned} \tag{4.8}$$

Thus, by (4.8) and Theorem 1.1, we can prove that the higher-order commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, whenever

$$1<p<\frac{n}{\alpha+m\beta_1}\quad \text{and}\quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha+m\beta_1}{n}.$$

This result was obtained by Mo and Lu in [19, Theorem 1.8]. Moreover, by using Theorem 1.3 and (4.8), we can also show that the higher-order commutator $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$ is a bounded operator from $\mathscr{M}^{p,\beta}(\mathbb{R}^n)$ to $\mathscr{M}^{q,\gamma}(\mathbb{R}^n)$, whenever

$$-\frac{n}{p}\leqslant\beta<-(\alpha+m\beta_1),\quad \frac{1}{q}=\frac{1}{p}-\frac{\alpha+m\beta_1}{n},\quad \text{and}\quad \gamma=\alpha+m\beta_1+\beta.$$

Inspired by the above results, it will be interesting to consider the boundedness of $\big[b,\mathscr{L}^{-\alpha/2}\big]^m$, when the symbol function $b(x)$ belongs to the space $\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ with $1\leqslant p_1<\infty$ and $-n/p_1\leqslant\beta_1<0$. Below we study the case $m=2$, the general case follows by using the same method.

THEOREM 4.3. *Let* $0<\alpha<\min\{n,2\}$, $1<p_2<n/\alpha$ *and* $-n/p_2\leqslant\beta_2<(-\alpha)$. *Suppose that* $b\in\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $2<p_1<\infty$ *and* $-n/p_1\leqslant\beta_1<0$. *Then for any* $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, *there exists a positive constant* $C>0$ *independent of* b *and* f *such that*

$$\big\|\big[b,\mathscr{L}^{-\alpha/2}\big]^2(f)\big\|_{\mathscr{C}^{q,\gamma}_{\mathscr{L}}}\leqslant C\|b\|^2_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}},$$

provided that

$$1/q=2/p_1+1/p_2-\alpha/n\quad \&\quad \gamma=2\beta_1+\beta_2+\alpha.$$

Before proving our main theorem, let us first establish the following result.

THEOREM 4.4. *Let* $0<\alpha<n$, $1<p_2<n/\alpha$ *and* $-n/p_2\leqslant\beta_2<(-\alpha)$. *Suppose that* $b\in\mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $1<p_1<\infty$ *and* $-n/p_1\leqslant\beta_1<0$. *Then for any* $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, *there exists a positive constant* $C>0$ *independent of* b *and* f *such that*

$$\big\|\big[b,\mathscr{L}^{-\alpha/2}\big](f)\big\|_{\mathscr{M}^{q,\gamma}}\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}},$$

provided that

$$1/q=1/p_1+1/p_2-\alpha/n\quad \&\quad \gamma=\beta_1+\beta_2+\alpha.$$

Proof of Theorem 4.4. Let $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ with $1<p_2<n/\alpha$ and $-n/p_2\leqslant\beta_2<(-\alpha)$. For any fixed ball $\mathscr{B}=B(x_0,r_{\mathscr{B}})\subset\mathbb{R}^n$, we decompose the function $f(x)$ into two parts.

$$f(x)=f(x)\cdot\chi_{2\mathscr{B}}+f(x)\cdot\chi_{(2\mathscr{B})^{\complement}}:=f_1(x)+f_2(x).$$

Observe that for any fixed ball $\mathscr{B}\subset\mathbb{R}^n$,

$$\begin{aligned}&\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x)\\&=\big[b(x)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}(f)(x)-\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_1\big)(x)-\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f_2\big)(x).\end{aligned}$$

Then we can write

$$\begin{aligned}
&\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\left[b,\mathscr{L}^{-\alpha/2}\right](f)(x)\right|^{q}dx\right)^{1/q}\\
&\leqslant \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\left[b(x)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}(f)(x)\right|^{q}dx\right)^{1/q}\\
&\quad+\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f_1\big)(x)\right|^{q}dx\right)^{1/q}\\
&\quad+\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f_2\big)(x)\right|^{q}dx\right)^{1/q}\\
&:=\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Arguing as in the proof of Theorem 4.1, we have

$$\mathrm{I}+\mathrm{II}\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.$$

