Mathematical
Inequalities
& Applications
Volume 29, Number 1 (2026), 75–97 doi:10.7153/mia-2026-29-04

NEW ASYMPTOTICS AND INEQUALITIES RELATED TO THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

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(*Communicated by N. Elezović*)

Abstract. Let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)\,(n \in \mathbb{N})$ denote the volume of the unit ball in $\mathbb{R}^n$. Define the function $I(x)$ by

$$I(x) = \frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} = \left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2,$$

where $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2}+1)$. In this paper, we present asymptotic expansions of the function $I(x)$, and then establish asymptotic expansions and inequalities of the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$. Also, we prove that the function $F(x) = (1+\frac{1}{x})^{1/4}/I(x)$ is logarithmically completely monotonic on $(0,\infty)$, which derives a double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$.

1. Introduction

In the recent past, several researchers have established interesting properties of the volume Ω_n of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \qquad n \in \mathbb{N} := \{1,2,\ldots\},$$

including monotonicity properties, inequalities and asymptotic expansions.

Böhm and Hertel [9, p. 264] pointed out that the sequence $\{\Omega_n\}_{n\geqslant 1}$ is not monotonic for $n \geqslant 1$. Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if} \quad 1 \leqslant n \leqslant 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if} \quad n \geqslant 5.$$

Anderson *et al.* [5] showed that $\{\Omega_n^{1/n}\}_{n\geqslant 1}$ is monotonically decreasing to zero, while Anderson and Qiu [4] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\geqslant 2}$ decreases to $e^{-1/2}$. Guo and Qi [18] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\geqslant 2}$ is logarithmically convex. Klain and Rota [21] proved that the sequence $\{n\Omega_n/\Omega_{n-1}\}_{n\geqslant 1}$ is increasing.

Mathematics subject classification (2020): 33B15, 41A60, 26D15.
Keywords and phrases: Volume of the unit n-dimensional ball, gamma function, asymptotic expansions, inequalities, logarithmically completely monotonic function.
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Diverse sharp inequalities for the volume of the unit ball in $\mathbb{R}^n$ have been established [2, 3, 7, 10, 12, 14, 22, 25, 26, 27, 28, 32, 33]. For example, Alzer [2] proved that for $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n}\right)^{a_3} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{b_3}, \tag{1}$$

with the best possible constants

$$a_3 = 2 - \frac{\ln \pi}{\ln 2} = 0.3485\ldots, \qquad b_3 = \frac{1}{2}.$$

Merkle [25] improved the left-hand side of (1) and obtained the following result:

$$\left(1+\frac{1}{n+1}\right)^{1/2} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \qquad n \in \mathbb{N}. \tag{2}$$

Chen and Lin [12, Theorem 3.1] developed (2) to produce the following symmetric double inequality:

$$\left(1+\frac{1}{n+1}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{1}{n+1}\right)^{\beta}, \qquad n \in \mathbb{N}, \tag{3}$$

with the best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln \pi}{\ln 3 - \ln 2} = 0.5957713\ldots.$$

Ban and Chen [7, Theorem 3.2] improved (3) and obtained the following result:

$$\left(1+\frac{1}{n+\theta_1}\right)^{1/2} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n+\theta_2}\right)^{1/2}, \qquad n \in \mathbb{N}, \tag{4}$$

with best possible constants

$$\theta_1 = \frac{2\pi^2 - 16}{16 - \pi^2} = 0.60994576\ldots \quad \text{and} \quad \theta_2 = \frac{1}{2}.$$

Recently, Mortici [28] constructed asymptotic series associated with some expressions involving the volume of the n-dimensional unit ball. New refinements and improvements of some old and recent inequalities for Ω_n are also presented. Lu and Zhang [22] established a general continued fraction approximation for the nth root of the volume of the unit n-dimensional ball, and then obtained related inequalities.

It is easy to see that

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \left(\frac{n}{2}+\frac{1}{2}\right)\left[\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}\right]^2. \tag{5}$$

Replacement of n by $2x$ in (5) yields

$$I(x) := \frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} = \left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2, \tag{6}$$

where $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2}+1)$.

In this paper, we present asymptotic expansions of the function $I(x)$ defined by (6), and then establish asymptotic expansions and inequalities of the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$. Also, we prove that the function $F(x) = (1+\frac{1}{x})^{1/4}/I(x)$ is logarithmically completely monotonic on $(0,\infty)$, which derives a double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$.

The numerical values given in this paper have been calculated via the computer program MAPLE 17.

2. Lemmas

The gamma function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called psi (or digamma) function, and $\psi^{(k)}(x)$ $(k \in \mathbb{N})$ are called polygamma functions.

The following lemmas are required in our present investigation.

LEMMA 1. ([13]) *Let $r \neq 0$ be a given real number and $\ell \geqslant 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(1+\sum_{j=1}^\infty \frac{p_j}{x^j}\right)^{x^\ell/r}, \qquad x \to \infty, \tag{7}$$

with the coefficients $p_j \equiv p_j(\ell,r)$ $(j \in \mathbb{N})$ given by

$$p_j = \sum \frac{r^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!}\left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1}\left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2}\cdots\left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j},$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})$ are the Bernoulli numbers defined by

$$\frac{z}{e^z-1} = \sum_{n=0}^\infty B_n\frac{z^n}{n!}, \qquad |z| < 2\pi,$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \cdots + (2j+\ell-1)k_j = j.$$

