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# NEW BOUNDS FOR THE IDENTRIC AND LOGARITHMIC MEANS

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*Abstract.* We assume that the numbers $x$ and $y$ are positive and unequal. Let $H(x,y)$, $G(x,y)$, $L(x,y)$, $I(x,y)$, and $A(x,y)$ be the harmonic, geometric, logarithmic, identric, and arithmetic means of $x$ and $y$, respectively. In this paper we present new bounds for the identric and logarithmic means. For example, we prove that the inequality $\frac{5}{6}A^p+\frac{1}{6}H^p<I^p$ holds for $0<p\leqslant 12/25$, and the reverse inequality holds for $p\in(-\infty,0)\cup[1,\infty)$. We prove $L^p<\frac{2}{3}G^p+\frac{1}{3}A^p$ for $p\in(-\infty,0)\cup[4/5,\infty)$.

## 1. Introduction

Throughout this paper we assume that the numbers $x$ and $y$ are positive and unequal. Let

$$H=\frac{2xy}{x+y},\quad G=\sqrt{xy},\quad L=\frac{x-y}{\ln x-\ln y},\quad I=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},\quad A=\frac{x+y}{2}$$

be the harmonic, geometric, logarithmic, identric, and arithmetic means of $x$ and $y$, respectively. The first Seiffert mean $P(x,y)$ [16] is defined by

$$P(x,y)=\frac{x-y}{2\arcsin\frac{x-y}{x+y}}.$$

It is known (see [17, 19]) that

$$H<G<L<P<I<A.$$

Sándor [13] proved that

$$\frac{2}{3}A+\frac{1}{3}G<I. \tag{1}$$

Alzer and Qiu [2] developed (1) to produce a double inequality. More precisely, these authors proved that the double inequality

$$\alpha A+(1-\alpha)G<I<\beta A+(1-\beta)G \tag{2}$$

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holds if and only if

$$\alpha \leqslant 2/3 \quad \text{and} \quad \beta \geqslant 2/e = 0.73575\dots.$$

Zhu [21, Theorem 2] (see also [22, Theorem 5.4, Eq. (5.5)]) established a more general result and proved, for $0 < p \leqslant 6/5$,

$$\alpha A^p + (1-\alpha)G^p < I^p < \beta A^p + (1-\beta)G^p \tag{3}$$

holds if and only if $\alpha \leqslant 2/3$ and $\beta \geqslant (2/e)^p$. The choice $p = 1$ in (3) yields (2).

Trif [18] proved, for $p \geqslant 2$,

$$\alpha A^p + (1-\alpha)G^p < I^p < \beta A^p + (1-\beta)G^p \tag{4}$$

holds if and only if $\alpha \leqslant (2/e)^p$ and $\beta \geqslant 2/3$. The choice $(p,\beta) = (2, 2/3)$ in the right-hand side of (4) yields

$$I^2 < \frac{2}{3}A^2 + \frac{1}{3}G^2, \tag{5}$$

which has been presented by Sándor and Trif [15].

Let $p > 0$. Kouba [8] proved that the inequality

$$I^p < \frac{2}{3}A^p + \frac{1}{3}G^p \tag{6}$$

holds if and only if $p \geqslant \ln(3/2)/\ln(e/2) = 1.3214\dots$, and the reverse inequality holds if and only if $p \leqslant 6/5$.

Zhu [20, Theorem 3] proved that the double inequality

$$\alpha A + (1-\alpha)L < I < \beta A + (1-\beta)L \tag{7}$$

holds if and only if $\alpha \leqslant 1/2$ and $\beta \geqslant 2/e$. Subsequently, Zhu [22, Theorem 1.3] established a more general result and proved, for $0 < p \leqslant 8/5$,

$$\alpha A^p + (1-\alpha)L^p < I^p < \beta A^p + (1-\beta)L^p \tag{8}$$

holds if and only if $\alpha \leqslant 1/2$ and $\beta \geqslant (2/e)^p$. The choice $p = 1$ in (8) yields (7).

Elezović [6], by using the asymptotic expansion method, proved some inequalities for means. The author also formulated several conjectures in connection with optimal inequalities. Recently, Chen [5] proved certain conjectures of Elezović. For example, in 2015 Elezović [6] conjectured, then Chen [5] proved

$$\frac{2}{e}A + \frac{e-2}{e}H < I < \frac{5}{6}A + \frac{1}{6}H. \tag{9}$$

The right-hand side of (9) motivated us to establish Theorem 1.