Let us now estimate the last term III. Note that if $x\in\mathscr{B}$ and $y\in 2^{k+1}\mathscr{B}\setminus 2^k\mathscr{B}$ with $k\in\mathbb{N}$, then $|y-x|\approx|y-x_0|$. Since for $0<\alpha<n$, the kernel of $\mathscr{L}^{-\alpha/2}$ is $\mathscr{K}_\alpha(x,y)$, by using the kernel estimate (1.5), we can deduce that for any $x\in\mathscr{B}$,

$$\begin{aligned}
&\left|\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f_2\big)(x)\right|\\
&\leqslant \int_{(2\mathscr{B})^{\complement}}\left|\mathscr{K}_\alpha(x,y)\right|\cdot\left|\left[b(y)-b_{2\mathscr{B}}\right]f(y)\right|dy\\
&\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}\mathscr{B}\setminus 2^k\mathscr{B}}\frac{1}{|x-y|^{n-\alpha}}\cdot\left|b(y)-b_{2\mathscr{B}}\right|\cdot\left|f(y)\right|dy\\
&\leqslant C\sum_{k=1}^{\infty}\frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\left|b(y)-b_{2\mathscr{B}}\right|\cdot\left|f(y)\right|dy,
\end{aligned}$$

which together with Lemma 3.1 implies that

$$\begin{aligned}
\mathrm{III}&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\left|b(y)-b_{2\mathscr{B}}\right|\left|f(y)\right|dy\\
&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}k\left[m\big(2^{k+1}\mathscr{B}\big)^{(\beta_1+\beta_2+\alpha)/n}\cdot\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\right].
\end{aligned}$$

Since $\beta_1<0$ and $\beta_2+\alpha<0$, we see that

$$\gamma=\beta_1+\beta_2+\alpha<0.$$

Consequently,

$$\begin{aligned}\mathrm{III} &\leqslant C\sum_{k=1}^{\infty} k\cdot\left[\frac{m(2^{k+1}\mathscr{B})}{m(\mathscr{B})}\right]^{\gamma/n}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\sum_{k=1}^{\infty}\frac{k}{(2^{-\gamma})^k}\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\big\|b\big\|_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}$$

Summing up the above estimates of I, II and III, and then taking the supremum over all balls $\mathscr{B}$ in $\mathbb{R}^n$, we conclude the proof of Theorem 4.4. □

We are now ready to show our main theorem of this section.

Proof of Theorem 4.3. Let $f\in\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ with $1<p_2<n/\alpha$ and $-n/p_2\leqslant\beta_2<(-\alpha)$. For any fixed ball $\mathscr{B}$ in $\mathbb{R}^n$, it suffices to verify that

$$\begin{aligned}\mathbf{W}:=&\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|\big[b,\mathscr{L}^{-\alpha/2}\big]^2(f)(x)\right.\\ &\left.-e^{-t_{\mathscr{B}}\mathscr{L}}\big(\big[b,\mathscr{L}^{-\alpha/2}\big]^2 f\big)(x)\Big|^q dx\right)^{1/q}\\ \leqslant& C\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}\tag{4.9}$$

To prove (4.9), we first observe that for any fixed ball $\mathscr{B}$ in $\mathbb{R}^n$,

$$\big[b(x)-b(y)\big]^2=\big[b(x)-b_{2\mathscr{B}}\big]^2-2\big[b(x)-b_{2\mathscr{B}}\big]\cdot\big[b(y)-b_{2\mathscr{B}}\big]+\big[b(y)-b_{2\mathscr{B}}\big]^2.$$

Let $\mathscr{B}=B(x_0,r_{\mathscr{B}})$ be a fixed ball centered at x_0 and with radius $r_{\mathscr{B}}$. As usual, we decompose the function $f(x)$ in the following way

$$f(x)=f(x)\cdot\chi_{2\mathscr{B}}+f(x)\cdot\chi_{(2\mathscr{B})^{\complement}}:=f_1(x)+f_2(x).$$

