ESTIMATES FOR MAXIMAL FOURIER MULTIPLIER
OPERATORS ON $\mathbb{R}^2$ VIA SQUARE FUNCTIONS

SHUICHI SATO

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Abstract. We consider certain Littlewood-Paley square functions on $\mathbb{R}^2$ and prove sharp estimates for them, from which we can deduce L^p boundedness of maximal functions defined by Fourier multipliers of Bochner-Riesz type on $\mathbb{R}^2$. This is a generalization of a result due to A. Carbery 1983.

1. Introduction

Let I be a compact interval in $\mathbb{R}$ and ψ a real valued function in $C^\infty(I)$. Let $a \in C_0^\infty(\mathbb{R}^2)$, $\mathrm{supp}(a) \subset I^\circ \times \mathbb{R}$, where I° denotes the interior of I. Put

$$\sigma_\lambda(\xi) = a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda \qquad (1.1)$$

for $\lambda > 0$, where $r_+ = \max(r,0)$ for $r \in \mathbb{R}$. Define

$$S_R^\lambda f(x) = \int_{\mathbb{R}^2} \sigma_\lambda(R^{-1}\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi, \quad R > 0, \qquad (1.2)$$

where $f \in \mathcal{S}(\mathbb{R}^2)$ (the Schwartz space of infinitely differentiable, rapidly decreasing functions), and

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx$$

is the Fourier transform on $\mathbb{R}^2$ with $\langle x,\xi\rangle = x_1\xi_1 + x_2\xi_2$, $x = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$. Put $S_*^\lambda f = \sup_{R>0}|S_R^\lambda f|$.

We assume that

$(A.1)$ $0 \notin \mathrm{supp}(a)$,

$(A.2)$ $\psi(t) - t\psi'(t) \neq 0$ on I, which means that the curve $\Gamma = \{(t,\psi(t)) : t \in I\}$ does not have a tangent line passing through the origin.

In this note we shall prove the following.

THEOREM 1.1. *Suppose that $\psi'' \neq 0$ on I. Let $\lambda > 0$. Then, there exists a positive constant C_λ such that*

$$\|S_*^\lambda f\|_4 \leqslant C_\lambda \|f\|_4.$$

This is a generalization of a result of Carbery [1]; weighted estimates can be found in [2]. See Sogge [16, Chap. 2] for related results. We refer to [3, 8, 13] for background results.

For the L^2 estimates we have the following.

THEOREM 1.2. *Let $\lambda > 0$. Then,*

$$\|S_*^\lambda f\|_2 \leqslant C_\lambda \|f\|_2.$$

Here we need not assume that $\psi'' \neq 0$ on I.

Interpolating between the estimates in Theorems 1.1 and 1.2, we have the following.

COROLLARY 1.3. *Let ψ satisfy the conditions in Theorem 1.1. Let $\lambda > 0$. Then*

$$\|S_*^\lambda f\|_p \leqslant C_\lambda \|f\|_p, \quad 2 \leqslant p \leqslant 4.$$

Let $\Phi \in C_0^\infty(\mathbb{R})$, $\mathrm{supp}(\Phi) \subset [-1,1]$. Let $b \in C_0^\infty(\mathbb{R}^2)$, $\mathrm{supp}(b) \subset I^\circ \times \mathbb{R}$. We assume that $b = 0$ near the origin. Put

$$\phi(\xi) = \phi^{(\delta)}(\xi) = b(\xi)\Phi(\delta^{-1}(\xi_2 - \psi(\xi_1))) \tag{1.3}$$

with a small number $\delta \in (0, 1/2]$. For appropriate functions h and f, define the Littlewood-Paley function

$$g_h(f) = \left(\int_0^\infty |h_t * f|^2 \frac{dt}{t} \right)^{1/2}, \quad h_t(x) = t^{-2}h(t^{-1}x).$$

Let $\eta = \mathcal{F}^{-1}(\phi^{(\delta)})$. To prove Theorems 1.1 and 1.2, we show the following two theorems on g_η.

THEOREM 1.4. *Let ψ be as in Theorem 1.1. Then we have*

$$\|g_\eta(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta} \right)^\tau \|f\|_4$$

for some $\tau > 0$, where the constant C is independent of Φ provided that $\|(d/dr)^m \Phi\|_\infty \leqslant 1$ for $0 \leqslant m \leqslant 3$.

THEOREM 1.5. *Let ψ be as in Theorem 1.2. Then*

$$\|g_\eta(f)\|_2 \leqslant C\delta^{1/2}\|\Phi\|_\infty\|f\|_2,$$

where the constant C is independent of Φ.

These results can be used to prove the following.

THEOREM 1.6. *Let* $r, \lambda, \theta \in \mathbb{R}$. *Define* $\varphi_{\lambda,\theta}(r) = r^{\lambda} (\log(2 + r^{-1}))^{-\theta}$ *if* $r > 0$ *and* $\varphi_{\lambda,\theta}(r) = 0$ *if* $r \leqslant 0$. *Let* ψ *be as in Theorem* 1.1 *and let* b *be as in* (1.3). *Put* $\Psi_{\lambda,\theta}(\xi) = b(\xi)\varphi_{\lambda,\theta}(\xi_2 - \psi(\xi_1))$. *Let* τ *be as in Theorem* 1.4. *Then we have*

$$\|g_{\mathscr{F}^{-1}(\Psi_{\lambda,\theta})}(f)\|_4 \leqslant C_{\lambda,\theta}\|f\|_4$$

if $\lambda = -1/2$ *and* $\theta > \tau + 1$; *this is also valid for every* $\theta \in \mathbb{R}$ *when* $\lambda > -1/2$.

THEOREM 1.7. *Let* $\varphi_{\lambda,\theta}$ *be as in Theorem* 1.6. *Let* ψ *be as in Theorem* 1.2 *and let* b *be as in* (1.3). *Put* $\Psi_{\lambda,\theta}(\xi) = b(\xi)\varphi_{\lambda,\theta}(\xi_2 - \psi(\xi_1))$. *Then*

$$\|g_{\mathscr{F}^{-1}(\Psi_{\lambda,\theta})}(f)\|_2 \leqslant C_{\lambda,\theta}\|f\|_2$$

if $\lambda = -1/2$ *and* $\theta > 1$; *this also holds true for every* $\theta \in \mathbb{R}$ *when* $\lambda > -1/2$.

Theorems 1.6 and 1.7 will be shown by applying Theorems 1.4 and 1.5, respectively. Considering the case when $\lambda > -1/2$ and $\theta = 0$ and applying interpolation, by Theorems 1.6 and 1.7 we have in particular the following.

COROLLARY 1.8. *Let* $\Psi_{\lambda,\theta}$ *be as in Theorem* 1.6. *Then if* $\lambda > -1/2$, *we have*

$$\|g_{\mathscr{F}^{-1}(\Psi_{\lambda,0})}(f)\|_p \leqslant C_{\lambda}\|f\|_p, \quad p \in [2,4].$$

Theorem 1.6 and a modification will be applied in proving Theorem 1.1. Similarly, Theorem 1.2 can be shown by Theorem 1.7 and a variant. See Stein-Weiss [17, §5, Chap. VII] for the relation between boundedness of square functions and boundedness of maximal Bochner-Riesz type operators. Theorems 1.4 and 1.6 generalize results of [1]. See also [4] and [14] for related results. A result analogous to Theorem 1.6 of the case $\lambda > -1/2$ and $\theta = 0$ is stated in [4] by considering square functions defined with homogeneous functions in a general setting.

In the proofs of the theorems we shall apply arguments of [1]. Also, we shall apply a reasoning in [6], which leads to (3.5) below. See also [5, 9] for background results in the ideas of the proofs. One purpose of this note is to provide proofs of the theorems with computations being accessible based on the condition $(A.2)$ and [13, Lemma 5.3].

By a partition of unity, the proofs of Theorems 1.1, 1.4 and 1.6 may be divided into several cases:

$(B.1)$ $\Gamma \subset (-b,b) \times (c,d)$, $0 < b$, $0 < c < d$, $I = [A,B] \subset (-b,b)$;

$(B.2)$ $\Gamma \subset (-b,b) \times (-d,-c)$, $0 < b$, $0 < c < d$, $I = [A,B] \subset (-b,b)$;

$(B.3)$ $\Gamma \subset (a,b) \times (-d,d)$, $0 < a < b$, $0 < d$, $I = [A,B] \subset (a,b)$;

$(B.4)$ $\Gamma \subset (-b,-a) \times (-d,d)$, $0 < a < b$, $0 < d$, $I = [A,B] \subset (-b,-a)$.

In Section 2 we shall review some preliminary results for Kakeya maximal functions and Littlewood-Paley inequalities. In Section 3 we shall prove Theorems 1.4 and 1.6 under the conditions of case $(B.1)$ with $\Psi' > 0$ on I, where $\Psi(t) = t/\psi(t)$; we note that the condition $\psi(t) \neq t\psi'(t)$ implies that $\Psi'(t) \neq 0$. We shall prove a vector valued inequality for a sequence of Fourier multiplier operators in Section 3 (Proposition 3.7), which will be applied to prove Theorem 1.4. To prove Theorem 1.4 we shall also need some results on geometry of support sets of certain Fourier multipliers, which will be taken from [13]. Theorem 1.6 follows from Theorem 1.4.

Theorem 1.1 will be shown in Section 4 by applying Theorem 1.6 under the conditions of the case $(B.1)$ with $\Psi' > 0$ on I. To apply Theorem 1.6 we need to introduce a homogeneous function of degree one in a cone. The function will be constructed by using the condition $(A, 2)$. (See Corollaries 4.2 and 4.4.)

Theorems 1.1, 1.4 and 1.6 will be shown in Section 5 in their full generalities claimed. The proof of Theorem 1.5 will be given in Section 6. The other results on L^2 boundedness can be shown in the same way as in the case of L^4 boundedness by applying the boundedness of g_η, as we can easily see.

2. Maximal functions and Littlewood-Paley inequalities

Let N be a positive integer and let Ω_N be a set of vectors v in $\mathbb{R}^2$ with $|v| = 1$ such that $\operatorname{card} \Omega_N \leqslant N$. Let $\mathcal{B}_N$ be the set of all rectangles in $\mathbb{R}^2$ one of whose sides is parallel to a vector in Ω_N. Let M_{Ω_N} be the maximal operator defined by

$$M_{\Omega_N} f(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)| \, dy,$$

where $|R|$ denotes the Lebesgue measure of R. Then we have the following (see [18, 11]).

LEMMA 2.1. *There exists a positive constant α independent of N such that*

$$\|M_{\Omega_N} f\|_2 \leqslant C(\log(2N))^\alpha \|f\|_2$$

for some $C > 0$.

Let $P_n = I_n \times \mathbb{R}$, $n \in \mathbb{Z}$ (the set of integers), where $\{I_n\}$ is a sequence of non-overlapping intervals in $\mathbb{R}$ such that $|I_n| = \tau$, $\forall n$. Let $T_m f = \mathcal{F}^{-1}(m\hat{f})$ for a bounded function m. If $m = \chi_E$ for a measurable set E, we also write T_E for T_{χ_E}. Then we have the following result (see [1, Lemma 2]).

LEMMA 2.2. *Let w be a non-negative function on $\mathbb{R}^2$. Let $s > 1$. Then there exists $\Theta > 0$ such that*

$$\int_{\mathbb{R}^2} \sum_{n \in \mathbb{Z}} |T_{P_n} f|^2 w \, dx \leqslant C \left(\frac{1}{s-1} \right)^\Theta \int_{\mathbb{R}^2} |f|^2 (M_1(w^s))^{1/s} \, dx,$$

where M_1 denotes the strong maximal operator and C is a constant independent of τ. This is also true if P_n is replaced by $Q_n = \mathbb{R} \times I_n$.

Proof. Let H be the Hilbert transform on $\mathbb{R}$. Put $M^{(s)}(w) = (M(w^s))^{1/s}$, $s > 1$, where M denotes the Hardy-Littlewood maximal operator on $\mathbb{R}$ and w a non-negative function on $\mathbb{R}$. Then it is known that

$$\int |Hf|^2 M^{(s)}(w)\,dx \leqslant C \left(\frac{1}{s-1}\right)^{\beta_0} \int |f|^2 M^{(s)}(w)\,dx \qquad (2.1)$$

for some $\beta_0 > 0$. It follows that if $J = [a,b]$ is an interval

$$\int |T_J f|^2 M^{(s)}(w)\,dx \leqslant C \left(\frac{1}{s-1}\right)^{2\beta_0} \int |f|^2 M^{(s)}(w)\,dx. \qquad (2.2)$$

This can be seen from (2.1) by the equation

$$T_J = (1/2)\,(I + iN_a HN_{-a})\,(1/2)\,(I - iN_b HN_{-b}),$$

where I denotes the identity operator and $N_a f(x) = e^{2\pi ixa} f(x)$.