In particular, setting $(\ell, r) = (0, -2)$ in (7) yields

$$x\left(\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}\right)^2 \sim \sum_{j=0}^{\infty} \frac{c_j}{x^j} = 1 - \frac{1}{4x} + \frac{1}{32x^2} + \frac{1}{128x^3} - \frac{5}{2048x^4} - \frac{23}{8192x^5} + \frac{53}{65536x^6} + \frac{593}{262144x^7} - \frac{5165}{8388608x^8} - \frac{110123}{33554432x^9} + \dots \tag{8}$$

as $x \to \infty$, where the coefficients $c_j \equiv p_j(0, -2)$ $(j \in \mathbb{N}_0)$ are given by

$$c_0 = 1,$$
$$c_j = \sum \frac{(-2)^{k_1+k_2+\cdots+k_j}}{k_1!k_2!\cdots k_j!} \left(\frac{(2^2-1)B_2}{1\cdot 1\cdot 2^2}\right)^{k_1} \left(\frac{(2^4-1)B_4}{2\cdot 3\cdot 2^4}\right)^{k_2} \cdots \left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}}\right)^{k_j} \tag{9}$$

for $j \in \mathbb{N}$, summed over all nonnegative integers k_j satisfying the equation

$$k_1 + 3k_2 + \cdots + (2j-1)k_j = j.$$

LEMMA 2. ([13]) *Let $m, n \in \mathbb{N}$. Then for $x > 0$,*

$$\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} < (-1)^n \left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x+\frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n} < \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}}\right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}}. \tag{10}$$

In particular, we obtain from (10) that

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} < \psi(x+1) - \psi\left(x+\frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4}, \quad x > 0, \tag{11}$$

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8} - \frac{31}{2048x^{10}} < \psi(x+1) - \psi\left(x+\frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} + \frac{17}{2048x^8}, \quad x > 0 \tag{12}$$

and

$$-\frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{16x^5} < \psi'(x+1) - \psi'\left(x+\frac{1}{2}\right), \quad x > 0. \tag{13}$$

LEMMA 3. ([29]) *Let* $-\infty \leqslant a < b \leqslant \infty$. *Let* f *and* g *be differentiable functions on an interval* (a,b). *Assume that either* $g' > 0$ *everywhere on* (a,b) *or* $g' < 0$ *on* (a,b). *Suppose that* $f(a+) = g(a+) = 0$ *or* $f(b-) = g(b-) = 0$. *Then*

(1) *if* $\frac{f'}{g'}$ *is increasing on* (a,b), *then* $\left(\frac{f}{g}\right)' > 0$ *on* (a,b);

(2) *if* $\frac{f'}{g'}$ *is decreasing on* (a,b), *then* $\left(\frac{f}{g}\right)' < 0$ *on* (a,b).

3. Asymptotic expansions for $I(x)$ and $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$

In this section, we establish the asymptotic expansions for the function $I(x)$ and the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$, which are based upon the Bell polynomials. The Bell polynomials, named in honor of Eric Temple Bell (1883–1960), are a triangular array of polynomials given by (see, for example, Comtet [15, pp. 133–134], Cvijović [16] and Masjed-Jamei *et al.* [24])

$$B_{n,k}(x_1,x_2,\cdots,x_{n-k+1}) = \sum \frac{n!}{j_1!\, j_2! \cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum is taken over all non-negative integers $j_1, j_2, j_3, \cdots, j_{n-k+1}$ such that

$$j_1 + j_2 + \cdots + j_{n-k+1} = k \qquad \text{and} \qquad j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n.$$

The following sum:

$$B_n(x_1,x_2,x_3,\cdots,x_n) = \sum_{k=1}^{n} B_{n,k}(x_1,x_2,x_3,\cdots,x_{n-k+1})$$

is sometimes called the nth complete Bell polynomial. These complete Bell polynomials satisfy the following identity:

$$B_n(x_1,x_2,x_3,\cdots,x_n) = \begin{vmatrix} x_1 & \binom{n-1}{1}x_2 & \binom{n-1}{2}x_3 & \binom{n-1}{3}x_4 & \cdots & \cdots & x_n \\ -1 & x_1 & \binom{n-2}{1}x_2 & \binom{n-2}{2}x_3 & \cdots & \cdots & x_{n-1} \\ 0 & -1 & x_1 & \binom{n-3}{1}x_2 & \cdots & \cdots & x_{n-2} \\ 0 & 0 & -1 & x_1 & \cdots & \cdots & x_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & x_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & x_1 \end{vmatrix}. \tag{14}$$

In order to contrast them with complete Bell polynomials, the polynomials $B_{n,k}$ defined above are sometimes called partial Bell polynomials. The complete Bell polynomials appear in the exponential of a formal power series:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n!}u^n\right)=\sum_{n=0}^{\infty}\frac{B_n(x_1,\cdots,x_n)}{n!}u^n. \tag{15}$$

The Bell polynomials are quite general polynomials and they have been found in many applications in combinatorics. In his monograph, Comtet [15] devoted much to a thorough presentation of the Bell polynomials in the chapter on identities and expansions. For more results, see the works by Charalambides [16, Chapter 11] and Riordan [24, Chapter 5].

We now state and prove the asymptotic expansions of the function $I(x)$ defined by (6), and then obtain the asymptotic expansions of the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$.