Then $\big[b,\mathscr{L}^{-\alpha/2}\big]^2(f)$ may be written as

$$\begin{aligned}\big[b,\mathscr{L}^{-\alpha/2}\big]^2(f)(x)=&\big[b(x)-b_{2\mathscr{B}}\big]^2\cdot\mathscr{L}^{-\alpha/2}(f)(x)\\ &-2\big[b(x)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f\big)(x)\\ &+\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_1\big)(x)+\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x),\end{aligned}$$

for any $x \in \mathbb{R}^n$. So we have

$$
\begin{aligned}
\mathbf{W} \leqslant & \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\left[b(x)-b_{2\mathscr{B}}\right]^2\cdot\mathscr{L}^{-\alpha/2}(f)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]^2 f_1\right)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|e^{-t_{\mathscr{B}}\mathscr{L}}\left(\left[b-b_{2\mathscr{B}}\right]^2\cdot\mathscr{L}^{-\alpha/2}(f)\right)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]^2 f_1\right)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|2\left[b(x)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]f\right)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|2e^{-t_{\mathscr{B}}\mathscr{L}}\Big(\left[b-b_{2\mathscr{B}}\right]\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]f\right)\Big)(x)\right|^q dx\right)^{1/q} \\
& + \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]^2 f_2\right)(x)\right.\right. \\
& \left.\left. - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\left(\left[b-b_{2\mathscr{B}}\right]^2 f_2\right)(x)\right|^q dx\right)^{1/q} \\
:= & \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}.
\end{aligned}
$$

Here $t_{\mathscr{B}} = r_{\mathscr{B}}^2$ and $r_{\mathscr{B}}$ is the radius of the ball $\mathscr{B}$. Let us now estimate I, II, III, IV, V, VI and VII, respectively. Let s be the same as in Theorem 4.1. Then

$$1/q = 1/p_1 + 1/p_1 + 1/s. \tag{4.10}$$

For the first term I, by Hölder's inequality and (4.10), we obtain

$$
\begin{aligned}
\mathrm{I} & \leqslant \frac{1}{m(\mathscr{B})^{\gamma/n+1/q}}\left(\int_{\mathscr{B}}\left|b(x)-b_{2\mathscr{B}}\right|^{p_1}dx\right)^{1/p_1}\left(\int_{\mathscr{B}}\left|b(x)-b_{2\mathscr{B}}\right|^{p_1}dx\right)^{1/p_1} \\
& \quad \times\left(\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}(f)(x)\right|^s dx\right)^{1/s} \\
& \leqslant \|b\|_{\mathscr{C}^{p_1,\beta_1}}^2 \times \frac{1}{m(\mathscr{B})^{\gamma/n+1/q-2\beta_1/n-2/p_1}}\left(\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}(f)(x)\right|^s dx\right)^{1/s} \\
& = \|b\|_{\mathscr{C}^{p_1,\beta_1}}^2 \times \frac{1}{m(\mathscr{B})^{(\gamma-2\beta_1)/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\mathscr{L}^{-\alpha/2}(f)(x)\right|^s dx\right)^{1/s}.
\end{aligned}
$$

Notice that

$$\gamma - 2\beta_1 = \beta_2 + \alpha < 0. \tag{4.11}$$

This fact, together with Theorem 1.4, implies that

$$\mathrm{I} \leqslant \|b\|_{\mathscr{C}^{p_1,\beta_1}}^2 \left\|\mathscr{L}^{-\alpha/2}(f)\right\|_{\mathscr{M}^{s,\alpha+\beta_2}} \leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}^2 \|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

For the second term II, we now choose a real number $\widetilde{p}>1$ so that

$$1/\widetilde{p}=1/p_1+1/p_1+1/p_2 \quad (\text{which forces } p_1>2,\ p_2>1).$$

Then we have

$$1/q=1/\widetilde{p}-\alpha/n. \tag{4.12}$$

By using Hölder's inequality and Theorem 1.2, we have

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}}\left(\int_{2\mathscr{B}}\Big|\big[b(x)-b_{2\mathscr{B}}\big]^2\cdot f(x)\Big|^{\widetilde{p}}dx\right)^{1/\widetilde{p}}\\
&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}}\left(\int_{2\mathscr{B}}\big|b(x)-b_{2\mathscr{B}}\big|^{p_1}dx\right)^{1/p_1}\left(\int_{2\mathscr{B}}\big|b(x)-b_{2\mathscr{B}}\big|^{p_1}dx\right)^{1/p_1}\\
&\quad\times\left(\int_{2\mathscr{B}}\big|f(x)\big|^{p_2}dx\right)^{1/p_2}.
\end{aligned}$$