We show the lemma with $\Theta = 2\beta_0 + 1$. Let $J_j = [(j-1/2)\tau, (j+1/2)\tau]$, $j \in \mathbb{Z}$. If $\varphi \in C^\infty(\mathbb{R})$, $\mathrm{supp}(\varphi) \subset [-2,2]$, $\varphi = 1$ on $[-1,1]$ and $\widehat{f_j}(\xi) = \varphi(2j - 2\tau^{-1}\xi)\hat{f}(\xi)$, then $T_{J_m} f = T_{J_m} f_m$ and

$$|T_{I_j} f|^2 = \left| \sum_{m:|J_m \cap I_j| \neq 0} T_{J_m \cap I_j} f \right|^2 \leqslant 2 \sum_{m:|J_m \cap I_j| \neq 0} \left| T_{J_m \cap I_j} f \right|^2$$

$$= 2 \sum_{m:|J_m \cap I_j| \neq 0} \left| T_{J_m \cap I_j} f_m \right|^2.$$

Using this and (2.2) with $J_m \cap I_j$ in place of J, we have,

$$\sum_j \int |T_{I_j} f|^2 M^{(s)}(w)\,dx \leqslant C \sum_j \sum_{m:|J_m \cap I_j| \neq 0} \left(\frac{1}{s-1}\right)^{2\beta_0} \int |f_m|^2 M^{(s)}(w)\,dx$$

$$= C \sum_m \sum_{j:|J_m \cap I_j| \neq 0} \left(\frac{1}{s-1}\right)^{2\beta_0} \int |f_m|^2 M^{(s)}(w)\,dx$$

$$\leqslant C \left(\frac{1}{s-1}\right)^{2\beta_0} \sum_m \int |f_m|^2 M^{(s)}(w)\,dx$$

$$\leqslant C \left(\frac{1}{s-1}\right)^{2\beta_0} \int |f|^2 MM^{(s)}(w)\,dx$$

$$\leqslant C \left(\frac{1}{s-1}\right)^{2\beta_0+1} \int |f|^2 M^{(s)}(w)\,dx,$$

where the penultimate inequality follows from the arguments in [7, pp. 297–298] and the last inequality is implied by the well-known estimate for the A_1 constant of $M^{(s)}(w)$. From this the conclusion of the Lemma 2.2 can be derived. $\quad\square$

3. Proofs of Theorems 1.4 and 1.6 under $(B.1)$ with $\Psi' > 0$

In this section we prove Theorems 1.4 and 1.6 under the conditions of $(B.1)$ stated in Section 1 with $\Psi' > 0$. We may assume that $\delta = 2^{-L}$ for some positive integer L. Let

$$W_k = \cup\{(2^n, 2^{n+1}] : n \equiv k \mod L\}, \quad k = 0, 1, \dots, L-1.$$

We have

$$g_h(f) \leqslant \sum_{k=0}^{L-1} g_h^{(k)}(f), \tag{3.1}$$

where

$$g_h^{(k)}(f) = \left(\int_{W_k} |h_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

We write W for W_0 and estimate $g_\eta^{(0)}(f)$. The other functions $g_\eta^{(k)}(f)$ will be estimated similarly.

We assume that δ is a small positive number and that $\delta^{-1/2}$ is an integer. Put $N = c_0 \delta^{-1/2}$, where $c_0 \delta^{-1/2}$ is a positive integer. We assume that $\omega_k = [a_{k-1}, a_k]$, $-b = a_0 < a_1 < \dots < a_N = b$, $a_k - a_{k-1} = \delta^{1/2}$, $1 \leqslant k \leqslant N$, $\cup_{k=1}^N \omega_k = [-b, b]$ and that $H_j = [b_{j-1}, b_j]$, $b_j - b_{j-1} = \delta^{1/2}$, $1 \leqslant j \leqslant N$, $\cup_{j=1}^N H_j = [c, d]$.

Recall that

$$\phi^{(\delta)}(\xi) = \phi(\xi) = b(\xi)\Phi(\delta^{-1}(\xi_2 - \psi(\xi_1))).$$

If δ is small enough, we have

$$\mathrm{supp}(\phi) \subset \left(\bigcup_{\omega_\ell \subset I} \omega_\ell\right) \times \mathbb{R}.$$

Decompose $\phi^{(\delta)}$ as

$$\phi^{(\delta)} = \sum_\ell \zeta_\ell \phi^{(\delta)} + \sum_\ell \widetilde{\zeta}_\ell \phi^{(\delta)} =: \phi_{(1)}^{(\delta)} + \phi_{(2)}^{(\delta)}, \tag{3.2}$$

where $\zeta_\ell, \widetilde{\zeta}_\ell \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(\zeta_\ell) \subset \omega_\ell$, $\mathrm{supp}(\widetilde{\zeta}_\ell) \subset \omega_\ell + (1/2)\delta^{1/2}$ and

$$|(d/ds)^\gamma \zeta_\ell| \leqslant C_\gamma \delta^{-\gamma/2}, \quad |(d/ds)^\gamma \widetilde{\zeta}_\ell| \leqslant C_\gamma \delta^{-\gamma/2}, \quad \forall \gamma \in \mathbb{Z} \cap [0, \infty).$$

We estimate $g_{(i)}(f)$ for $i = 1$, where $g_{(i)}(f) = g_{h_{(i)}}(f)$ with $h_{(i)} = \mathcal{F}^{-1}(\phi_{(i)}^{(\delta)})$, $i = 1, 2$. The function $g_{(2)}(f)$ can be estimated similarly. Also, let $g_{(i)}^{(k)}(f) = g_{h_{(i)}}^{(k)}(f)$, $0 \leqslant k \leqslant L-1$, $i = 1, 2$.

To prove Theorem 1.4, we apply Lemma 5.3 of [13] (see (3.5) below). For this reason it is convenient to assume that $\mathrm{supp}(\Phi) \subset [0, c_*]$, where c_* is a sufficiently

small positive number depending on ψ. The result for Φ with support in $[-1, 1]$ easily follows by change of variables. We divide the intervals $\{\omega_\ell\}$ into 4 families as in [13]:

$$\mathcal{F}_1 = \{\omega_1, \omega_5, \omega_9, \ldots\}, \quad \mathcal{F}_2 = \{\omega_2, \omega_6, \omega_{10}, \ldots\},$$
$$\mathcal{F}_3 = \{\omega_3, \omega_7, \omega_{11}, \ldots\}, \quad \mathcal{F}_4 = \{\omega_4, \omega_8, \omega_{12}, \ldots\}.$$

We focus on the family $\mathcal{F}_1$; the other families can be treated similarly. Let ζ_ℓ, $\omega_\ell \subset I$, be as above. Let $\nu(\ell) = 4(\ell - 1) + 1$. Put

$$s^{(\delta)}(\xi) = \sum_{\ell : \omega_{\nu(\ell)} \subset I} s^{(\delta)}_{\nu(\ell)}(\xi), \quad s^{(\delta)}_\ell(\xi) = \phi(\xi)\zeta_\ell(\xi_1), \quad s^{(\delta,t)}_\ell(\xi) = s^{(\delta)}_\ell(t\xi),$$

$$s^{(\delta,t)}(\xi) = s^{(\delta)}(t\xi).$$

We consider $g^{(k)}_{\mathcal{F}_1}(f) := g^{(k)}_{\mathcal{F}^{-1}(s^{(\delta)})}(f)$, $0 \leqslant k \leqslant L - 1$ and we define $g^{(k)}_{\mathcal{F}_i}(f)$, $2 \leqslant i \leqslant 4$, similarly to $g^{(k)}_{\mathcal{F}_1}(f)$ by using $\mathcal{F}_i$. We prove

$$\|g^{(k)}_{\mathcal{F}_i}(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta}\right)^{(\beta/2)+2\alpha} \|f\|_4, \quad \beta = 3\Theta + 2\beta_0, \tag{3.3}$$

for $1 \leqslant i \leqslant 4$, $0 \leqslant k \leqslant L - 1$, where Θ and β_0 are as in Lemma 2.2 and (2.1), respectively, and α is as in Lemma 2.1. It follows that

$$\|g^{(k)}_{(i)}(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta}\right)^{(\beta/2)+2\alpha} \|f\|_4$$

for $i = 1$. By this and (3.1) we see that

$$\|g_{(i)}(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta}\right)^{(\beta/2)+2\alpha+1} \|f\|_4$$

for $i = 1$. This estimate for $i = 2$ can be shown similarly. Thus we have

$$\|g_\eta(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta}\right)^{(\beta/2)+2\alpha+1} \|f\|_4. \tag{3.4}$$

This will complete the proof of Theorem 1.4 with $\tau = (\beta/2) + 2\alpha + 1$ under the conditions of the case $(B.1)$ with $\Psi' > 0$.

Now we prove (3.3) for $k = 0$ and $i = 1$; the other estimates can be shown similarly. We observe that if $t, u \in W$ and $t > u$, then $t < 2u$ or $u < 2\delta t$. Put

$$\Xi = \{(u,t) \in W \times W : u < t < 2u\}, \quad \Lambda = \{(u,t) \in W \times W : u < 2\delta t\}.$$

By applying the Plancherel theorem we see that

$$
\begin{aligned}
\|g_{\mathcal{F}_1}^{(0)}(f)\|_4^4 &\leqslant 2 \iint_{W \times W, t > u} \int_{\mathbb{R}^2} \left| s^{(\delta,t)} \hat{f} * s^{(\delta,u)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&= 2 \iint_{\Xi} \int_{\mathbb{R}^2} \left| s^{(\delta,t)} \hat{f} * s^{(\delta,u)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&\quad + 2 \iint_{\Lambda} \int_{\mathbb{R}^2} \left| s^{(\delta,t)} \hat{f} * s^{(\delta,u)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&= 2 \iint_{\Xi} \int_{\mathbb{R}^2} \left| \sum_\ell \sum_m s_{\nu(\ell)}^{(\delta,t)} \hat{f} * s_{\nu(m)}^{(\delta,u)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&\quad + 2 \iint_{\Lambda} \int_{\mathbb{R}^2} \left| \sum_\ell s_{\nu(\ell)}^{(\delta,u)} \hat{f} * s^{(\delta,t)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&=: Q_1 + Q_2.
\end{aligned}
$$

Arguing similarly to [13, pp. 319–320] with application of [13, Lemma 5.3] for Q_1, where we need the condition that $\psi'' \neq 0$, we see that

$$
\begin{aligned}
Q_1 + Q_2 &\leqslant C \iint_{\Xi} \int_{\mathbb{R}^2} \sum_\ell \sum_m \left| s_{\nu(\ell)}^{(\delta,t)} \hat{f} * s_{\nu(m)}^{(\delta,u)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u} \\
&\quad + C \iint_{\Lambda} \int_{\mathbb{R}^2} \sum_\ell \left| s_{\nu(\ell)}^{(\delta,u)} \hat{f} * s^{(\delta,t)} \hat{f} \right|^2 d\xi \, \frac{dt}{t} \frac{du}{u}
\end{aligned}
\tag{3.5}
$$

(see also [6] for these arguments). Define

$$
\mathcal{F}(S_\ell^{(\delta,t)} f)(\xi) = s_\ell^{(\delta,t)}(\xi) \hat{f}(\xi), \quad V(f)(x) = \left(\int_0^\infty \sum_\ell |S_\ell^{(\delta,t)} f|^2 \, \frac{dt}{t} \right)^{1/2}.
$$

Applying the Plancherel theorem on the right hand side of (3.5), we see that

$$
\|g_{\mathcal{F}_1}^{(0)}(f)\|_4^4 \leqslant C \int V(f)(x)^4 \, dx + C \int \left(g_{\mathcal{F}_1}^{(0)}(f)(x) V(f)(x) \right)^2 dx,
$$

which implies that

$$
\|g_{\mathcal{F}_1}^{(0)}(f)\|_4 \leqslant C \|V(f)\|_4.
\tag{3.6}
$$

Let $[a_{\ell-2}, a_{\ell+1}] \subset I$. Define

$$
\begin{aligned}
\Delta_\ell &= \{(\xi_1, \xi_2) : \Psi(a_{\ell-1}) \xi_2 \leqslant \xi_1 \leqslant \Psi(a_\ell) \xi_2, \, \xi_2 > 0\} = \{r(t, \psi(t)) : t \in \omega_\ell, r > 0\}, \\
S_h^\ell &= \{(\xi_1, \xi_2) : \psi'(a_{\ell-1}) \xi_1 + h\delta \leqslant \xi_2 \leqslant \psi'(a_{\ell-1}) \xi_1 + (h+1)\delta\}, \quad h \in \mathbb{Z}, \\
P_h^\ell &= S_h^\ell \cap \widetilde{\Delta}_\ell, \quad P_h^{\ell,k,j} = S_h^\ell \cap (\omega_k \times H_j) \cap \widetilde{\Delta}_\ell \cap (\mathbb{R} \times [c,d]), \quad \widetilde{\Delta}_\ell = \cup_{|\ell' - \ell| \leqslant 1} \Delta_{\ell'}.
\end{aligned}
$$

To prove (3.4), we may assume that $\operatorname{supp}(\hat{f}) \subset \cup_{1 \leqslant \ell \leqslant N} \Delta_\ell$. To estimate $V(f)$ we need Lemmas 3.2 and 3.3 below. We first observe the following.