We find from (8) that, as $x\to\infty$,

$$\begin{aligned} I(x)&=\left(1+\frac{1}{2x}\right)x\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\sim\left(1+\frac{1}{2x}\right)\left(1+\sum_{j=1}^{\infty}\frac{c_j}{x^j}\right)=1+\sum_{j=1}^{\infty}\frac{\lambda_j}{x^j}\\ &=1+\frac{1}{4x}-\frac{3}{32x^2}+\frac{3}{128x^3}+\frac{3}{2048x^4}-\frac{33}{8192x^5}-\frac{39}{65536x^6}+\ldots, \end{aligned} \tag{16}$$

where

$$\lambda_j=c_j+\frac{1}{2}c_{j-1},\qquad j\in\mathbb{N}, \tag{17}$$

and c_j are given in (9). Replacement of x by $n/2$ in (16) then produces the following asymptotic expansion for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$:

$$\begin{aligned} \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}&=I\left(\frac{n}{2}\right)\sim 1+\sum_{j=1}^{\infty}\frac{d_j}{n^j}\\ &=1+\frac{1}{2n}-\frac{3}{8n^2}+\frac{3}{16n^3}+\frac{3}{128n^4}-\frac{33}{256n^5}-\frac{39}{1024n^6}+\ldots \end{aligned} \tag{18}$$

as $n\to\infty$, where $d_j=2^j\lambda_j$, and λ_j are given in (17).

Mortici [28, Theorem 15] provided a recurrence relation for successively determining the coefficient d_j in expansion (18).

In view of (1), we introduce the approximations family

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\sim\left(1+\frac{a}{n}\right)^b, \tag{19}$$

where $a,b\in\mathbb{R}$ are parameters. By the computer program MAPLE 17, we find, as

$n \to \infty$,

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} - \left(1+\frac{a}{n}\right)^b = \frac{-ab+\frac{1}{2}}{n} + \frac{-\frac{1}{2}a^2b^2+\frac{1}{2}a^2b-\frac{3}{8}}{n^2} + \frac{\frac{3}{16}-\frac{1}{6}a^3b^3+\frac{1}{2}a^3b^2-\frac{1}{3}a^3b}{n^3} + O\left(\frac{1}{n^4}\right). \tag{20}$$

This produces the best approximation from (20):

$$\begin{cases} -ab+\frac{1}{2}=0 \\ -\frac{1}{2}a^2b^2+\frac{1}{2}a^2b-\frac{3}{8}=0 \end{cases}$$

so

$$a=2, \quad b=\frac{1}{4}.$$

We then find that

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{2}{n}\right)^{1/4}, \qquad n\to\infty \tag{21}$$

is the best approximation among all approximations given by (19), namely,

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \left(1+\frac{2}{n}\right)^{1/4} + O\left(\frac{1}{n^3}\right), \qquad n\to\infty. \tag{22}$$

Replacement of n by $2x$ in (21) and (22) yields

$$\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} \sim \left(1+\frac{1}{x}\right)^{1/4}, \qquad x\to\infty \tag{23}$$

and

$$\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}} = \left(1+\frac{1}{x}\right)^{1/4} + O\left(\frac{1}{x^3}\right), \qquad x\to\infty. \tag{24}$$

Theorem 1 develops the approximation formula (23) to produce a complete asymptotic expansion.

THEOREM 1. *The function $I(x)$, defined by (6), has the following asymptotic expansion:*

$$I(x) \sim \left(1+\frac{1}{x}\right)^{1/4} \exp\left\{\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right\} = \left(1+\frac{1}{x}\right)^{1/4} \exp\left\{-\frac{1}{32x^3}+\frac{3}{64x^4}-\frac{3}{64x^5}+\frac{5}{128x^6}-\frac{33}{1024x^7}+\dots\right\} \tag{25}$$

as $x \to \infty$, with the coefficients a_j given by

$$a_1 = 0, \ \ a_2 = 0, \ \ a_j = \frac{2\big[(-1)^{j+1}(2^{-j}-1)-1\big]B_{j+1}}{j(j+1)} + \frac{(-1)^{j-1}}{j}\left(\frac{1}{2^j}-\frac{1}{4}\right), \ \ j \geqslant 3, \tag{26}$$

where B_n are the Bernoulli numbers.

Proof. In view of (24), we can let

$$I(x) \sim \left(1+\frac{1}{x}\right)^{1/4} \exp\left\{\sum_{j=1}^{\infty}\frac{a_j}{x^j}\right\},$$

or alternatively

$$\ln I(x) \sim \frac{1}{4}\ln\left(1+\frac{1}{x}\right) + \sum_{j=1}^{\infty}\frac{a_j}{x^j} \tag{27}$$

as $x \to \infty$, where a_j are real numbers to be determined. Taking logarithm on the both sides of (6), we have

$$\ln I(x) = \ln\left(1+\frac{1}{2x}\right) + \ln\left(x\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\right). \tag{28}$$

We then obtain from (27) and (28) that

$$\sum_{j=1}^{\infty}\frac{a_j}{x^j} \sim \ln\left(x\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\right) + \ln\left(1+\frac{1}{2x}\right) - \frac{1}{4}\ln\left(1+\frac{1}{x}\right). \tag{29}$$

The logarithm of gamma function has asymptotic expansion (see [23, p. 32]):

$$\ln\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \tag{30}$$

as $x \to \infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1} = \sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}. \tag{31}$$

From (30), we obtain, as $x \to \infty$,

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x\exp\left(\frac{1}{t-s}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\big(B_{j+1}(t)-B_{j+1}(s)\big)}{j(j+1)}\frac{1}{x^j}\right). \tag{32}$$

Setting $(s,t) = (1, \frac{1}{2})$ and noting that

$$B_n(0) = (-1)^n B_n(1) = B_n \quad \text{and} \quad B_n\left(\tfrac{1}{2}\right) = (2^{1-n} - 1)B_n \quad \text{for} \quad n \in \mathbb{N}_0$$