Moreover, it follows from (4.12) that

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}}\cdot m(2\mathscr{B})^{2\beta_1/n+2/p_1}\cdot m(2\mathscr{B})^{\beta_2/n+1/p_2}\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&= \frac{C}{m(\mathscr{B})^{\gamma/n+1/q}}\cdot m(2\mathscr{B})^{(2\beta_1+\beta_2+\alpha)/n+1/q}\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\\
&\leqslant C\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}},
\end{aligned}$$

where in the last step we have used the fact that $\gamma=2\beta_1+\beta_2+\alpha$. Arguing as in the proof of Theorem 4.1, we can also prove that

$$\mathrm{III}+\mathrm{IV}\leqslant C\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.$$

Let us now turn to deal with the term V. It is easy to see that for any $x\in\mathscr{B}$,

$$\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x)=\big[b(x)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}(f)(x)-\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f\big)(x),$$

and then we obtain

$$\begin{aligned}
&2\big[b(x)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f\big)(x)\\
&=2\big[b(x)-b_{2\mathscr{B}}\big]^2\cdot\mathscr{L}^{-\alpha/2}(f)(x)-2\big[b(x)-b_{2\mathscr{B}}\big]\cdot\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x).
\end{aligned}\tag{4.13}$$

Consequently, one can write

$$\begin{aligned}
\mathrm{V} &\leqslant \frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|2\big[b(x)-b_{2\mathscr{B}}\big]^2\cdot\mathscr{L}^{-\alpha/2}(f)(x)\Big|^q dx\right)^{1/q}\\
&\quad+\frac{1}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|2\big[b(x)-b_{2\mathscr{B}}\big]\cdot\big[b,\mathscr{L}^{-\alpha/2}\big](f)(x)\Big|^q dx\right)^{1/q}\\
&:=\mathrm{V}^{(1)}+\mathrm{V}^{(2)}.
\end{aligned}$$

By the estimate of I, we see that

$$\mathrm{V}^{(1)} \leqslant C\|b\|^2_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

On the other hand, we choose a real number $\widetilde{q} > 1$ so that

$$1/\widetilde{q} = 1/p_1 + 1/p_2 - \alpha/n \Longrightarrow 1/q = 1/p_1 + 1/\widetilde{q}. \tag{4.14}$$

It follows from Hölder's inequality and (4.14) that

$$\begin{aligned}
\mathrm{V}^{(2)} &\leqslant \frac{2}{m(\mathscr{B})^{\gamma/n+1/q}}\left(\int_{\mathscr{B}}\left|b(x)-b_{2\mathscr{B}}\right|^{p_1}dx\right)^{1/p_1} \times \left(\int_{\mathscr{B}}\left|\left[b,\mathscr{L}^{-\alpha/2}\right](f)(x)\right|^{\widetilde{q}}dx\right)^{1/\widetilde{q}}\\
&\leqslant \frac{2}{m(\mathscr{B})^{\gamma/n+1/q}}\cdot m(2\mathscr{B})^{\beta_1/n+1/p_1}\|b\|_{\mathscr{C}^{p_1,\beta_1}}\left(\int_{\mathscr{B}}\left|\left[b,\mathscr{L}^{-\alpha/2}\right](f)(x)\right|^{\widetilde{q}}dx\right)^{1/\widetilde{q}}\\
&\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\frac{1}{m(\mathscr{B})^{(\gamma-\beta_1)/n}}\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\left|\left[b,\mathscr{L}^{-\alpha/2}\right](f)(x)\right|^{\widetilde{q}}dx\right)^{1/\widetilde{q}}.
\end{aligned}$$

Moreover, by using Theorem 4.4, we thus obtain

$$\begin{aligned}
\mathrm{V}^{(2)} &\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\left\|\left[b,\mathscr{L}^{-\alpha/2}\right](f)\right\|_{\mathscr{M}^{\widetilde{q},\beta_1+\beta_2+\alpha}}\\
&\leqslant C\|b\|^2_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Summing up the estimates of $\mathrm{V}^{(1)}$ and $\mathrm{V}^{(2)}$, we conclude that

$$\mathrm{V} \leqslant C\|b\|^2_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