LEMMA 3.1. *It holds that* $\operatorname{supp}(s_\ell^{(\delta)}) \subset \widetilde{\Delta}_\ell$.

Proof. (1) Suppose that $a_{\ell-1} \geqslant 0$. Recall that Φ is supported in $[0, c_*]$, $c_* < 1$. If $\xi \in \operatorname{supp}(s_\ell^{(\delta)}) \setminus \Delta_\ell$, then $\xi = (s, u + \kappa)$ with $s/u = \Psi(a_{\ell-1})$ for some $s \in [a_{\ell-1}, a_\ell]$ and $\kappa \in [0, \delta]$.

Note that $|s/(u + \kappa) - s/u| \leqslant C\delta$ and that $|\Psi(a_{\ell-1}) - \Psi(a_{\ell-2})| \geqslant c\delta^{1/2}$, $s/u = \Psi(a_{\ell-1})$. Thus we see that $s/(u + \kappa) \geqslant \Psi(a_{\ell-2})$ if δ is small enough, and hence $\Psi(a_{\ell-2}) \leqslant s/(u + \kappa) \leqslant \Psi(a_{\ell-1})$, which implies that $\xi = (s, u + \kappa) \in \Delta_{\ell-1}$.

(2) Suppose that $a_\ell \leqslant 0$. If $\xi \in \operatorname{supp}(s_\ell^{(\delta)}) \setminus \Delta_\ell$, then $\xi = (s, u + \kappa)$ with $s/u = \Psi(a_\ell)$ for some $s \in [a_{\ell-1}, a_\ell]$ and $\kappa \in [0, \delta]$. Therefore, arguing as in part (1), we see that $\xi \in \Delta_{\ell+1}$ if δ is small enough.

(3) Suppose that $a_{\ell-1} < 0 < a_\ell$. Then we can see that $\operatorname{supp}(s_\ell^{(\delta)}) \setminus \Delta_\ell = \emptyset$.

Combining results in (1), (2) and (3), we finish the proof of the lemma. $\square$

LEMMA 3.2. *Let* $t \in [1,2]$ *and*

$$\mathcal{F}_{\ell,t} = \{h \in \mathbb{Z} : S_h^\ell \cap \operatorname{supp}(s_\ell^{(\delta,t)}) \neq \emptyset\},$$

$$\mathcal{G}_{\ell,t} = \{(k,j) : (\omega_k \times H_j) \cap \operatorname{supp}(s_\ell^{(\delta,t)}) \neq \emptyset\}.$$

Then $\operatorname{card}(\mathcal{F}_{\ell,t}) \leqslant C$, $\operatorname{card}(\mathcal{G}_{\ell,t}) \leqslant C$ *with a constant* C *independent of* ℓ, t *and*

$$s_\ell^{(\delta,t)} = \sum_{(k,j)\in\mathcal{G}_{\ell,t}} \sum_{h\in\mathcal{F}_{\ell,t}} \chi_{P_h^{\ell,k,j}} s_\ell^{(\delta,t)}.$$

Proof. Since $\operatorname{supp}(s_\ell^{(\delta,t)}) \subset B(t^{-1}(a_{\ell-1}, \psi(a_{\ell-1})), c\delta^{1/2})$ for some $c > 0$, we have $\operatorname{card}(\mathcal{G}_{\ell,t}) \leqslant C$, where $B(\xi, r)$ denotes a ball of radius r centered at ξ.

The slope of the tangent line to the curve $t^{-1}\Gamma$ at $t^{-1}(a_{\ell-1}, \psi(a_{\ell-1}))$ is $\psi'(a_{\ell-1})$. Thus $\operatorname{supp}(s_\ell^{(\delta,t)})$ is contained in a parallelogram centered at $t^{-1}(a_{\ell-1}, \psi(a_{\ell-1}))$ with side lengths $c\delta$ and $c\delta^{1/2}$, where the longer sides are parallel to the vector $(1, \psi'(a_{\ell-1}))$. From this we can deduce that $\operatorname{card}(\mathcal{F}_{\ell,t}) \leqslant C$.

By Lemma 3.1, $\operatorname{supp}(s_\ell^{(\delta,t)}) = t^{-1}\operatorname{supp}(s_\ell^{(\delta)}) \subset t^{-1}\widetilde{\Delta}_\ell = \widetilde{\Delta}_\ell$. Also, $\operatorname{supp}(s_\ell^{(\delta,t)}) \subset \cup_{(k,j)\in\mathcal{G}_{\ell,t}}(\omega_k \times H_j)$, $\operatorname{supp}(s_\ell^{(\delta,t)}) \subset \cup_{h\in\mathcal{F}_{\ell,t}} S_h^\ell$. These observations prove the equation for decomposing $s_\ell^{(\delta,t)}$, since $P_h^{\ell,k,j} = (\omega_k \times H_j) \cap S_h^\ell \cap \widetilde{\Delta}_\ell$ and the sets $P_h^{\ell,k,j}$ are mutually non-overlapping when ℓ is fixed. $\square$

LEMMA 3.3. *Let*

$$E(n,\ell,h) = \{t \in [2^n, 2^{n+1}] : (2^{-n}P_h^\ell) \cap \operatorname{supp}(\phi^{(\delta,t)}) \neq \emptyset\}$$
$$= \{t \in [2^n, 2^{n+1}] : (t2^{-n}P_h^\ell) \cap \operatorname{supp}(\phi^{(\delta)}) \neq \emptyset\},$$

where $\phi^{(\delta,t)}(\xi) = \phi^{(\delta)}(t\xi)$. *Then there exists a constant* C *independent of* n, ℓ, h *such that*

$$\int_0^\infty \chi_{E(n,\ell,h)}(t) \frac{dt}{t} \leqslant C\delta.$$

The proof of this requires the following results; Lemmas 3.4, 3.5 and 3.6 below. In Lemmas 3.4 and 3.5, we consider the cases $(B.i)$, $1 \leqslant i \leqslant 4$, together assuming $(A.2)$.

LEMMA 3.4. *Let I' be a compact interval such that $I' \subset I^\circ$. Then there exists a positive constant B_1 such that*

$$\{(\xi_1, \xi_2) : |\xi_2 - \psi(\xi_1)| \leqslant \delta, \, \xi_1 \in I'\} \subset \cup\{s\Gamma : |1 - s| \leqslant B_1\delta\},$$

if δ is small enough.

Proof. Here we give the proof for the case $(B.1)$. The other cases can be handled similarly. Let $t \in I'$ and $\kappa \in [-\delta, \delta]$. Let $m_0 = \min(\Psi(A), \Psi(B))$, $m_1 = \max(\Psi(A), \Psi(B))$. Then $m_0 + \varepsilon_0 < t/\psi(t) < m_1 - \varepsilon_0$ for some $\varepsilon_0 > 0$. (In the cases $(B.3)$, $(B.4)$, we use $\Psi_*(t) = \psi(t)/t$ instead of Ψ.) Thus, if δ is small enough, we have $t/(\psi(t) + \kappa) \in [m_0, m_1]$. So, by the intermediate value theorem there exists $s \in I = [A, B]$ such that $\Psi(s) = s/\psi(s) = t/(\psi(t) + \kappa)$.

If $s \neq 0$, put $r = t/s = (\psi(t) + \kappa)/\psi(s)$. Then we have $r(s, \psi(s)) = (t, \psi(t) + \kappa)$ and

$$\left| \frac{t}{\psi(t)} - \frac{s}{\psi(s)} \right| = \left| \frac{t\kappa}{\psi(t)(\psi(t) + \kappa)} \right| \leqslant C\delta.$$

Thus we have $|t - s| \leqslant C\delta$, since $c_1 \leqslant |\Psi'| \leqslant c_2$ with positive constants c_1, c_2, which follows by the assumption that $\Psi' \neq 0$ on I. This implies that

$$(s, \psi(s)), (t, \psi(t) + \kappa) \in B((t, \psi(t)), c\delta).$$

It follows that

$$|(s, \psi(s)) - r(s, \psi(s))| = |1 - r||(s, \psi(s))| \leqslant C\delta,$$

and hence $|1 - r| \leqslant C\delta$. This proves the assertion of the lemma when $s \neq 0$.

If $s = 0$, then $t = 0$ and $R(0, \psi(0)) = (0, \psi(0) + \kappa)$ with $R = (\psi(0) + \kappa)/\psi(0) = 1 + \kappa/\psi(0)$, $|\kappa/\psi(0)| \leqslant C\delta$. This applies to the case $s = 0$. $\square$

LEMMA 3.5. *If $r, s > 0$ and $r \neq s$, then $r\Gamma \cap s\Gamma = \emptyset$.*

Proof. We prove the lemma for the case $(B.1)$. The results in the other cases can be shown analogously. The proof is by contradiction. Suppose that $r\Gamma \cap s\Gamma \neq \emptyset$ for some $r, s > 0$ with $r \neq s$. We may assume that $r\Gamma \cap \Gamma \neq \emptyset$ for some $r > 1$. Then $(t, \psi(t)) = r(u, \psi(u))$ for some $t, u \in I$. We see that $u \neq 0$, since if $u = 0$, then $t = 0$ and $\psi(0) = r\psi(0)$, which implies that $r = 1$. So, $u \neq 0$, $u \neq t$ and $\Psi(t) = \Psi(u)$. By Rolle's theorem there exists t_0 between t and u such that $\Psi'(t_0) = 0$. This contradicts the assumption $(A.2)$, since $\Psi'(t) = (\psi(t) - t\psi'(t))/\psi(t)^2$. $\square$

LEMMA 3.6. *Let $\tau \in [1/2, 1]$. If $P_h^\ell \cap (\tau \operatorname{supp}(\phi^{(\delta)})) \neq \emptyset$, then there exists a positive constant D such that*

$$P_h^\ell \subset \cup\{s\tau\Gamma : 1 - D\delta \leqslant s \leqslant 1 + D\delta\}.$$

Proof. We have $\tau^{-1}P_h^{\ell} \cap (\text{supp}(\phi^{(\delta)})) \neq \emptyset$. So by Lemma 3.4, $\tau^{-1}P_h^{\ell} \cap v\Gamma \neq \emptyset$ for some $v \in [1 - B_1\delta, 1 + B_1\delta]$. Thus $u^{-1}P_h^{\ell} \cap \Gamma \neq \emptyset$ with $u = v\tau$. We note that

$$u^{-1}S_h^{\ell} = \{(\xi_1, \xi_2) : \psi'(a_{\ell-1})\xi_1 + u^{-1}h\delta \leqslant \xi_2 \leqslant \psi'(a_{\ell-1})\xi_1 + u^{-1}(h+1)\delta\}.$$

So, there exists $h_1 \in [h, h+1]$ such that, if L_{h_1} is the line defined by the equation $\xi_2 = \psi'(a_{\ell-1})\xi_1 + u^{-1}h_1\delta$, $\Gamma \cap u^{-1}\widetilde{\Delta}_{\ell} \cap L_{h_1} = \Gamma \cap \widetilde{\Delta}_{\ell} \cap L_{h_1} \neq \emptyset$.