(see [1, p. 805]), we obtain from (32), as $x \to \infty$,

$$\ln\left(x\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2\right) \sim \sum_{j=1}^{\infty} \frac{2\left[(-1)^{j+1}(2^{-j}-1)-1\right]B_{j+1}}{j(j+1)} \frac{1}{x^j}. \tag{33}$$

By using the Maclaurin expansion of $\ln(1+t)$,

$$\ln(1+t) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} t^j \quad \text{for} \quad -1 < t \leqslant 1,$$

we obtain, as $x \to \infty$,

$$\ln\left(1+\frac{1}{2x}\right) - \frac{1}{4}\ln\left(1+\frac{1}{x}\right) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j}\left(\frac{1}{2^j}-\frac{1}{4}\right)\frac{1}{x^j}. \tag{34}$$

Substitution of (33) and (34) into (29) yields

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{a_j}{x^j} &\sim \sum_{j=1}^{\infty}\left\{\frac{2\left[(-1)^{j+1}(2^{-j}-1)-1\right]B_{j+1}}{j(j+1)} + \frac{(-1)^{j-1}}{j}\left(\frac{1}{2^j}-\frac{1}{4}\right)\right\}\frac{1}{x^j} \\ &= \sum_{j=3}^{\infty}\left\{\frac{2\left[(-1)^{j+1}(2^{-j}-1)-1\right]B_{j+1}}{j(j+1)} + \frac{(-1)^{j-1}}{j}\left(\frac{1}{2^j}-\frac{1}{4}\right)\right\}\frac{1}{x^j}.\end{aligned}$$

This gives

$$a_1 = 0, \quad a_2 = 0, \quad a_j = \frac{2\left[(-1)^{j+1}(2^{-j}-1)-1\right]B_{j+1}}{j(j+1)} + \frac{(-1)^{j-1}}{j}\left(\frac{1}{2^j}-\frac{1}{4}\right), \quad j \geqslant 3.$$

The proof of Theorem 1 is complete. □

REMARK 1. Replacement of x by $n/2$ in (25) yields

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{2}{n}\right)^{1/4} \exp\left\{-\frac{1}{4n^3}+\frac{3}{4n^4}-\frac{3}{2n^5}+\frac{5}{2n^6}-\frac{33}{8n^7}+\dots\right\}, \quad n \to \infty. \tag{35}$$

Theorem 2 develops the approximation formula (23) to produce an alternative asymptotic expansion.

THEOREM 2. *Let $I(x)$ be defined by (6). As $x \to \infty$, we have*

$$I(x) \sim \left(1+\frac{1}{x}\right)^{1/4}\left\{\sum_{j=0}^{\infty}\frac{b_j}{x^j}\right\}$$
$$= \left(1+\frac{1}{x}\right)^{1/4}\left\{1-\frac{1}{32x^3}+\frac{3}{64x^4}-\frac{3}{64x^5}+\frac{81}{2048x^6}-\frac{69}{2048x^7}+\dots\right\}, \tag{36}$$

with the coefficients b_j given by the recursive formula

$$b_0 = 1, \quad b_j = \sum_{\ell=0}^{j-1}\frac{j-\ell}{j}a_{j-\ell}b_\ell, \qquad j \in \mathbb{N}, \tag{37}$$

where a_j are given in (26).

Proof. It follows from (25) and (15) that

$$\frac{I(x)}{\left(1+\frac{1}{x}\right)^{1/4}} \sim \exp\left(\sum_{j=1}^{\infty}\frac{j!\,a_j}{j!}\frac{1}{x^j}\right) \sim \sum_{j=0}^{\infty}\frac{b_j}{x^j},$$

where

$$b_j = \frac{B_j\Big(1!\,a_1, 2!\,a_2, \cdots, j!\,a_j\Big)}{j!}. \tag{38}$$

Bulò *et al.* [11, Theorem 1] proved that the complete Bell polynomials can be expressed by using the following recursive relation:

$$B_n(x_1,x_2,\cdots,x_n) = \begin{cases} \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}x_{n-\ell}B_\ell(x_1,x_2,\cdots,x_\ell) & (n>0) \\ 1 & (\text{otherwise}). \end{cases} \tag{39}$$

Therefore, by employing (39), the formula (38) can be rewritten as follows:

$$\begin{aligned} b_0 &= 1 \quad \text{and} \\ b_j &= \frac{1}{j!}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(j-\ell)!\,a_{j-\ell}B_\ell\Big(1!\,a_1, 2!\,a_2\cdots, \ell!\,a_\ell\Big) \\ &= \frac{1}{j!}\sum_{\ell=0}^{j-1}\binom{j-1}{\ell}(j-\ell)!\,a_{j-\ell}\ell!b_\ell \\ &= \sum_{\ell=0}^{j-1}\frac{j-\ell}{j}\,a_{j-\ell}b_\ell, \qquad j \in \mathbb{N}. \end{aligned}$$

This completes the proof of Theorem 2. □

REMARK 2. We can calculate the coefficients b_j in (36) by using the formulas (38) and (14). We thus find that

$$b_n = \frac{1}{n!}\begin{vmatrix} 1!\,a_1 & \binom{n-1}{1}2!\,a_2 & \binom{n-1}{2}3!\,a_3 & \binom{n-1}{3}4!\,a_4 & \cdots & \cdots & n!\,a_n \\ -1 & 1!\,a_1 & \binom{n-2}{1}2!\,a_2 & \binom{n-2}{2}3!\,a_3 & \cdots & \cdots & (n-1)!\,a_{n-1} \\ 0 & -1 & 1!\,a_1 & \binom{n-3}{1}2!\,a_2 & \cdots & \cdots & (n-2)!\,a_{n-2} \\ 0 & 0 & -1 & 1!\,a_1 & \cdots & \cdots & (n-3)!\,a_{n-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & (n-4)!\,a_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1!\,a_1 \end{vmatrix}. \tag{40}$$

The representation using a recursive algorithm for the coefficients b_j in (37) is more practical for numerical evaluation than the expression in (40).