As for the term VI, it follows from the estimates (4.5) and (4.6) that for any $x \in \mathscr{B}$,

$$\begin{aligned}
&\left|e^{-t_{\mathscr{B}}\mathscr{L}}\Big(\left[b-b_{2\mathscr{B}}\right]\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)\Big)(x)\right|\\
&=\left|\int_{\mathbb{R}^n}\mathscr{P}_{t_{\mathscr{B}}}(x,y)\cdot\left[b(y)-b_{2\mathscr{B}}\right]\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)(y)\,dy\right|\\
&\leqslant \frac{C}{m(2\mathscr{B})}\int_{2\mathscr{B}}\left|\left[b(y)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)(y)\right|dy\\
&\quad+C\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\left|\left[b(y)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)(y)\right|dy.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
\mathrm{VI} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\left|\left[b(y)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)(y)\right|dy\right)\\
&\quad+\frac{C}{m(\mathscr{B})^{\gamma/n}}\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\left|\left[b(y)-b_{2\mathscr{B}}\right]\cdot\mathscr{L}^{-\alpha/2}\big(\left[b-b_{2\mathscr{B}}\right]f\big)(y)\right|dy\right)\\
&:=\mathrm{VI}^{(1)}+\mathrm{VI}^{(2)}.
\end{aligned}$$

As in the estimate of V, we also obtain that

$$\begin{aligned}\mathrm{VI}^{(1)} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\left(\frac{1}{m(2\mathscr{B})}\int_{2\mathscr{B}}\Big|\big[b(y)-b_{2\mathscr{B}}\big]\cdot\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]f\big)(y)\Big|^{q}dy\right)^{1/q}\\ &\leqslant C\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}$$

By using equation (4.13), the term $\mathrm{VI}^{(2)}$ can be further divided into two parts.

$$\begin{aligned}\mathrm{VI}^{(2)} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\big[b(y)-b_{2\mathscr{B}}\big]^{2}\cdot\mathscr{L}^{-\alpha/2}(f)(y)\Big|dy\right)\\ &\quad+\frac{C}{m(\mathscr{B})^{\gamma/n}}\left(\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\Big|\big[b(y)-b_{2\mathscr{B}}\big]\cdot\big[b,\mathscr{L}^{-\alpha/2}\big](f)(y)\Big|dy\right)\\ &:=\mathrm{VI}^{(3)}+\mathrm{VI}^{(4)}.\end{aligned}$$

Applying Theorem 4.4 again, we know that $\big[b,\mathscr{L}^{-\alpha/2}\big](f)$ belongs to the space $\mathscr{M}^{\widetilde{q},\beta_1+\beta_2+\alpha}(\mathbb{R}^n)$, and

$$\big\|\big[b,\mathscr{L}^{-\alpha/2}\big](f)\big\|_{\mathscr{M}^{\widetilde{q},\beta_1+\beta_2+\alpha}}\leqslant C\|b\|_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

Here the number $\widetilde{q}$ is the same as above. This fact together with Lemma 3.1 gives us that

$$\begin{aligned}\mathrm{VI}^{(4)} &\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{1}{2^{kn}}\cdot k\Big[m\big(2\mathscr{B}\big)^{(2\beta_1+\beta_2+\alpha)/n}\cdot\|b\|_{\mathscr{C}^{p_1,\beta_1}}\big\|\big[b,\mathscr{L}^{-\alpha/2}\big](f)\big\|_{\mathscr{M}^{\widetilde{q},\beta_1+\beta_2+\alpha}}\Big]\\ &\leqslant C\sum_{k=1}^{\infty}\frac{k}{2^{kn}}\left[\frac{m(2\mathscr{B})}{m(\mathscr{B})}\right]^{\gamma/n}\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}\\ &\leqslant C\sum_{k=1}^{\infty}\frac{k}{2^{kn}}\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}\leqslant C\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.\end{aligned}$$

Similarly, we obtain that

$$\mathrm{VI}^{(3)}\leqslant C\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

Summing up the above estimates, we conclude that

$$\mathrm{VI}\leqslant C\|b\|^{2}_{\mathscr{C}^{p_1,\beta_1}}\|f\|_{\mathscr{M}^{p_2,\beta_2}}.$$