Let $l_{a_{\ell-1}}$ be the tangent line to Γ at $(a_{\ell-1}, \psi(a_{\ell-1}))$. Let $\Gamma_{\ell} = \Gamma \cap \widetilde{\Delta}_{\ell} = \{(t, \psi(t)) : t \in [a_{\ell-2}, a_{\ell+1}]\}$ and $l_{a_{\ell-1}}^* = l_{a_{\ell-1}} \cap \widetilde{\Delta}_{\ell}$. Then we see that

$$l_{a_{\ell-1}}^* \subset \Gamma_{\ell}(c_1) := \{(\xi_1, \xi_2) \in \widetilde{\Delta}_{\ell} : |\xi_2 - \psi(\xi_1)| < c_1\delta, |a_{\ell-1} - \xi_1| \leqslant c_1\delta^{1/2}\}$$

for some $c_1 > 0$. Here we give the proof of this. Let (α_k, β_k) be the point of intersection of lines $l_{a_{\ell-1}} + (0, d)$, $|d| \leqslant c\delta$, and $a_k\xi_2 = \psi(a_k)\xi_1$, $k = \ell + 1$, $\ell - 2$. It suffices to show that $|\alpha_k - a_{\ell-1}| \leqslant c\delta^{1/2}$ for some $c > 0$. For this we note that

$$\begin{aligned}
\alpha_k - a_{\ell-1} &= \frac{a_k\psi(a_{\ell-1}) - \psi(a_k)a_{\ell-1} + a_kd}{\psi(a_k) - a_k\psi'(a_{\ell-1})} \qquad\qquad (3.7) \\
&= \frac{(a_k - a_{\ell-1})\psi(a_{\ell-1}) + a_{\ell-1}(\psi(a_{\ell-1}) - \psi(a_k)) + a_kd}{\psi(a_k) - a_k\psi'(a_k) + a_k(\psi'(a_k) - \psi'(a_{\ell-1}))}.
\end{aligned}$$

From this with $d = 0$ we have the inequality claimed, since $|\psi(a_k) - a_k\psi'(a_k)| \geqslant c > 0$ by $(A.2)$.

Similarly we have

$$\Gamma_{\ell} \subset l_{a_{\ell-1}}(c_2) := \{(\xi_1, \xi_2) \in \widetilde{\Delta}_{\ell} : \exists \xi_2' \text{ such that } |\xi_2 - \xi_2'| < c_2\delta, (\xi_1, \xi_2') \in l_{a_{\ell-1}}\}$$

for some $c_2 > 0$. Since $\Gamma \cap \widetilde{\Delta}_{\ell} \cap L_{h_1} \neq \emptyset$ and

$$\Gamma \cap \widetilde{\Delta}_{\ell} \cap L_{h_1} \subset \Gamma_{\ell} \subset l_{a_{\ell-1}}(c_2)$$

and the slopes of the lines L_{h_1} and $l_{a_{\ell-1}}$ are the same, the distance between ξ_2-intercepts of the lines L_{h_1} and $l_{a_{\ell-1}}$ is less than $c\delta$. Thus $u^{-1}S_h^{\ell} \cap \widetilde{\Delta}_{\ell} \subset l_{a_{\ell-1}}(c_3)$ for some $c_3 > 0$ and hence

$$u^{-1}P_h^{\ell} \subset l_{a_{\ell-1}}(c_3) \subset \Gamma_{\ell}(c_4)$$

for some $c_4 > 0$, where the second inclusion relation can be shown by using (3.7). By applying Lemma 3.4 we have

$$\begin{aligned}
P_h^{\ell} &\subset \cup\{v\tau s\Gamma : 1 - c_4B_1\delta \leqslant s \leqslant 1 + c_4B_1\delta\} \\
&= \cup\{\tau s\Gamma : v(1 - c_4B_1\delta) \leqslant s \leqslant v(1 + c_4B_1\delta)\} \\
&\subset \cup\{\tau s\Gamma : (1 - B_1\delta)(1 - c_4B\delta) \leqslant s \leqslant (1 + B_1\delta)(1 + c_4B_1\delta)\} \\
&\subset \cup\{\tau s\Gamma : 1 - D\delta \leqslant s \leqslant 1 + D\delta\}
\end{aligned}$$

for some positive constant D. This completes the proof. $\quad\square$

Proof of Lemma 3.3. We may assume that $P_h^\ell \cap (\cup_{1 \leqslant t \leqslant 2} t^{-1} \operatorname{supp}(\phi^{(\delta)})) \neq \emptyset$. Then we can find $\tau \in [1/2, 1]$ such that $P_h^\ell \cap (\tau \operatorname{supp}(\phi^{(\delta)})) \neq \emptyset$, which implies by Lemma 3.6 that for $t \in [2^n, 2^{n+1}]$

$$t 2^{-n} P_h^\ell \subset \cup \{st 2^{-n} \tau \Gamma : 1 - D\delta \leqslant s \leqslant 1 + D\delta\}.$$

So, if $t \in E(n, \ell, h)$, by Lemmas 3.4 and 3.5,

$$\frac{1 - B_1 \delta}{2^{-n} \tau (1 + D\delta)} \leqslant t \leqslant \frac{1 + B_1 \delta}{2^{-n} \tau (1 - D\delta)}.$$

It follows that

$$\int_0^\infty \chi_{E(n,\ell,h)}(t) \frac{dt}{t} \leqslant \int_{\frac{1 - B_1 \delta}{2^{-n} \tau (1 + D\delta)}}^{\frac{1 + B_1 \delta}{2^{-n} \tau (1 - D\delta)}} \frac{dt}{t} \leqslant \log \frac{1 + B_1 \delta}{1 - B_1 \delta} + \log \frac{1 + D\delta}{1 - D\delta} \leqslant C\delta. \quad \square$$

Let $t \in [2^n, 2^{n+1}]$. Then $2^{-n} t \in [1, 2]$ and by Lemma 3.2 we have

$$\begin{aligned}
s_\ell^{(\delta,t)}(\xi) &= s_\ell^{(\delta, 2^{-n} t)}(2^n \xi) \\
&= \sum_{(k,j) \in \mathcal{G}_{\ell, 2^{-n} t}} \sum_{h \in \mathcal{F}_{\ell, 2^{-n} t}} \chi_{P_h^{\ell,k,j}}(2^n \xi) s_\ell^{(\delta, 2^{-n} t)}(2^n \xi) \\
&= \sum_{(k,j) \in \mathcal{G}_{\ell, 2^{-n} t}} \sum_{h \in \mathcal{F}_{\ell, 2^{-n} t}} \chi_{2^{-n} P_h^{\ell,k,j}}(\xi) s_\ell^{(\delta,t)}(\xi).
\end{aligned}$$

Using this and applying Lemma 3.3, we see that

$$\begin{aligned}
V(f)^2 &= \sum_\ell \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} |S_\ell^{(\delta,t)} f|^2 \frac{dt}{t} \\
&\leqslant C \sum_\ell \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} \sum_{(k,j) \in \mathcal{G}_{\ell, 2^{-n} t}} \sum_{h \in \mathcal{F}_{\ell, 2^{-n} t}} \left| T_{2^{-n} P_h^{\ell,k,j}} S_\ell^{(\delta,t)} f \right|^2 \frac{dt}{t} \\
&\leqslant C \sum_n \sum_k \sum_j \sum_\ell \sum_h \int \chi_{E(n,\ell,h)}(t) \left| T_{2^{-n} P_h^{\ell,k,j}} S_\ell^{(\delta,t)} f \right|^2 \frac{dt}{t} \\
&\leqslant C \sum_{n,k,j,\ell,h} \int \chi_{E(n,\ell,h)}(t) \sup_{t \in [2^n, 2^{n+1}]} \left| S_\ell^{(\delta,t)} T_{2^{-n} P_h^{\ell,k,j}} f \right|^2 \frac{dt}{t} \\
&\leqslant C\delta \sum_{n,k,j,\ell,h} \sup_{t \in [2^n, 2^{n+1}]} \left| S_\ell^{(\delta,t)} T_{2^{-n} P_h^{\ell,k,j}} f \right|^2.
\end{aligned}$$

Thus we have

$$V(f)(x) \leqslant C\delta^{1/2} \left(\sum_{n,k,j,\ell,h} \sup_{t \in [2^n, 2^{n+1}]} \left| S_\ell^{(\delta,t)} T_{2^{-n} P_h^{\ell,k,j}} f \right|^2 \right)^{1/2}. \tag{3.8}$$

Observe that $S_\ell^{(\delta,t)}(f) = K_{\ell,t} * f$ with

$$K_{\ell,t}(x) = \int_{\mathbb{R}^2} \zeta_\ell(t\xi_1)\phi(t\xi)e^{2\pi i \langle x,\xi \rangle}\,d\xi. \tag{3.9}$$

We can see that

$$|(\partial/\partial t_\ell)^r(\partial/\partial n_\ell)^s \zeta_\ell(t\xi_1)\phi(t\xi)| \leqslant C_{r,s}(t^{-1}\delta^{1/2})^{-r}(t^{-1}\delta)^{-s}, \tag{3.10}$$

where $\partial/\partial t_\ell$ and $\partial/\partial n_\ell$ denote differentiations in the directions t_ℓ and n_ℓ, where $t_\ell = (t_\ell^{(1)}, t_\ell^{(2)}) = (1, \psi'(a_{\ell-1}))'$ ($\xi' = |\xi|^{-1}\xi$), $n_\ell = (-t_\ell^{(2)}, t_\ell^{(1)})$, and we recall that $\phi(\xi) = \Phi(\delta^{-1}(\xi_2 - \psi(\xi_1)))b(\xi)$. The estimates (3.10) follow by the observations as in [13, p. 310]. Let $O_\ell \in O(2)$ be such that $O_\ell e_1 = t_\ell$ and $O_\ell e_2 = n_\ell$ with $e_1 = (1,0)$, $e_2 = (0,1)$. Then applying integration by parts in (3.9) and using (3.10), we see that

$$|K_{\ell,t}(O_\ell x)| \leqslant C_{\alpha,\beta} t^{-2}\delta^{3/2}|t^{-1}\delta^{1/2}x_1|^{-\alpha}|t^{-1}\delta x_2|^{-\beta} \tag{3.11}$$

for $\alpha,\beta \in \mathbb{Z} \cap [0,3]$. Taking $(\alpha,\beta) = (3,0),(0,3)$ in (3.11), we have

$$\sup_{t \in [2^n, 2^{n+1}]} |S_\ell^{(\delta,t)}f| \leqslant C \sum_{\nu=0}^\infty 2^{-\nu}|E_{\ell,n,\nu}|^{-1}\chi_{E_{\ell,n,\nu}} * |f|,$$

where

$$E_{\ell,n,\nu} = \{O_\ell x : |x_1| \leqslant 2^\nu 2^n \delta^{-1/2}, |x_2| \leqslant 2^\nu 2^n \delta^{-1}\}.$$

Therefore by (3.8) we see that

$$\|V(f)\|_4 \leqslant C\delta^{1/2}\sum_{\nu=0}^\infty 2^{-\nu}\left\|\left(\sum_{n,k,j,\ell,h}\left||E_{\ell,n,\nu}|^{-1}\chi_{E_{\ell,n,\nu}} * \left|T_{2^{-n}P_h^{\ell,k,j}}f\right|\right|^2\right)^{1/2}\right\|_4. \tag{3.12}$$

Applying Lemmas 2.1 and 2.2, we have the following result which will be used in estimating the L^4 norms on the right hand side of (3.12).

PROPOSITION 3.7. *There exists a constant $C > 0$ such that*

$$\left\|\left(\sum_{n,k,j,\ell,h}\left|T_{2^{-n}P_h^{\ell,k,j}}f\right|^2\right)^{1/2}\right\|_4 \leqslant C\left(\log\frac{1}{\delta}\right)^{(\beta+3\alpha)/2}\|f\|_4, \quad \beta = 3\Theta + 2\beta_0,$$

where Θ, β_0 and α are as in (3.3).

To prove this we also need the following.

LEMMA 3.8. *Let $\omega_k \times H_j \subset [-b,b] \times [c,d]$. Then there exists a constant C independent of k,j such that*

$$\text{card}\{\ell : \widetilde{\Delta}_\ell \cap (\omega_k \times H_j) \neq \emptyset\} \leqslant C.$$

Proof. Let $\Delta_\ell \cap (\omega_k \times H_j) \neq \emptyset$. Then $L_{\alpha_\ell} \cap (\omega_k \times H_j) \neq \emptyset$ for some $\alpha_\ell \in [a_{\ell-1}, a_\ell]$, where $L_{\alpha_\ell} = \{\xi : \xi_1 = \Psi(\alpha_\ell)\xi_2\}$, $\Psi(t) = t/\psi(t)$.

Suppose that $a_{k-1} \geqslant 0$. If $\omega_k = [a_{k-1}, a_k]$, $H_j = [b_{j-1}, b_j]$, then

$$a_{k-1}/b_j \leqslant \Psi(\alpha_\ell) \leqslant a_k/b_{j-1}.$$

If also $\Delta_m \cap (\omega_k \times H_j) \neq \emptyset$, then we take $\alpha_m \in [a_{m-1}, a_m]$ such that

$$a_{k-1}/b_j \leqslant \Psi(\alpha_m) \leqslant a_k/b_{j-1}.$$

Thus

$$|\Psi(\alpha_\ell) - \Psi(\alpha_m)| \leqslant a_k/b_{j-1} - a_{k-1}/b_j = \frac{\delta^{1/2}(b_j + a_{k-1})}{b_j b_{j-1}}. \tag{3.13}$$

Since $\psi'(t) \neq \psi(t)/t$ on I, we see that $|\Psi'(t)| \geqslant c$ on I with a constant $c > 0$. Thus by (3.13) we have

$$c|\alpha_\ell - \alpha_m| \leqslant C\delta^{1/2},$$

which implies that $|\ell - m| \leqslant C$. From this we can derive what is claimed.