REMARK 3. Replacement of x by $n/2$ in (36) yields

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{1}{4n^3}+\frac{3}{4n^4}-\frac{3}{2n^5}+\frac{81}{32n^6}-\frac{69}{16n^7}+\ldots\right) \tag{41}$$

as $n \to \infty$.

4. Inequalities for $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$

In view of (24) it is natural to ask: what are the largest number α and the smallest number β such that the inequalities

$$\left(1+\frac{\alpha}{n}\right)^{1/4} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{\beta}{n}\right)^{1/4}$$

are valid for all $n \in \mathbb{N}_0$? Theorem 3 answers this question.

THEOREM 3. *For $n \in \mathbb{N}$, the following double inequality holds:*

$$\left(1+\frac{\alpha}{n}\right)^{1/4} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{\beta}{n}\right)^{1/4}, \tag{42}$$

where the constants

$$\alpha = \frac{256}{\pi^4} - 1 = 1.628091457199\ldots \quad \textit{and} \quad \beta = 2$$

are the best possible.

Proof. If we write (42) as

$$\alpha \leqslant x_n < \beta,, \qquad x_n = n\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^4 - 1,$$

we find that

$$x_1 = \frac{256}{\pi^4} - 1$$

and

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left\{n\left(\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right)^4 - 1\right\}\\ &= \lim_{n\to\infty}\left\{n\left[1+\frac{1}{2n}-\frac{3}{8n^2}+\frac{3}{16n^3}+O\left(\frac{1}{n^4}\right)\right]^4 - 1\right\}\\ &= \lim_{n\to\infty}\left\{2+O\left(\frac{1}{n^2}\right)\right\} = 2.\end{aligned}$$

This limit is obtained by using the asymptotic expansion (18).

In order prove Theorem 3, it suffices to show that the sequence $\{x_n\}$ is strictly increasing for $n \geqslant 1$. The monotonicity property of $\{x_n\}$ is obtained by considering the function $J(x)$ defined by

$$J(x) = 2x\left(\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}}\right)^4 - 1 = 2xI^4(x) - 1,$$

where $I(x)$ is given in (6). Differentiating $J(x)$ and applying the right-hand side of (11), we obtain that for $x \geqslant \frac{1}{2}$,

$$\begin{aligned}\frac{J'(x)}{J(x)+1} &= \frac{1}{x} + 4\frac{I'(x)}{I(x)} = \frac{1}{x} - 8\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{2}{2x+1}\right]\\ &= \frac{18x+1}{x(2x+1)} - 8\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right]\\ &> \frac{18x+1}{x(2x+1)} - 8\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4}\right)\\ &= \frac{4+40(x-\frac{1}{2})+116(x-\frac{1}{2})^2+152(x-\frac{1}{2})^3+80(x-\frac{1}{2})^4}{8x^4(2x+1)} > 0.\end{aligned}$$

Hence, $J(x)$ is strictly decreasing for $x \geqslant \frac{1}{2}$. We then obtain that the sequence $\{x_n\} = \{J(n/2)\}$ is strictly decreasing for $n \geqslant 1$. The proof of Theorem 3 is complete. □

In view of (24) it is natural to ask: what are the largest number p and the smallest number q such that the inequalities

$$\left(1+\frac{2}{n}\right)^p \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leqslant \left(1+\frac{2}{n}\right)^q$$

are valid for all $n \in \mathbb{N}_0$? Theorem 4 answers this question.

THEOREM 4. *For $n \in \mathbb{N}$, the following double inequality holds:*

$$\left(1+\frac{2}{n}\right)^p \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{2}{n}\right)^q, \tag{43}$$

where the constants

$$p = \frac{\ln(4/\pi)}{\ln 3} = 0.21988\ldots \quad \textit{and} \quad q = \frac{1}{4}$$

are the best possible.

Proof. Inequality (43) can be written as

$$p \leqslant y_n < q,$$

where the sequence $\{y_n\}_{n\in\mathbb{N}}$ is defined by

$$y_n = \frac{\ln\left(\left(\frac{n}{2}+\frac{1}{2}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}\right)^2\right)}{\ln\left(1+\frac{2}{n}\right)}.$$

We are now in a position to show that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is strictly increasing. To this end, we consider the function $f(x)$ defined by

$$f(x) = \frac{2\ln\Gamma\left(x+\frac{1}{2}\right) - 2\ln\Gamma(x+1) + \ln\left(x+\frac{1}{2}\right)}{\ln\left(1+\frac{1}{x}\right)} = \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = 2\ln\Gamma\left(x+\frac{1}{2}\right) - 2\ln\Gamma(x+1) + \ln\left(x+\frac{1}{2}\right)$$

and

$$f_2(x) = \ln\left(1+\frac{1}{x}\right).$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [1, p. 257, Eq. (6.1.41)]) that

$$f_1(\infty) = \lim_{x\to\infty} f_1(x) = 0.$$

Elementary calculations show that

$$\frac{f_1'(x)}{2f_2'(x)} = x(x+1)\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x+1}\right] =: f_3(x).$$