It remains to estimate the last term VII. Applying Lemma 3.2, we obtain that for any $x \in \mathscr{B}$,

$$\begin{aligned}
&\Big|\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x) - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x)\Big| \\
&= \Big|\big(I-e^{-t_{\mathscr{B}}\mathscr{L}}\big)\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x)\Big| \\
&\leqslant \int_{(2\mathscr{B})^{\complement}} \big|\widetilde{K}_{\alpha,t_{\mathscr{B}}}(x,y)\big| \cdot \big|\big[b(y)-b_{2\mathscr{B}}\big]^2 f(y)\big|\,dy \\
&\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}\mathscr{B}\setminus 2^k\mathscr{B}} \frac{1}{|x-y|^{n-\alpha}} \cdot \frac{r_{\mathscr{B}}^2}{|x-y|^2}\big|b(y)-b_{2\mathscr{B}}\big|^2 \cdot \big|f(y)\big|\,dy \\
&\leqslant C\sum_{k=1}^{\infty}\frac{1}{2^{2k}} \cdot \frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^2 \cdot \big|f(y)\big|\,dy.
\end{aligned}$$

Moreover, by using the same arguments as in the proof of Lemma 3.1, we can prove that the following estimate

$$\frac{1}{m(2^{k+1}\mathscr{B})}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^2\big|f(y)\big|\,dy \leqslant C\cdot k^2\Big[m\big(2\mathscr{B}\big)^{(2\beta_1+\beta_2)/n}\cdot\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}\Big]$$

holds for any $f \in \mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$ with $1 < p_2 < n/\alpha$ and $-n/p_2 \leqslant \beta_2 < (-\alpha)$, and $b \in \mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ with $1 < p_1 < \infty$ and $-n/p_1 \leqslant \beta_1 < 0$. This in turn implies that

$$\begin{aligned}
\mathrm{VII} &= \frac{1}{m(\mathscr{B})^{\gamma/n}}\bigg(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}}\Big|\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x) \\
&\quad - e^{-t_{\mathscr{B}}\mathscr{L}}\mathscr{L}^{-\alpha/2}\big(\big[b-b_{2\mathscr{B}}\big]^2 f_2\big)(x)\Big|^q dx\bigg)^{1/q} \\
&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{1}{2^{2k}} \cdot \frac{1}{m(2^{k+1}\mathscr{B})^{1-\alpha/n}}\int_{2^{k+1}\mathscr{B}}\big|b(y)-b_{2\mathscr{B}}\big|^2 \cdot \big|f(y)\big|\,dy \\
&\leqslant \frac{C}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{k^2}{2^{2k}} \cdot m\big(2\mathscr{B}\big)^{(2\beta_1+\beta_2)/n} m\big(2^{k+1}\mathscr{B}\big)^{\alpha/n}\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Therefore, by the facts that $\gamma = 2\beta_1 + \beta_2 + \alpha$ and $\alpha < 2$, we further obtain

$$\begin{aligned}
\mathrm{VII} &\leqslant C\cdot\frac{m(2\mathscr{B})^{\gamma/n}}{m(\mathscr{B})^{\gamma/n}}\sum_{k=1}^{\infty}\frac{k^2}{2^{2k}}\cdot\left[\frac{m(2^{k+1}\mathscr{B})}{m(2\mathscr{B})}\right]^{\alpha/n}\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}} \\
&\leqslant C\sum_{k=1}^{\infty}\frac{k^2}{2^{(2-\alpha)k}}\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}} \\
&\leqslant C\big\|b\big\|^2_{\mathscr{C}^{p_1,\beta_1}}\big\|f\big\|_{\mathscr{M}^{p_2,\beta_2}}.
\end{aligned}$$

Combining the estimates of I, II, III, IV, V, VI with VII, we get the desired result (4.9). Hence, the proof of Theorem 4.3 is complete. □

By induction on m, we can also prove the following general result.