Suppose that $a_k \leqslant 0$. Then

$$a_{k-1}/b_{j-1} \leqslant \Psi(\alpha_\ell) \leqslant a_k/b_j.$$

Since

$$a_k/b_j - a_{k-1}/b_{j-1} = \frac{\delta^{1/2}(b_{j-1} - a_{k-1})}{b_j b_{j-1}},$$

arguing as above, we can handle this case.

Suppose that $a_{k-1} < 0 < a_k$. Then

$$a_{k-1}/b_{j-1} \leqslant \Psi(\alpha_\ell) \leqslant a_k/b_{j-1}.$$

Using this we can also get the desired result. $\square$

Proof of Proposition 3.7. Let $\Omega_{N_*} = \{(1, \psi'(a_k))', (a_m, \psi(a_m))' : 0 \leqslant k, m \leqslant N\}$, $N_* = 2(N+1)$. Let w be a bounded, non-negative function with compact support. Then by Lemma 2.2 and Lemma 3.8 we see that

$$\int \sum_{n,k,j,\ell,h} \left| T_{2^{-n}P_h^{\ell,k,j}} f \right|^2 w\, dx$$

$$\leqslant C\left(\frac{1}{s-1}\right)^\Theta \sum_{n,k,j,\ell} \int \left| T_{2^{-n}\Delta_\ell} T_{2^{-n}(\omega_k \times H_j)} T_{2^{-n}(\mathbb{R} \times [c,d])} f \right|^2 (M_{\Omega_{N_*}}(w^s))^{1/s}\, dx$$

$$\leqslant C\left(\frac{1}{s-1}\right)^{\Theta + 2\beta_0} \sum_{n,k,j} \int \left| T_{2^{-n}(\omega_k \times H_j)} T_{2^{-n}(\mathbb{R} \times [c,d])} f \right|^2 (M_{\Omega_{N_*}}^3(w^s))^{1/s}\, dx$$

$$\leqslant C\left(\frac{1}{s-1}\right)^{2\Theta + 2\beta_0} \sum_{n,k} \int \left| T_{2^{-n}(\omega_k \times \mathbb{R})} T_{2^{-n}(\mathbb{R} \times [c,d])} f \right|^2 (M_1 M_{\Omega_{N_*}}^3(w^s))^{1/s}\, dx$$

$$\leqslant C\left(\frac{1}{s-1}\right)^{3\Theta + 2\beta_0} \sum_n \int \left| T_{2^{-n}(\mathbb{R} \times [c,d])} f \right|^2 (M_1^2 M_{\Omega_{N_*}}^3(w^s))^{1/s}\, dx,$$

where the second inequality follows by Lemma 3.8 and estimates shown similarly to (2.2) by applying (2.1), and the other inequalities are derived from Lemma 2.2.

Thus, applying the Schwarz inequality, we have

$$\int \sum_{n,k,j,\ell,h} \left| T_{2^{-n} P_h^{\ell,k,j}} f \right|^2 w\, dx \tag{3.14}$$

$$\leqslant C\left(\frac{1}{s-1}\right)^{3\Theta+2\beta_0} \left\| \left(\sum_n \left| T_{2^{-n}(\mathbb{R}\times[c,d])} f \right|^2 \right)^{1/2} \right\|_4^2 \left\| (M_\mathrm{I}^2 M_{\Omega_{N_*}}^3 (w^s))^{1/s} \right\|_2$$

$$\leqslant C\left(\frac{1}{s-1}\right)^{3\Theta+2\beta_0} \|f\|_4^2 \left\| (M_\mathrm{I}^2 M_{\Omega_{N_*}}^3 (w^s))^{1/s} \right\|_2,$$

where the last inequality follows by the Littlewood-Paley estimates.

Let $\beta = 3\Theta + 2\beta_0$. We estimate $A_0 := (1/(s-1))^\beta \left\| (M_\mathrm{I}^2 M_{\Omega_{N_*}}^3 (w^s))^{1/s} \right\|_2$ as follows. Let $s = 1 + (\log N)^{-1}$. Interpolating between the estimates $\|M_{\Omega_{N_*}}(f)\|_{4/3} \leqslant CN\|f\|_{4/3}$ and $\|M_{\Omega_{N_*}}(f)\|_2 \leqslant C(\log N)^\alpha \|f\|_2$ (Lemma 2.1), we have $\|M_{\Omega_{N_*}}(f)\|_{2/s} \leqslant C_N \|f\|_{2/s}$, where $C_N \leqslant CN^{1-\theta}(\log N)^{\theta\alpha}$ with $1-\theta = 2(\log N)^{-1}$, which implies that

$$C_N \leqslant CN^{2(\log N)^{-1}} (\log N)^{\alpha(1-2(\log N)^{-1})} \leqslant C(\log N)^\alpha.$$

Also, $(1/(s-1))^\beta = (\log N)^\beta$. Thus

$$A_0 \leqslant C(\log N)^\beta \left\| M_{\Omega_{N_*}}^3 (w^s) \right\|_{2/s}^{1/s}$$

$$\leqslant C(\log N)^\beta (\log N)^{3\alpha/s} \|w\|_2$$

$$\leqslant C(\log N)^{\beta+3\alpha} \|w\|_2.$$

Taking the supremum in (3.14) over w with $\|w\|_2 \leqslant 1$, we have the conclusion of Proposition 3.7. $\quad\square$

Take a non-negative $w \in C_0^\infty(\mathbb{R}^2)$. By Lemma 2.1 and Proposition 3.7 we see that

$$\int \sum_{n,k,j,\ell,h} \left| |E_{\ell,n,v}|^{-1} \chi_{E_{\ell,n,v}} * \left| T_{2^{-n} P_h^{\ell,k,j}} f \right| (x) \right|^2 w(x)\, dx$$

$$\leqslant \int \sum_{n,k,j,\ell,h} |E_{\ell,v}|^{-1} \chi_{E_{\ell,n,v}} * \left| T_{2^{-n} P_h^{\ell,k,j}} f \right|^2 (x) w(x)\, dx$$

$$= \int \sum_{n,k,j,\ell,h} \left| T_{2^{-n} P_h^{\ell,k,j}} f(y) \right|^2 |E_{\ell,n,v}|^{-1} \chi_{E_{\ell,n,v}} * w(y)\, dy$$

$$\leqslant \left\| \left(\sum_{n,k,j,\ell,h} \left| T_{2^{-n} P_h^{\ell,k,j}} f \right|^2 \right)^{1/2} \right\|_4^2 \|M_{\Omega_{N_*}} w\|_2$$

$$\leqslant C\left(\log \frac{1}{\delta}\right)^{\beta+4\alpha} \|f\|_4^2 \|w\|_2.$$

Taking the supremum over w with $\|w\|_2 \leqslant 1$, we have

$$\left\| \left(\sum_{n,k,j,\ell,h} \left| |E_{\ell,n,\nu}|^{-1} \chi_{E_{\ell,n,\nu}} * \left| T_{2^{-n}P_h^{\ell,k,j}} f \right| \right|^2 \right)^{1/2} \right\|_4 \leqslant C \left(\log \frac{1}{\delta} \right)^{(\beta/2)+2\alpha} \|f\|_4.$$

Using this in (3.12), we see that

$$\|V(f)\|_4 \leqslant C\delta^{1/2} \left(\log \frac{1}{\delta} \right)^{(\beta/2)+2\alpha} \|f\|_4.$$

By (3.6) this proves (3.3) for $k=0$ and $i=1$. As we have already seen, this will lead to the estimate (3.4). This completes the proof of Theorem 1.4 under $(B.1)$ with $\Psi' > 0$.

  Proof of Theorem 1.6 *in the case* $(B.1)$ *with* $\Psi' > 0$. Let $\phi_0 \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$ and $\sum_{n=0}^\infty \phi_0(2^n t) = 1$ for $0 < t < 1$. Decompose

$$\begin{aligned}
\Psi_{\lambda,\theta}(\xi) &= b(\xi)\varphi_{\lambda,\theta}(\xi_2 - \psi(\xi_1)) \\
&= r(\xi) + \sum_{n=1}^\infty b(\xi)\varphi_{\lambda,\theta}(\xi_2 - \psi(\xi_1))\phi_0(2^n(\xi_2 - \psi(\xi_1))) \\
&= r(\xi) + \sum_{n=1}^\infty 2^{-n\lambda} n^{-\theta} b(\xi) \left(2^{n\lambda} n^\theta \varphi_{\lambda,\theta}(\xi_2 - \psi(\xi_1)) \right) \phi_0(2^n(\xi_2 - \psi(\xi_1))) \\
&= r(\xi) + \sum_{n=1}^\infty 2^{-n\lambda} n^{-\theta} b(\xi)\varphi_{n,\lambda,\theta}(2^n(\xi_2 - \psi(\xi_1))),
\end{aligned}$$

where $r \in C_0^\infty(\mathbb{R}^2)$, $0 \notin \operatorname{supp}(r)$, and $\varphi_{n,\lambda,\theta}(u) = u^\lambda n^\theta (\log(2 + 2^n/u))^{-\theta} \phi_0(u)$. Put $\Psi_{n,\lambda,\theta}(\xi) = b(\xi)\varphi_{n,\lambda,\theta}(2^n(\xi_2 - \psi(\xi_1)))$. Applying Theorem 1.4 and a triangle inequality, if λ and θ satisfy the conditions of Theorem 1.6, we see that

$$\begin{aligned}
\|g_{\mathscr{F}^{-1}(\Psi_{\lambda,\theta})}(f)\|_4 &\leqslant \|g_{\mathscr{F}^{-1}(r)}(f)\|_4 + \sum_{n=1}^\infty 2^{-n\lambda} n^{-\theta} \|g_{\mathscr{F}^{-1}(\Psi_{n,\lambda,\theta})}(f)\|_4 \\
&\leqslant C\|f\|_4 + C \sum_{n=1}^\infty 2^{-n\lambda} n^{-\theta} n^\tau 2^{-n/2} \|f\|_4 \\
&\leqslant C\|f\|_4.
\end{aligned}$$

This completes the proof of Theorem 1.6 for the case $(B.1)$ with $\Psi' > 0$. $\square$

4. Proof of Theorem 1.1 for case $(B.1)$ with $\Psi' > 0$

  In this section we prove Theorem 1.1 under the conditions that $\Gamma \subset (-b, b) \times (c, d)$, $0 < b$, $0 < c < d$, $I = [A, B] \subset (-b, b)$ and $\Psi' > 0$.
  We first prepare some results involving homogeneous functions of degree one in a cone, which will be used in what follows. Let H be an interval in $\mathbb{R}$ with $|H| < \pi$. Define a cone C_H by

$$C_H = \{r(\cos\theta, \sin\theta) : \theta \in H, r > 0\}.$$

For an appropriate function f, let

$$B_R^\lambda f(x) = \int_{\rho(\xi)<R} \hat{f}(\xi) b(R^{-1}\xi)(1 - R^{-1}\rho(\xi))_+^\lambda e^{2\pi i \langle x,\xi\rangle}\, d\xi$$

$$= R^{-\lambda} \int_{\rho(\xi)<R} \hat{f}(\xi) b(R^{-1}\xi)(R - \rho(\xi))_+^\lambda e^{2\pi i \langle x,\xi\rangle}\, d\xi,$$

where ρ is a non-negative function on $\mathbb{R}^2$ and $b \in C_0^\infty(\mathbb{R}^2)$ such that $0 \notin \text{supp}(b) \subset C_H$ with an open interval H, $\text{supp}(b) \subset \{0 < r_1 < |\xi| < r_2\}$. We assume that $\rho(t\xi) = t\rho(\xi)$ if $\xi \in C_H$ and $t > 0$. Then we have the following (see [17, §5, Chap. VII]) and also [12, Lemma 4] for relevant results).