By using inequalities (11) and (13), we obtain, for $x \geqslant 2$,

$$\begin{aligned} f_3'(x) &= (2x+1)\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2x+1}\right] \\ &\quad + x(x+1)\left[\psi'(x+1) - \psi'\left(x+\frac{1}{2}\right) + \frac{2}{(2x+1)^2}\right] \\ &> (2x+1)\left[\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} - \frac{1}{2x+1}\right] \\ &\quad + x(x+1)\left[-\frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{16x^5} + \frac{2}{(2x+1)^2}\right] \\ &= \frac{779 + 2562(x-2) + 3030(x-2)^2 + 1692(x-2)^3 + 456(x-2)^4 + 48(x-2)^5}{128x^6(2x+1)^2} \\ &> 0. \end{aligned}$$

Hence, $f_3(x)$ and $\frac{f_1'(x)}{f_2'(x)}$ are both strictly increasing for $x \geqslant 2$. By Lemma 3, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(\infty)}{f_2(x) - f_2(\infty)}$$

is strictly increasing for $x \geqslant 2$. Therefore, the sequence $\{y_n\}$ is strictly increasing for $n \geqslant 4$. Direct computation would yield

$$y_1 = 0.21988\ldots, \quad y_2 = 0.23645\ldots, \quad y_3 = 0.24231\ldots, \quad y_4 = 0.24505\ldots.$$

Consequently, the sequence $\{y_n\}_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$\frac{\ln(4/\pi)}{\ln 3} = y_1 \leqslant y_n < \lim_{n\to\infty} y_n \quad \text{for} \quad n \in \mathbb{N}.$$

It remains to prove that

$$\lim_{n\to\infty} y_n = \frac{1}{4}. \tag{44}$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [1, p. 257, Eq. (6.1.41)]) that

$$y_n = \frac{\frac{1}{2n} - \frac{1}{2n^2} + O(n^{-3})}{\frac{2}{n} - \frac{2}{n^2} + O(n^{-3})} = \frac{\frac{1}{2} + O(n^{-1})}{2 + O(n^{-1})} \to \frac{1}{4} \quad \text{as} \quad n \to \infty.$$

Hence, (44) holds. This completes the proof of Theorem 4. □

Theorem 5 below improves Theorems 3 and 4.

THEOREM 5. *For $n \in \mathbb{N}$, the following double inequality holds:*

$$\begin{aligned} \left(1+\frac{2}{n}\right)^{1/4}\left(1 - \frac{2}{8n^3+24n^2+24n+a}\right) &\leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \\ &< \left(1+\frac{2}{n}\right)^{1/4}\left(1 - \frac{2}{8n^3+24n^2+24n+b}\right), \end{aligned} \tag{45}$$

where the constants

$$a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}=5.449298\ldots \quad \textit{and} \quad b=9$$

are the best possible.

Proof. First of all, we show that the double inequality (45) with $a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}$ and $b=9$ is valid for $n=1,2,3,4$ and 5. For $n\in\mathbb{N}$, let

$$L_n=\left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{2}{8n^3+24n^2+24n+\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}}\right),$$
$$U_n=\left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{2}{8n^3+24n^2+24n+9}\right).$$

Direct computation yields

$$L_1=\frac{4}{\pi},\qquad \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=1}=\frac{4}{\pi}=1.2732\ldots,\qquad U_1=1.2755\ldots,$$

$$L_2=1.178064357\ldots,\ \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=2}=1.17809724510\ldots,\ U_2=1.178246681\ldots,$$

$$L_3=1.131758795\ldots,\ \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=3}=1.13176848421\ldots,\ U_3=1.131789661\ldots,$$

$$L_4=1.104462901\ldots,\ \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=4}=1.10446616728\ldots,\ U_4=1.104470767\ldots,$$

$$L_5=1.086496467\ldots,\ \left[\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}\right]_{n=5}=1.08649774484\ldots,\ U_5=1.086499056\ldots.$$

Clearly, the double inequality (45) with $a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}$ and $b=9$ is valid for $n=1,2,3,4$ and 5. For $n=1$, the equal sign on the left-hand side of (45) holds.

We now prove that the double inequality (45) with $a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}$ and $b=9$ is valid for $n\geqslant 6$. It suffices to show that for $x\geqslant 3$,

$$\left(1+\frac{1}{x}\right)^{1/4}\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+a}\right)\leqslant\frac{\Omega_{2x}^2}{\Omega_{2x-1}\Omega_{2x+1}}$$
$$<\left(1+\frac{1}{x}\right)^{1/4}\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+9}\right),$$

which can be written as

$$\left(1+\frac{1}{x}\right)^{1/4}\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+a}\right)\leqslant\left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)}\right]^2$$
$$<\left(1+\frac{1}{x}\right)^{1/4}\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+9}\right). \tag{46}$$

In order to prove the double inequality (46) for $x\geqslant 3$, it suffices to show that

$$F(x)>0 \quad \text{and} \quad G(x)<0 \quad \text{for} \quad x\geqslant 3,$$

where

$$F(x)=2\left[\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\Gamma(x+1)\right]+\ln\left(x+\frac{1}{2}\right)-\frac{1}{4}\ln\left(1+\frac{1}{x}\right)$$
$$-\ln\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+a}\right),$$

$$G(x)=2\left[\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\Gamma(x+1)\right]+\ln\left(x+\frac{1}{2}\right)-\frac{1}{4}\ln\left(1+\frac{1}{x}\right)$$
$$-\ln\left(1-\frac{2}{8(2x)^3+24(2x)^2+24(2x)+9}\right).$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ (see [1, p. 257, Eq. (6.1.41)]) that