THEOREM 4.5. *Let* $2 \leqslant m \in \mathbb{N}$, $0 < \alpha < \min\{n,2\}$, $1 < p_2 < n/\alpha$ *and* $-n/p_2 \leqslant \beta_2 < (-\alpha)$. *Suppose that* $b \in \mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $m < p_1 < \infty$ *and* $-n/p_1 \leqslant \beta_1 < 0$. *Then for any* $f \in \mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n)$, *there exists a positive constant* $C > 0$ *independent of* b *and* f *such that*

$$\big\| \big[b, \mathscr{L}^{-\alpha/2}\big]^m (f) \big\|_{\mathscr{C}^{q,\gamma}_{\mathscr{L}}} \leqslant C \|b\|^m_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{\mathscr{M}^{p_2,\beta_2}},$$

provided that

$$1/q = m/p_1 + 1/p_2 - \alpha/n \quad \& \quad \gamma = m\beta_1 + \beta_2 + \alpha.$$

In particular, we have

COROLLARY 4.6. *Let* $2 \leqslant m \in \mathbb{N}$ *and* $0 < \alpha < \min\{n,2\}$. *Suppose that* $b \in \mathscr{C}^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $m < p_1 < \infty$ *and* $-n/p_1 \leqslant \beta_1 < 0$. *Then for any* $f \in L^{p_2}(\mathbb{R}^n)$ *with* $1 < p_2 < n/\alpha$, *there exists a positive constant* $C > 0$ *independent of* b *and* f *such that*

$$\big\| \big[b, \mathscr{L}^{-\alpha/2}\big]^m (f) \big\|_{\mathscr{C}^{q,\gamma}_{\mathscr{L}}} \leqslant C \|b\|^m_{\mathscr{C}^{p_1,\beta_1}} \|f\|_{L^{p_2}},$$

provided that

$$1/q = m/p_1 + 1/p_2 - \alpha/n \quad \& \quad \gamma = m\beta_1 + \alpha - n/p_2.$$

In fact, Corollary 4.6 is a straightforward consequence of Theorem 4.5, since $\mathscr{M}^{p_2,\beta_2}(\mathbb{R}^n) = L^{p_2}(\mathbb{R}^n)$ if we take $\beta_2 = -n/p_2$.

We remark that when $q \geqslant 1$ and $2 \leqslant m \in \mathbb{N}$,

$$1/q = m/p_1 + 1/p_2 - \alpha/n \Longrightarrow m < p_1 < \infty.$$
$$\gamma = m\beta_1 + \beta_2 + \alpha \Longrightarrow -n/q \leqslant \gamma < 0.$$

REMARK 4.1. Let $2 \leqslant m \in \mathbb{N}$ and $b(x)$ be a locally integrable function on $\mathbb{R}^n$. The higher-order commutator $\big[b, I_\alpha\big]^m$ generated by I_α and b is defined by

$$\big[b, I_\alpha\big]^m (f)(x) := \big[b, \ldots \big[b, \big[b, I_\alpha\big]\big]\big](f)(x), \quad 0 < \alpha < n.$$

That is, these commutators $\big[b, I_\alpha\big]^m$ $(m = 1, 2, \ldots)$ can be defined by recurrence:

$$\big[b, I_\alpha\big]^m (f) = \big[b, \big[b, I_\alpha\big]^{m-1}(f)\big],$$

where

$$\big[b, I_\alpha\big]^1 (f) := \big[b, I_\alpha\big](f).$$

Then we have

$$\big[b, I_\alpha\big]^m (f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]^m}{|x-y|^{n-\alpha}} f(y)\, dy.$$

Following [26, 27], we will say that a locally integrable function $b(x)$ belongs to the reverse Hölder class RH_∞, if there exists a constant $C > 0$ such that for any ball $\mathscr{B} \subset \mathbb{R}^n$,

$$\sup_{x\in\mathscr{B}} \left|b(x) - b_{\mathscr{B}}\right| \leqslant C\left(\frac{1}{m(\mathscr{B})}\int_{\mathscr{B}} \left|b(x) - b_{\mathscr{B}}\right| dx\right). \tag{4.15}$$

When $L = -\Delta$, from Theorem 4.5 and Corollary 4.6, we can obtain the corresponding results for $\left[b, I_\alpha\right]^m$, which have been proved by Shi and Lu, under the assumption that $b(x)$ satisfies (4.15), see [27, Theorem 1.1]. It should be pointed out that the condition (4.15) assumed on symbol functions has been removed from Theorems 4.3 and 4.5 in this paper. We improve and extend Shi and Lu's result [27] about the higher-order commutator $\left[b, I_\alpha\right]^m$ with $2 \leqslant m \in \mathbb{N}$.

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