LEMMA 4.1. *If $\beta > 1/2$ and $\delta > -1/2$, then*

$$\left| B_R^{\delta+\beta} f(x) \right| \leqslant C_{\delta,\beta} \left(\int_0^1 (1-s)^{2(\beta-1)} s^{2\delta}\, ds \right)^{1/2} \left(R^{-1} \int_0^R \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{B}_s^\delta f(x) \right|^2 ds \right)^{1/2}.$$

Here $C_{\delta,\beta} = \Gamma(\delta + \beta + 1)/(\Gamma(\delta+1)\Gamma(\beta))$ and

$$\widetilde{B}_s^\delta f(x) = \int_{\rho(\xi)<s} \hat{f}(\xi) \widetilde{b}(s^{-1}\xi)(1 - s^{-1}\rho(\xi))_+^\delta e^{2\pi i \langle x,\xi\rangle}\, d\xi,$$

where $\widetilde{b}$ is a function in $C^\infty(\mathbb{R}^2)$ such that $0 \notin \text{supp}(\widetilde{b}) \subset C_H$ and such that there exists a compact subinterval H' of H for which we have $\text{supp}(b) \subset C_{H'}$ and $\widetilde{b}(\xi) = 1$ for all $\xi \in C_{H'}$ with $|\xi| \geqslant r_1$.

Proof. Using the formula

$$(R - \rho(\xi))_+^{\delta+\beta} = C_{\delta,\beta} \int_0^{(R-\rho(\xi))_+} ((R-\rho(\xi))_+ - u)^{\beta-1} u^\delta\, du,$$

we have

$$B_R^{\delta+\beta} f(x)$$
$$= C_{\delta,\beta} R^{-\delta-\beta} \int_{\rho(\xi)<R} \int_0^{(R-\rho(\xi))_+} ((R-\rho(\xi))_+ - u)^{\beta-1} u^\delta\, du\, b(R^{-1}\xi) \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}\, d\xi.$$

Changing variables $u = s - \rho(\xi)$, we see that this is equal to

$$C_{\delta,\beta} R^{-\delta-\beta} \int_{\rho(\xi)<R} \int_{\rho(\xi)}^R (R-s)^{\beta-1}(s-\rho(\xi))^\delta\, ds\, b(R^{-1}\xi) \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}\, d\xi$$

$$= C_{\delta,\beta} R^{-\delta-\beta} \int_0^R \int_{\rho(\xi)<s} (s-\rho(\xi))^\delta b(R^{-1}\xi) \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}\, d\xi\, (R-s)^{\beta-1}\, ds$$

$$= C_{\delta,\beta} R^{-\delta-\beta}$$
$$\times \int_0^R (R-s)^{\beta-1} s^\delta \int_{\rho(\xi)<s} (1 - \rho(s^{-1}\xi))^\delta \widetilde{b}(s^{-1}\xi) b(R^{-1}\xi) \hat{f}(\xi) e^{2\pi i \langle x,\xi\rangle}\, d\xi\, ds$$

$$= C_{\delta,\beta} R^{-\delta-\beta} \int_0^R (R-s)^{\beta-1} s^\delta (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{B}_s^\delta f(x)\, ds.$$

Thus by applying the Schwarz inequality, we have

$$\left| B_R^{\delta+\beta} f(x) \right|$$

$$\leqslant C_{\delta,\beta} R^{-\delta-\beta} \left(\int_0^R (R-s)^{2(\beta-1)} s^{2\delta}\, ds \right)^{1/2} \left(\int_0^R \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{B}_s^{\delta} f(x) \right|^2 ds \right)^{1/2}$$

$$= C_{\delta,\beta} R^{-1/2} \left(\int_0^1 (1-s)^{2(\beta-1)} s^{2\delta}\, ds \right)^{1/2} \left(\int_0^R \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{B}_s^{\delta} f(x) \right|^2 ds \right)^{1/2}.$$

This completes the proof of Lemma 4.1. $\square$

COROLLARY 4.2. *Let* β, δ, ρ, b, $\widetilde{b}$, B_R^{λ} *and* $\widetilde{B}_R^{\lambda}$ *be as in Lemma* 4.1. *Let* $B_*^{\lambda} f(x) = \sup_{R>0} |B_R^{\lambda} f(x)|$ *and* $\psi^{\delta}(\xi) = \widetilde{b}(\xi)(1-\rho(\xi))_+^{\delta}$. *Then*

$$B_*^{\delta+\beta} f(x) \leqslant C M(g_{\mathcal{F}^{-1}(\psi^{\delta})} f)(x),$$

where M denotes the Hardy-Littlewood maximal operator.

Proof. By Lemma 4.1 and Minkowski's inequality we have

$$\left| B_R^{\delta+\beta} f(x) \right|$$

$$\leqslant C_{\delta,\beta} \left(\int_0^1 (1-s)^{2(\beta-1)} s^{2\delta}\, ds \right)^{1/2} \left| (\mathcal{F}^{-1}b)_{R^{-1}} \right| * \left(R^{-1} \int_0^R \left| \widetilde{B}_s^{\delta} f(\cdot) \right|^2 ds \right)^{1/2}(x).$$

Thus

$$B_*^{\delta+\beta} f(x) \leqslant C \sup_{R>0} \left(\left| (\mathcal{F}^{-1}b)_{R^{-1}} \right| * g_{\mathcal{F}^{-1}(\psi^{\delta})} f(x) \right) \leqslant C M(g_{\mathcal{F}^{-1}(\psi^{\delta})} f)(x).   \square$$

We also consider

$$C_R^{\lambda} f(x) = \int_{\rho(\xi)>R} \hat{f}(\xi) b(R^{-1}\xi)(R^{-1}\rho(\xi)-1)_+^{\lambda} e^{2\pi i \langle x,\xi \rangle}\, d\xi$$

$$= R^{-\lambda} \int_{\rho(\xi)>R} \hat{f}(\xi) b(R^{-1}\xi)(\rho(\xi)-R)_+^{\lambda} e^{2\pi i \langle x,\xi \rangle}\, d\xi,$$

where ρ and $b \in C_0^{\infty}(\mathbb{R}^2)$ are as in the definition of B_R^{λ}. Then we have the following.

LEMMA 4.3. *Let* $\beta > 1/2$, $\delta > -1/2$. *Suppose that* $c_0 = \sup_{\xi' \in C_H} \rho(\xi') < \infty$, *with* $\xi' = \xi/|\xi|$. *Put* $d_0 = c_0 r_2$. *Then,* $C_R^{\delta+\beta} f = 0$ *if* $d_0 < 1$ *and if* $d_0 \geqslant 1$ *we have*

$$\left| C_R^{\delta+\beta} f(x) \right|$$

$$\leqslant C_{\delta,\beta} \left(\int_1^{d_0} (s-1)^{2(\beta-1)} s^{2\delta}\, ds \right)^{1/2} \left(R^{-1} \int_R^{d_0 R} \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{C}_s^{\delta} f(x) \right|^2 ds \right)^{1/2},$$

where the constant $C_{\delta,\beta}$ is as in Lemma 4.1 *and*

$$\widetilde{C}_s^{\delta} f(x) = \int_{\rho(\xi)>s} \hat{f}(\xi)\widetilde{b}(s^{-1}\xi)(s^{-1}\rho(\xi)-1)_+^{\delta} e^{2\pi i\langle x,\xi\rangle} \, d\xi,$$

where $\widetilde{b}$ is a function in $C_0^{\infty}(\mathbb{R}^2)$ such that $0 \notin \mathrm{supp}(\widetilde{b}) \subset C_H$ and such that there exists a compact subinterval H' of H for which we have $\mathrm{supp}(b) \subset C_{H'}$ and $\widetilde{b}(\xi) = 1$ if $d_0^{-1}r_1 \leqslant |\xi| \leqslant r_2$ and $\xi \in C_{H'}$.

Proof. As in the case of B_R^{λ}, using the formula

$$(\rho(\xi) - R)_+^{\delta+\beta} = C_{\delta,\beta} \int_0^{(\rho(\xi)-R)_+} ((\rho(\xi) - R)_+ - u)^{\beta-1} u^{\delta} \, du,$$

we have

$$C_R^{\delta+\beta} f(x)$$
$$= C_{\delta,\beta} R^{-\delta-\beta} \int_{\rho(\xi)>R} \int_0^{(\rho(\xi)-R)_+} ((\rho(\xi)-R)_+ - u)^{\beta-1} u^{\delta} \, du \, b(R^{-1}\xi)\hat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} \, d\xi.$$

Changing variables $u = \rho(\xi) - s$, we see that this is equal to

$$C_{\delta,\beta} R^{-\delta-\beta} \int_{\rho(\xi)>R} \int_R^{\rho(\xi)} (s-R)^{\beta-1}(\rho(\xi)-s)^{\delta} \, ds \, b(R^{-1}\xi)\hat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} \, d\xi$$

$$= C_{\delta,\beta} R^{-\delta-\beta} \int_R^{d_0 R} \int_{\rho(\xi)>s} (\rho(\xi)-s)^{\delta} b(R^{-1}\xi)\hat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} \, d\xi \, (s-R)^{\beta-1} \, ds$$

$$= C_{\delta,\beta} R^{-\delta-\beta}$$
$$\times \int_R^{d_0 R} (s-R)^{\beta-1} s^{\delta} \int_{\rho(\xi)>s} (\rho(s^{-1}\xi)-1)^{\delta}\widetilde{b}(s^{-1}\xi) b(R^{-1}\xi)\hat{f}(\xi) e^{2\pi i\langle x,\xi\rangle} \, d\xi \, ds$$

$$= C_{\delta,\beta} R^{-\delta-\beta} \int_R^{d_0 R} (s-R)^{\beta-1} s^{\delta} (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{C}_s^{\delta} f(x) \, ds.$$

By the Schwarz inequality, it follows that

$$\left| C_R^{\delta+\beta} f(x) \right|$$

$$\leqslant C_{\delta,\beta} R^{-\delta-\beta} \left(\int_R^{d_0 R} (s-R)^{2(\beta-1)} s^{2\delta} \, ds \right)^{1/2} \left(\int_R^{d_0 R} \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{C}_s^{\delta} f(x) \right|^2 \, ds \right)^{1/2}$$

$$= C_{\delta,\beta} R^{-1/2} \left(\int_1^{d_0} (s-1)^{2(\beta-1)} s^{2\delta} \, ds \right)^{1/2} \left(\int_R^{d_0 R} \left| (\mathcal{F}^{-1}b)_{R^{-1}} * \widetilde{C}_s^{\delta} f(x) \right|^2 \, ds \right)^{1/2}.$$

This completes the proof of Lemma 4.3. □

COROLLARY 4.4. *Let* β, δ, ρ, b, $\widetilde{b}$, C_R^λ *and* $\widetilde{C}_R^\lambda$ *be as in Lemma* 4.3. *Let* $C_*^\lambda f(x) = \sup_{R>0} |C_R^\lambda f(x)|$. *Put* $\varphi^\delta(\xi) = \widetilde{b}(\xi)(\rho(\xi) - 1)_+^\delta$. *Then*

$$C_*^{\delta+\beta} f(x) \leqslant CM(g_{\mathcal{F}^{-1}(\varphi^\delta)} f)(x),$$

where M denotes the Hardy-Littlewood maximal operator as above.

Proof. As in the proof of Corollary 4.2, by Lemma 4.3 and Minkowski's inequality we see that

$$\left| C_R^{\delta+\beta} f(x) \right|$$

$$\leqslant C_{\delta,\beta} \left(\int_1^{d_0} (s-1)^{2(\beta-1)} s^{2\delta} \, ds \right)^{1/2} \left| (\mathcal{F}^{-1} b)_{R^{-1}} \right| * \left(R^{-1} \int_R^{d_0 R} \left| \widetilde{C}_s^\delta f(\cdot) \right|^2 \, ds \right)^{1/2} (x).$$

Thus

$$C_*^{\delta+\beta} f(x) \leqslant C_{d_0} \sup_{R>0} \left(\left| (\mathcal{F}^{-1} b)_{R^{-1}} \right| * g_{\mathcal{F}^{-1}(\varphi^\delta)} f(x) \right) \leqslant CM(g_{\mathcal{F}^{-1}(\varphi^\delta)} f)(x). \quad \square$$

Now we can start the proof of the theorem in the situation stated. For $a_1 < a_2$ let

$$C(a_1, a_2) = \{\xi \in \mathbb{R}^2 \setminus \{0\} : a_1 \xi_2 < \xi_1 < a_2 \xi_2, \xi_2 > 0\}.$$

Take intervals $I_1 = [\sigma, \tau]$, $I_2 = [\sigma', \tau']$, $I_3 = [\sigma'', \tau'']$, $-b < \sigma'' < \sigma' < \sigma < \tau < \tau' < \tau'' < b$ such that $\text{supp}(\sigma_\lambda) \subset I_1 \times \mathbb{R}$ and $\Psi' > 0$ on I_3. Let a function ρ be defined on $C(a_1'', a_2'')$, $a_1'' = \sigma'' / \psi(\sigma'')$, $a_2'' = \tau'' / \psi(\tau'')$, by

$$\rho(u, v) = \frac{u}{\Psi^{-1}(\frac{u}{v})},$$

if $u \neq 0$, where $\Psi : I_3 \to J := \Psi(I_3)$, $\Psi^{-1} : J \to I_3$; also let $\rho(0, v) = v \Psi'(0) = v/\psi(0)$ if $0 \in I_3$. We note that ρ is non-negative and homogeneous of degree one on $C(a_1'', a_2'')$, and $\rho(u, \psi(u)) = 1$ for $u \in I_3$. We also note that $\Psi^{-1}(s) = (\Psi'(0))^{-1} s + O(s^2)$ near $s = 0$ when $0 \in J$. We can see that ρ is infinitely differentiable in $C(a_1'', a_2'')$. Further, by taking account of a suitable partition of unity, we may assume that $\text{supp}(a) \subset C(a_1, a_2) \cap (I_1 \times \mathbb{R})$, $a_1 = \sigma / \psi(\sigma)$, $a_2 = \tau / \psi(\tau)$, where a is as in Theorem 1.1 (see (1.1)).