$$\lim_{x\to\infty}F(x)=\lim_{x\to\infty}G(x)=0.$$

Differentiating $F(x)$ and applying the left-hand side of (12), and noting that

$$a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}<\frac{11}{2},$$

we obtain for $x\geqslant 3$,

$$F'(x)=-2\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right]+\frac{8x^2+10x+1}{4x(2x+1)(x+1)}$$
$$-\frac{96(4x^2+4x+1)}{(64x^3+96x^2+48x+a-2)(64x^3+96x^2+48x+a)}$$
$$<-2\left(\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{64x^4}-\frac{1}{128x^6}+\frac{17}{2048x^8}-\frac{31}{2048x^{10}}\right)+\frac{8x^2+10x+1}{4x(2x+1)(x+1)}$$
$$-\frac{96(4x^2+4x+1)}{(64x^3+96x^2+48x+\frac{11}{2}-2)(64x^3+96x^2+48x+\frac{11}{2})}$$
$$=-\frac{P_{11}(x-3)}{1024x^{10}(2x+1)(x+1)(128x^3+192x^2+96x+7)(128x^3+192x^2+96x+11)},$$

where

$$\begin{aligned}P_{11}(x) =&33554257720+140555666982x+255136256226x^2+271392544335x^3\\&+190236248058x^4+92827792368x^5+32266565984x^6+7997603040x^7\\&+1385315904x^8+159645696x^9+11010048x^{10}+344064x^{11}.\end{aligned}$$

Hence, $F'(x)<0$ for $x\geqslant 3$. So, $F(x)$ is strictly decreasing for $x\geqslant 3$, and we have

$$F(x)>\lim_{t\to\infty}F(t)=0,\quad x\geqslant 3.$$

Therefore, the left-hand side of (45) with $a=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}$ is valid for $n\in\mathbb{N}$.

Differentiating $G(x)$ and applying the right-hand side of (12), we obtain for $x\geqslant 3$,

$$\begin{aligned}G'(x)&=-2\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right]+\frac{8x^2+10x+1}{4x(2x+1)(x+1)}\\&\quad-\frac{96(4x^2+4x+1)}{(64x^3+96x^2+48x+7)(64x^3+96x^2+48x+9)}\\&>-2\left(\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{64x^4}-\frac{1}{128x^6}+\frac{17}{2048x^8}\right)+\frac{8x^2+10x+1}{4x(2x+1)(x+1)}\\&\quad-\frac{96(4x^2+4x+1)}{(64x^3+96x^2+48x+7)(64x^3+96x^2+48x+9)}\\&=-\frac{P_8(x-3)}{1024x^8(2x+1)(x+1)(64x^3+96x^2+48x+7)(64x^3+96x^2+48x+9)},\end{aligned}$$

where

$$\begin{aligned}P_8(x) =&601103772+2153916975x+2973701346x^2+2191678864x^3+967623264x^4\\&+265598688x^5+44615488x^6+4214784x^7+172032x^8.\end{aligned}$$

Hence, $G'(x)<0$ for $x\geqslant 3$. So, $G(x)$ is strictly increasing for $x\geqslant 3$, and we have

$$G(x)<\lim_{t\to\infty}G(t)=0,\quad x\geqslant 3.$$

Therefore, the right-hand side of (45) with $b=9$ is valid for $n\in\mathbb{N}$.

If we write (45) as

$$a\leqslant z_n<b,\qquad z_n=\frac{2}{1-\frac{\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}}{(1+\frac{2}{n})^{1/4}}}-(8n^3+24n^2+24n),$$

we find that

$$z_1=\frac{2(112\cdot 3^{3/4}-81\pi)}{3\pi-4\cdot 3^{3/4}}$$

and

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left\{\frac{2}{1-\frac{\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}}{(1+\frac{2}{n})^{1/4}}} - (8n^3+24n^2+24n)\right\}$$
$$= \lim_{n\to\infty}\left\{\frac{2}{\frac{1}{4n^3}-\frac{3}{4n^4}+\frac{3}{2n^5}-\frac{81}{32n^6}+\frac{69}{16n^7}+O\left(\frac{1}{n^8}\right)} - (8n^3+24n^2+24n)\right\}$$
$$= \lim_{n\to\infty}\left\{9+O\left(\frac{1}{n}\right)\right\} = 9.$$

This limit is obtained by using the asymptotic expansion (41).

Hence, the double inequality (45) holds for $n \in \mathbb{N}$, and the constants

$$a = \frac{2(112\cdot 3^{3/4} - 81\pi)}{3\pi - 4\cdot 3^{3/4}} \quad \text{and} \quad b = 9$$

are the best possible. The proof of Theorem 5 is complete. □

It follows form (1), (2) and (4), (42) and (45) that

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{1}{n}\right)^{1/2} = u_n \qquad \text{(Alzer [2])}, \tag{47}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{1}{n+1}\right)^{1/2} = v_n \qquad \text{(Merkle [25])}, \tag{48}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{1}{n+\frac{1}{2}}\right)^{1/2} = w_n \qquad \text{(Ban and Chen [7])}, \tag{49}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{2}{n}\right)^{1/4} = p_n \qquad \text{(New)}, \tag{50}$$

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{2}{8n^3+24n^2+24n+9}\right) = q_n \quad \text{(New)}. \tag{51}$$

We here offer some numerical computations (see Table 1) to show the superiority of the sequence $\{q_n\}_{n\geqslant 1}$ over the sequences $\{u_n\}_{n\geqslant 1}$, $\{v_n\}_{n\geqslant 1}$, $\{w_n\}_{n\geqslant 1}$ and $\{p_n\}_{n\geqslant 1}$.