We have

$$\widetilde{a}(\xi)(\rho(\xi) - 1)_+^\lambda = a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda \tag{4.1}$$

for some $\widetilde{a} \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(a) = \text{supp}(\widetilde{a})$. This can be seen as follows. Since

$\partial_2 \rho > 0$ on $C(a_1'', a_2'')$, where $\partial_2 = \partial/\partial v$, for $\xi \in \mathrm{supp}(a)$ we see that

$$a(\xi)(\rho(\xi) - 1)_+^\lambda = a(\xi)(\rho(\xi) - \rho(\xi_1, \psi(\xi_1)))_+^\lambda$$
$$= a(\xi) \left(\int_0^1 \frac{\partial}{\partial t} \rho(\xi_1, t(\xi_2 - \psi(\xi_1)) + \psi(\xi_1)) \, dt \right)_+^\lambda$$
$$= a(\xi) \left((\xi_2 - \psi(\xi_1)) \int_0^1 \partial_2 \rho(\xi_1, t(\xi_2 - \psi(\xi_1)) + \psi(\xi_1)) \, dt \right)_+^\lambda$$
$$= a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda \left(\int_0^1 \partial_2 \rho(\xi_1, t(\xi_2 - \psi(\xi_1)) + \psi(\xi_1)) \, dt \right)^\lambda$$
$$= a(\xi)g(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda$$

for some $g \in C_0^\infty(\mathbb{R}^2)$ such that $g \neq 0$ on $\mathrm{supp}(a)$. This implies (4.1) by setting $\widetilde{a} = a/g$.

Thus to prove Theorem 1.1 it suffices to show that $\|T_*^\lambda(f)\|_4 \leqslant C\|f\|_4$, where

$$T_*^\lambda(f)(x) = \sup_{R>0} \left| \int \widetilde{a}(R^{-1}\xi)(\rho(R^{-1}\xi) - 1)_+^\lambda \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi \right|.$$

We have $T_*^\lambda(f) \leqslant CM(g_{\mathscr{F}^{-1}(\varphi^\delta)}(f))$ with some $\delta > -1/2$ by Corollary 4.4, where $\varphi^\delta(\xi) = A(\xi)(\rho(\xi) - 1)_+^\delta$ and A is related to $\widetilde{a}$ as $\widetilde{b}$ is related to b in Corollary 4.4; also we can assume that $\mathrm{supp}(A) \subset C(a_1, a_2)$. Therefore, to prove Theorem 1.1 it suffices to show that $\|g_{\mathscr{F}^{-1}(\varphi^\delta)}(f)\|_4 \leqslant C\|f\|_4$ for $\delta > -1/2$.

Let $b_1(\xi_1) \in C_0^\infty(\mathbb{R})$ satisfy that $b_1 = 1$ on I_2 and $\mathrm{supp}(b_1) \subset I_3$. Decompose $A(\xi) = b_1(\xi_1)A(\xi) + b_2(\xi_1)A(\xi) =: A_1(\xi) + A_2(\xi)$, where $b_2 = 1 - b_1$. We observe that the function $B_2(\xi) := A_2(\xi)(\rho(\xi) - 1)_+^\delta$ belongs to $C_0^\infty(\mathbb{R}^2)$, since $|\rho(\xi) - 1| \geqslant c > 0$ on $\mathrm{supp}(B_2)$, and vanishes near the origin. Thus $g_{\mathscr{F}^{-1}(B_2)}$ is bounded on L^p, $1 < p < \infty$. Since $\mathrm{supp}(A_1) \subset C(a_1, a_2) \cap (I_3 \times \mathbb{R})$, in the same way as in (4.1) we see that

$$B_1(\xi) := A_1(\xi)(\rho(\xi) - 1)_+^\delta = \widetilde{A_1}(\xi)(\xi_2 - \psi(\xi_1))_+^\delta,$$

where $\widetilde{A_1} \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(A_1) = \mathrm{supp}(\widetilde{A_1})$. Therefore, applying Theorem 1.6 for the case $\lambda > -1/2$ and $\theta = 0$, we have the boundedness of $g_{\mathscr{F}^{-1}(B_1)}$, which concludes the proof of Theorem 1.1 in the case $(B.1)$ with $\Psi' > 0$.

5. Proofs of Theorems 1.1, 1.4, 1.6 in full generality

First we prove Theorem 1.4 under the general conditions stated in the theorem. We have already proved the theorem in the case $(B.1)$ when $\Psi' > 0$ on I.

Proof of Theorem 1.4 for the case $(B.1)$ with $\Psi' < 0$ on I. We argue similarly to the case $\Psi' > 0$. Recall that $\omega_k = [a_{k-1}, a_k]$, $a_k - a_{k-1} = \delta^{1/2}$, $\cup \omega_k = [-b, b]$, $H_j = [b_{j-1}, b_j]$, $b_j - b_{j-1} = \delta^{1/2}$, $\cup H_j = [c, d]$. Define, for ℓ with $[a_{\ell-2}, a_{\ell+1}] \subset I = [A, B]$,

$$\Delta_\ell = \{(\xi_1, \xi_2) : \Psi(a_\ell)\xi_2 \leqslant \xi_1 \leqslant \Psi(a_{\ell-1})\xi_2, \, \xi_2 > 0\}, \quad \widetilde{\Delta}_\ell = \cup_{|\ell' - \ell| \leqslant 1} \Delta_{\ell'}$$

and recall S_h^ℓ. Also define P_h^ℓ, $P_h^{\ell,k,j}$ as in Section 3 by using Δ_ℓ.

Decompose $\phi^{(\delta)}$ as in (3.2). Arguing as in Section 3, by considering $g_{\mathcal{F}_i}^{(k)}(f)$, $1 \leqslant i \leqslant 4$, $0 \leqslant k \leqslant L-1$, we need to prove estimates (3.3), from which we can deduce the desired estimates for $g_\eta(f)$ (see (3.4)) by reasoning as in Section 3.

We give more specific arguments in the following, focusing on the case $k = 0$ and $i = 1$ in (3.3). We can estimate $g_{\mathcal{F}_1}^{(0)}(f)$ as in Section 3 and we have an analogue of (3.6):

$$\|g_{\mathcal{F}_1}^{(0)}(f)\|_4 \leqslant C\|V(f)\|_4, \tag{5.1}$$

where $V(f)$ is defined as in Section 3. Also, we have analogues of Lemma 3.2 and Lemma 3.3 with similar proofs.

Let $t \in [2^n, 2^{n+1}]$. Then $2^{-n}t \in [1,2]$ and by the analogue of Lemma 3.2 we have

$$s_\ell^{(\delta,t)}(\xi) = \sum_{k,j,h} \chi_{2^{-n}P_h^{\ell,k,j}}(\xi)s_\ell^{(\delta,t)}(\xi)$$

as in Section 3. Using this and applying an analogue of Lemma 3.3, as in (3.8) we see that

$$V(f)(x) \leqslant C\delta^{1/2}\left(\sum_{n,k,j,\ell,h} \sup_{t \in [2^n, 2^{n+1}]} \left|S_\ell^{(\delta,t)}T_{2^{-n}P_h^{\ell,k,j}}f\right|^2\right)^{1/2}, \tag{5.2}$$

where $S_\ell^{(\delta,t)}$ is defined in the same way as in Section 3.

We also have the following result by applying Lemmas 2.1 and 2.2.

PROPOSITION 5.1. *Let Θ and β_0 be as in Lemma* 2.2 *and* (2.1)*, respectively, and let α be as in Lemma* 2.1*. Then*

$$\left\|\left(\sum_{n,k,\ell,j,h}\left|T_{2^{-n}P_h^{\ell,k,j}}f\right|^2\right)^{1/2}\right\|_4 \leqslant C\left(\log\frac{1}{\delta}\right)^{(\beta+3\alpha)/2}\|f\|_4, \quad \beta = 3\Theta + 2\beta_0.$$

The following result is also needed in proving Proposition 5.1.

LEMMA 5.2. *Let $\omega_k \times H_j \subset [-b,b] \times [c,d]$. Suppose that $\Psi' < 0$ on I. Then there exists a constant C independent of k, j such that*

$$\text{card}\{\ell : \widetilde{\Delta}_\ell \cap (\omega_k \times H_j) \neq \emptyset\} \leqslant C.$$

Proof. The proof is similar to the one for Lemma 3.8. $\square$

By (5.2), Lemma 2.1 and Proposition 5.1 we have

$$\|V(f)\|_4 \leqslant C\delta^{1/2}\left(\log\frac{1}{\delta}\right)^{(\beta/2)+2\alpha}\|f\|_4.$$

By this and (5.1), we have an analogue of (3.3) for $k = 0$ and $i = 1$. Thus we have estimates for g_η analogous to (3.4) under the conditions of $(B.1)$ and $\Psi' < 0$.

Proof of Theorem 1.4 *for the case* $(B.2)$. We consider $-\psi$ in place of ψ in the definition of g_η and apply the result of case $(B.1)$. We can see that this proves the result in the case $(B.2)$ by changing variables $\xi_2 \to -\xi_2$ in ϕ.

Proof of Theorem 1.4 *for the case* $(B.3)$. Similarly to Section 3, we decompose $[a,b]$ and $[-d,d]$: $[a,b] = \cup \omega_k$, $\omega_k = [a_{k-1}, a_k]$, $|\omega_k| = \delta^{1/2}$; $[-d,d] = \cup H_j$, $H_j = [b_{j-1}, b_j]$, $|H_j| = \delta^{1/2}$. Recall that $\Psi_*(t) = \psi(t)/t$. Then we also have $\Psi_*' \neq 0$ on I by $(A.2)$. We use Ψ_* in the definition of Δ_ℓ in place of Ψ:

$$\Delta_\ell = \{(\xi_1, \xi_2) : \Psi_*(a_{\ell-1})\xi_1 \leqslant \xi_2 \leqslant \Psi_*(a_\ell)\xi_1, \xi_1 > 0\}$$

if $\Psi_*' > 0$ on I and

$$\Delta_\ell = \{(\xi_1, \xi_2) : \Psi_*(a_\ell)\xi_1 \leqslant \xi_2 \leqslant \Psi_*(a_{\ell-1})\xi_1, \xi_1 > 0\}$$

if $\Psi_*' < 0$ on I. Similarly, we also define $\widetilde{\Delta}_\ell$ and

$$S_h^\ell = \{(\xi_1, \xi_2) : \psi'(a_{\ell-1})\xi_1 + h\delta \leqslant \xi_2 \leqslant \psi'(a_{\ell-1})\xi_1 + (h+1)\delta\}, \quad h \in \mathbb{Z},$$
$$P_h^\ell = S_h^\ell \cap \widetilde{\Delta}_\ell, \quad P_h^{\ell,k,j} = S_h^\ell \cap (\omega_k \times H_j) \cap \widetilde{\Delta}_\ell \cap ([a,b] \times \mathbb{R}).$$

Then, arguing similarly to the case $(B.1)$, we can reach the conclusion of the theorem in the case $(B.3)$.

Proof of Theorem 1.4 *for the case* $(B.4)$. Apply the case $(B.3)$ with $\widetilde{\psi}(\xi_1) = \psi(-\xi_1)$ in place of ψ. Then by change of variables: $\xi_1 \to -\xi_1$ we have the desired result.

This completes the proof of Theorem 1.4 in its full generality.

Theorem 1.6 follows from Theorem 1.4 in the same way as the theorem was shown in Section 3 for the case $(B.1)$ with $\Psi' > 0$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1 *for the case* $(B.1)$. We have already proved the theorem in the case $(B.1)$ when $\Psi' > 0$ on I. Suppose that $\Psi' < 0$ on I. We note that $0 \notin I$, since if $0 \in I$, then $\Psi'(0) = 1/\psi(0) > 0$.