Table 1. Comparison among approximation formulas (47)–(51).

n	$\frac{u_n-V_n}{V_n}$	$\frac{V_n-v_n}{V_n}$	$\frac{w_n-V_n}{V_n}$	$\frac{p_n-V_n}{V_n}$	$\frac{q_n-V_n}{V_n}$
10	2.2651×10^{-3}	1.885×10^{-3}	9.3351×10^{-5}	1.8786×10^{-4}	1.8585×10^{-8}
100	2.4751×10^{-5}	1.885×10^{-5}	1.2131×10^{-7}	2.4264×10^{-7}	3.4961×10^{-15}
1000	2.4975×10^{-7}	2.4925×10^{-7}	1.2462×10^{-10}	2.4925×10^{-10}	3.7238×10^{-22}
10000	2.4997×10^{-9}	2.4992×10^{-9}	1.2496×10^{-13}	2.4992×10^{-13}	3.7473×10^{-29}

Here $V_n := \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$. In fact, we have, as $n\to\infty$,

$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = u_n + O\left(\frac{1}{n^2}\right),$$
$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = v_n + O\left(\frac{1}{n^2}\right),$$
$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = w_n + O\left(\frac{1}{n^3}\right),$$
$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = p_n + O\left(\frac{1}{n^3}\right),$$
$$\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = q_n + O\left(\frac{1}{n^7}\right).$$

These formulas are obtained by using the computer program MAPLE 17.

REMARK 4. The formula (41) motivated us to present the following inequalities:

$$\left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{1}{4n^3}\right) < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{1}{4n^3}+\frac{3}{4n^4}\right), \quad n\in\mathbb{N} \tag{52}$$

and

$$\left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{1}{4n^3}+\frac{3}{4n^4}-\frac{3}{2n^5}\right) < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{2}{n}\right)^{1/4}\left(1-\frac{1}{4n^3}+\frac{3}{4n^4}-\frac{3}{2n^5}+\frac{81}{32n^6}\right), \quad n\in\mathbb{N}. \tag{53}$$

Following the same method as was used in the proof of Theorem 5, we can prove the inequalities (52) and (53). Here we omit the proof.

5. Logarithmically complete monotonicity of the function $(1+\frac{1}{x})^{1/4}/I(x)$

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad \text{for} \quad x \in I \quad \text{and} \quad n \in \mathbb{N}_0. \tag{54}$$

Dubourdieu [17, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I=(a,\infty)$, then strict inequality holds true in (54). See also [20] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x>0$. See [31, p. 161].

Recall [19] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geqslant 0 \quad (x \in I;\, k \in \mathbb{N}).$$

A logarithmically completely monotonic function f on I must be completely monotonic on I (see, e.g., [6, 8, 30]).

THEOREM 6. *The function*

$$h(x) = \frac{(1+\frac{1}{x})^{1/4}}{I(x)} = \frac{(1+\frac{1}{x})^{1/4}}{(x+\frac{1}{2})}\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \tag{55}$$

is logarithmically completely monotonic on $(0,\infty)$.

Proof. The logarithm of the gamma function has the following integral representation (see [1, p. 258]):

$$\ln\Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-zt}-e^{-t}}{1-e^{-t}}\right]\frac{\mathrm{d}t}{t}. \tag{56}$$

Using (56) and

$$\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}\mathrm{d}t,$$

we obtain

$$\begin{aligned}
\ln h(x) &= \frac{1}{4}\ln\left(1+\frac{1}{x}\right) - \ln\left(x+\frac{1}{2}\right) + 2\left[\ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right)\right] \\
&= \int_0^\infty \left(\frac{1}{4} - \frac{1}{4e^t} + \frac{1}{e^{t/2}} - \frac{2}{e^{t/2}+1}\right)\frac{e^{-xt}}{t}\mathrm{d}t \\
&= \int_0^\infty p(t)e^{-xt}\mathrm{d}t,
\end{aligned} \tag{57}$$

where

$$p(t) = \left(\frac{1}{4} - \frac{1}{4e^t} + \frac{1}{e^{t/2}} - \frac{2}{e^{t/2}+1}\right)\frac{1}{t} = \frac{(e^{t/2}-1)^3}{4te^t(e^{t/2}+1)} > 0, \qquad t > 0.$$

We conclude from (57) that

$$(-1)^n\big(\ln h(x)\big)^{(n)} = \int_0^\infty t^n p(t)e^{-xt}\mathrm{d}t > 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad n \in \mathbb{N}.$$

The proof of Theorem 6 is complete. □

REMARK 5. The function $h(x)$, defined by (55), is completely monotonic on $(0,\infty)$. In particular, the sequence $\{h(n/2)\}$ is strictly decreasing for $n \in \mathbb{N}$, and we have

$$1 = h(\infty) < h\left(\frac{n}{2}\right) = \frac{(1+\frac{2}{n})^{1/4}}{I(\frac{n}{2})} \leqslant h\left(\frac{1}{2}\right) = \frac{3^{1/4}\cdot\pi}{4}, \qquad n \in \mathbb{N},$$

which derives the following double inequality for the quantity $\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}$:

$$p\left(1+\frac{2}{n}\right)^{1/4} \leqslant \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < q\left(1+\frac{2}{n}\right)^{1/4}, \qquad n \in \mathbb{N}, \tag{58}$$

with the best possible constants

$$p = \frac{4}{3^{1/4}\cdot\pi} = 0.96745284\ldots \quad \text{and} \quad q = 1.$$

Acknowledgements. This work was supported by the Key Science Research Project in the Universities of the Henan Province (20B110007) and by the Fundamental Research Funds for the Universities of the Henan Province (NSFRF210446). The authors would like to express their deep-felt gratitude for the reviewer's valuable comments to improve this paper as in the present form.

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(Received December 8, 2022)

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