Recall that $I_1 = [\sigma, \tau]$, $I_2 = [\sigma', \tau']$, $I_3 = [\sigma'', \tau'']$, $-b < \sigma'' < \sigma' < \sigma < \tau < \tau' < \tau'' < b$. We may assume that $\text{supp}(\sigma_\lambda) \subset I_1 \times \mathbb{R}$ and $\Psi' < 0$ on I_3, $\Psi(t) = t/\psi(t)$. Let a function ρ be defined on $C(a_1'', a_2'')$, $a_1'' = \tau''/\psi(\tau'')$, $a_2'' = \sigma''/\psi(\sigma'')$, by

$$\rho(u,v) = \frac{u}{\Psi^{-1}\left(\frac{u}{v}\right)},$$

where $\Psi : I_3 \to J := \Psi(I_3)$, $\Psi^{-1} : J \to I_3$. We note that $\rho(u, \psi(u)) = 1$ for $u \in I_3$. Also, we may assume that $\text{supp}(a) \subset C(a_1, a_2) \cap (I_1 \times \mathbb{R})$, $a_1 = \tau/\psi(\tau)$, $a_2 = \sigma/\psi(\sigma)$, where a is as in Theorem 1.1.

Then we have

$$\widetilde{a}(\xi)(1-\rho(\xi))_+^\lambda = a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda \qquad (5.3)$$

for some $\widetilde{a} \in C_0^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(a) = \mathrm{supp}(\widetilde{a})$. To see this, note that $\partial_2 \rho < 0$ on $C(a_1'', a_2'')$. So, for $\xi \in \mathrm{supp}(a)$ we have

$$
\begin{aligned}
a(\xi)(1-\rho(\xi))_+^\lambda &= a(\xi)(\rho(\xi_1, \psi(\xi_1)) - \rho(\xi))_+^\lambda \\
&= a(\xi)\left(\int_0^1 \frac{\partial}{\partial t} \rho(\xi_1, t(\psi(\xi_1) - \xi_2) + \xi_2)\, dt \right)_+^\lambda \\
&= a(\xi)\left((\psi(\xi_1) - \xi_2) \int_0^1 \partial_2 \rho(\xi_1, t(\psi(\xi_1) - \xi_2) + \xi_2)\, dt \right)_+^\lambda \\
&= a(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda \left(\int_0^1 (-1)\partial_2 \rho(\xi_1, t(\psi(\xi_1) - \xi_2) + \xi_2)\, dt \right)^\lambda \\
&= a(\xi) h(\xi)(\xi_2 - \psi(\xi_1))_+^\lambda
\end{aligned}
$$

for some $h \in C_0^\infty(\mathbb{R}^2)$ such that $h \neq 0$ on $\mathrm{supp}(a)$. This implies (5.3) by setting $\widetilde{a} = a/h$.

Thus to prove Theorem 1.1 it suffices to show that $\|U_*^\lambda(f)\|_4 \leqslant C\|f\|_4$, where

$$U_*^\lambda(f)(x) = \sup_{R>0} \left| \int \widetilde{a}(R^{-1}\xi)(1-\rho(R^{-1}\xi))_+^\lambda \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}\, d\xi \right|.$$

We have $U_*^\lambda(f) \leqslant CM(g_{\mathcal{F}^{-1}(\varphi^\delta)}(f))$ with some $\delta > -1/2$ by Corollary 4.2, where $\varphi^\delta(\xi) = A(\xi)(1-\rho(\xi))_+^\delta$ and A is related to $\widetilde{a}$ as $\widetilde{b}$ is related to b in Corollary 4.2 with $\mathrm{supp}(A) \subset C(a_1, a_2)$. Thus Theorem 1.1 follows from the estimates $\|g_{\mathcal{F}^{-1}(\varphi^\delta)}(f)\|_4 \leqslant C\|f\|_4$ for $\delta > -1/2$. To prove this we may replace $A(\xi)$ with $A_0(\xi) = w(\xi)A(\xi)$ for a suitable $w \in C_0^\infty(\mathbb{R}^2)$, since we may assume that

$$\{\rho(\xi) < s,\ \xi \in C(a_1, a_2)\} \subset \{|\xi| < Cs\}, \quad \forall s > 0,$$

with some positive constant C.

Choose $b_1(\xi_1) \in C_0^\infty(\mathbb{R})$ such that $b_1 = 1$ on I_2 and $\mathrm{supp}(b_1) \subset I_3$ and let $A_0(\xi) = b_1(\xi_1)A_0(\xi) + b_2(\xi_1)A_0(\xi) =: A_1(\xi) + A_2(\xi)$ with $b_2 = 1 - b_1$. Let $B_i(\xi) := A_i(\xi)(1 - \rho(\xi))_+^\delta$, $i = 1, 2$. Then $B_2 \in C_0^\infty(\mathbb{R}^2)$ and $\mathrm{supp}(B_2) \subset \mathbb{R}^2 \setminus \{0\}$. Thus $g_{\mathcal{F}^{-1}(B_2)}$ is bounded on L^p, $1 < p < \infty$. On the other hand, since $\mathrm{supp}(A_1) \subset C(a_1, a_2) \cap (I_3 \times \mathbb{R})$, arguing in the same way as in the proof of (5.3), we see that

$$B_1(\xi) = \widetilde{A_1}(\xi)(\xi_2 - \psi(\xi_1))_+^\delta,$$

where $\widetilde{A_1}$ is in $C_0^\infty(\mathbb{R}^2)$ and $\mathrm{supp}(A_1) = \mathrm{supp}(\widetilde{A_1})$. Therefore, by Theorem 1.6 for the case $\lambda > -1/2$ and $\theta = 0$, the boundedness of $g_{\mathcal{F}^{-1}(B_1)}$ follows. This completes the proof of Theorem 1.1 in the case $(B.1)$ with $\Psi' < 0$.

Proof of Theorem 1.1 in the case $(B.2)$. We have the following.

PROPOSITION 5.3. *Let* ψ, b, $\varphi_{\lambda,\theta}$ *be as in Theorem* 1.6.

Put $\Phi_{\lambda,\theta}(\xi) = b(\xi)\varphi_{\lambda,\theta}(\psi(\xi_1) - \xi_2)$. *Then, under the same conditions on* λ, θ *as in Theorem* 1.6 *we have*

$$\|g_{\mathcal{F}^{-1}(\Phi_{\lambda,\theta})}(f)\|_4 \leqslant C\|f\|_4.$$

This can be shown by using Theorem 1.4 in the same way as Theorem 1.6 is proved. Arguing as in the proof of Theorem 1.1 for the case $(B.1)$ and using Proposition 5.3 for the case $\lambda > -1/2$ and $\theta = 0$, we can prove the following.

PROPOSITION 5.4. *In the case* $(B.1)$ *it holds that*

$$\|\widetilde{S}_*^{\lambda} f\|_4 \leqslant C_{\lambda} \|f\|_4$$

for $\lambda > 0$, *where*

$$\widetilde{S}_*^{\lambda} f(x) = \sup_{R>0} \left|\widetilde{S}_R^{\lambda} f(x)\right|, \quad \widetilde{S}_R^{\lambda} f(x) = \int_{\mathbb{R}^2} \widetilde{\sigma}_{\lambda}(R^{-1}\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi,$$

$\widetilde{\sigma}_{\lambda}(\xi) = a(\xi)(\psi(\xi_1) - \xi_2)_+^{\lambda}$ *and* a *is as in Theorem* 1.1.

By applying Proposition 5.4 to $-\psi$ and by changing variables $\xi_2 \to -\xi_2$ we can prove Theorem 1.1 in the case $(B.2)$.

Proof of Theorem 1.1 *in the case* $(B.3)$. The proof is similar to that for the case $(B.1)$. In the proof we apply Theorem 1.6 and the homogeneous function ρ needed in applying Corollaries 4.2, 4.4 is defined by using $\Psi_*(t) = \psi(t)/t$; it is of the form

$$\rho(u,v) = \frac{u}{\Psi_*^{-1}\left(\frac{v}{u}\right)}.$$

Proof of Theorem 1.1 *in the case* $(B.4)$. We apply results of the case $(B.3)$ to $\widetilde{\psi}(\xi_1) = \psi(-\xi_1)$ and apply the change of variables: $\xi_1 \to -\xi_1$.

6. Proof of Theorem 1.5

It is sufficient to treat separately the cases $(B.i)$, $1 \leqslant i \leqslant 4$. In each case we argue similarly as follows.

By the Plancherel theorem we see that

$$\|g_{\eta}(f)\|_2^2 = \int_{\mathbb{R}^2} \int_0^{\infty} |\phi(t\xi)|^2 \frac{dt}{t}|\hat{f}(\xi)|^2\,d\xi. \tag{6.1}$$

Fix $\xi \in \mathbb{R}^2$. Suppose that $t_0\xi \in \operatorname{supp}(\phi)$. Then by Lemma 3.4 there exists $s_0 \in [1 - B_1\delta, 1 + B_1\delta]$ such that $t_0\xi \in s_0\Gamma$. Thus $\xi \in t_0^{-1}s_0\Gamma$. If a positive number t satisfies that $t\xi \in \operatorname{supp}(\phi)$, Lemma 3.4 implies that $t\xi \in s\Gamma$ for some $s \in [1 - B_1\delta, 1 + B_1\delta]$.

Since we have also $t\xi \in tt_0^{-1}s_0\Gamma$, it follow that $tt_0^{-1}s_0 = s$ by Lemma 3.5. Thus we see that $tt_0^{-1}s_0 \in [1 - B_1\delta, 1 + B_1\delta]$, which implies that $t \in t_0 s_0^{-1}[1 - B_1\delta, 1 + B_1\delta]$. Therefore we see that

$$
\begin{aligned}
\int_0^\infty |\phi(t\xi)|^2 \frac{dt}{t} &\leqslant \int_{t_0 s_0^{-1}(1-B_1\delta)}^{t_0 s_0^{-1}(1+B_1\delta)} |\phi(t\xi)|^2 \frac{dt}{t} \\
&\leqslant \|b\|_\infty^2 \|\Phi\|_\infty^2 \log \frac{1 + B_1\delta}{1 - B_1\delta} \\
&\leqslant C\|b\|_\infty^2 \|\Phi\|_\infty^2 \delta.
\end{aligned}
$$

Thus

$$
\int_0^\infty |\phi(t\xi)|^2 \frac{dt}{t} \leqslant C\|b\|_\infty^2 \|\Phi\|_\infty^2 \delta. \tag{6.2}
$$

This is also true when there is no t_0 such that $t_0\xi \in \mathrm{supp}(\phi)$. Thus (6.2) holds for all $\xi \in \mathbb{R}^2$. Using (6.2) in (6.1) and applying the Plancherel theorem again, we can obtain the conclusion of Theorem 1.5.

To conclude this note, we recall some results from [13]. Let σ_λ, $\lambda > 0$, and $S_R^\lambda f$ be as in (1.1) and (1.2), respectively, where ψ'' is allowed to have a finite number of zeros of finite order in I. The conditions $(A.1)$, $(A,2)$ need not be assumed. Then the following vector valued inequality was proved in [13].

THEOREM 6.1. *Let $\{R_\ell\}_{\ell=1}^\infty$ be a sequence of positive numbers and let $p \in [4/3, 4]$. Then we have*

$$
\left\| \left(\sum_{\ell=1}^\infty \left| S_{R_\ell}^\lambda f_\ell \right|^2 \right)^{1/2} \right\|_p \leqslant C_\lambda \left\| \left(\sum_{\ell=1}^\infty |f_\ell|^2 \right)^{1/2} \right\|_p.
$$

Using a special case of this, we can prove the following lacunary maximal theorem (see [13, Remark 6.2]).

COROLLARY 6.2. *Let $\{R_\ell\}_{\ell=-\infty}^\infty$ be a sequence of positive numbers such that $1 < q \leqslant \inf_\ell R_{\ell+1}/R_\ell$. Suppose that $S_R^\lambda f$ and σ_λ are as in Theorem 6.1 and that the condition $(A.1)$ holds. Then we have*

$$
\left\| \sup_\ell \left| S_{R_\ell}^\lambda f \right| \right\|_p \leqslant C_\lambda \|f\|_p, \quad 4/3 \leqslant p \leqslant 4.
$$

Theorem 6.1 was shown in [13] from the results for the case when $\psi'' \neq 0$ by an idea of Hörmander in [10] and it was applied to show Corollary 6.2 (see [15] for related results). Unfortunately, we cannot apply the idea to prove an analogue of Theorem 1.1 for $S_R^\lambda f$ with σ_λ defined under the conditions of Corollary 6.2.

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Shuichi Sato
Kanazawa, Japan
e-mail: shuichipm@gmail.com

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com