THEOREM 1. *For $0 < p \leqslant 12/25$, we have*

$$\frac{5}{6}A^p + \frac{1}{6}H^p < I^p. \tag{10}$$

*The reverse inequality holds for $p \in (-\infty, 0) \cup [1, \infty)$.*

It is known that

$$G^{2/3}A^{1/3} < L < \frac{2G+A}{3}. \tag{11}$$

The first inequality was proved by Carlson [4] (see also Burk [3]), and the second inequality was proved by Leach and Sholander [9]. Sándor [13] provided a simple proof of the double inequality (11). The double inequality (11) was also proved by Fechner [7, Corollary 3].

The right-hand side of (11) motivated us to establish Theorem 2.

THEOREM 2. *For* $p \in (-\infty, 0) \cup [4/5, \infty)$, *we have*

$$L^p < \frac{2}{3}G^p + \frac{1}{3}A^p. \tag{12}$$

Neuman and Sándor [10, Corollary 3.2] obtained that

$$L^2 < \frac{G^2+P^2}{2}, \quad \text{equivalently,} \quad L < \left(\frac{G^2+P^2}{2}\right)^{1/2}. \tag{13}$$

It is known that the power mean of two positive numbers

$$A_r(x,y) = \begin{cases} \left(\frac{x^r+y^r}{2}\right)^{1/r}, & r \neq 0 \\ \sqrt{xy}, & r = 0 \end{cases}$$

is strictly increasing in $r \in \mathbb{R}$. Clearly, Theorem 3 refines (13).

THEOREM 3. *The following inequality holds true:*

$$L < \frac{G+P}{2}. \tag{14}$$

REMARK 1. Alzer [1] proved

$$L < \frac{I+G}{2}. \tag{15}$$

Sándor [14] proved

$$(A^2G)^{1/3} < P < \frac{2A+G}{3} < I. \tag{16}$$

We find that the following inequality chain holds:

$$L < \frac{G+P}{2} < \frac{2G+A}{3} < \frac{I+G}{2}. \tag{17}$$

The second inequality of (17) follows by the right side of (16). The last inequality of (17) follows by the last inequality of (16). Clearly, the inequality (14) refines the right side of (11) and Alzer's inequality (15).

Noting that $HA = G^2$ holds, from (16) and the right side of (9) we obtain the following inequality chain:

$$A^{5/6}H^{1/6} < P < \frac{2A+G}{3} < I < \frac{5}{6}A + \frac{1}{6}H. \tag{18}$$

CONJECTURE 1. For $r \geqslant 4/5$, we have

$$L < \left(\frac{G^r + P^r}{2}\right)^{1/r}. \tag{19}$$

The reverse inequality holds for $r < 0$.

The following propositions will be used frequently in this paper.

PROPOSITION 1. Let $\sqrt{x/y} = e^t$, and suppose $x > y$. Then $t > 0$, and the following identities hold true:

$$\frac{H(x,y)}{G(x,y)} = \frac{1}{\cosh t}, \quad \frac{L(x,y)}{G(x,y)} = \frac{\sinh t}{t}, \quad \frac{I(x,y)}{G(x,y)} = e^{t\coth t - 1}, \quad \frac{A(x,y)}{G(x,y)} = \cosh t.$$

PROPOSITION 2. Let $(x-y)/(x+y) = z$, and suppose $x > y$. Then $z \in (0,1)$, and the following identities hold true:

$$\frac{P(x,y)}{A(x,y)} = \frac{z}{\arcsin z}, \quad \frac{G(x,y)}{A(x,y)} = \sqrt{1-z^2}, \quad \frac{L(x,y)}{A(x,y)} = \frac{2z}{\ln\frac{1+z}{1-z}}.$$

The following lemma is required in our present investigation.

LEMMA 1. (see [11, 12]) *Let* $-\infty \leqslant a < b \leqslant \infty$. *Let* $f$ *and* $g$ *be differentiable functions on an interval* $(a,b)$. *Assume that either* $g' > 0$ *everywhere on* $(a,b)$ *or* $g' < 0$ *on* $(a,b)$. *Suppose that* $f(a+) = g(a+) = 0$ *or* $f(b-) = g(b-) = 0$ *and* $\frac{f'}{g'}$ *is increasing (decreasing) on* $(a,b)$. *Then* $\frac{f}{g}$ *is increasing (respectively, decreasing) on* $(a,b)$.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

## 2. Proofs of Theorems 1–3

*Proof of Theorem 1.* For $t > 0$, let

$$f(t) = (t\coth t - 1) - \frac{1}{p}\ln\left(\frac{5}{6}(\cosh t)^p + \frac{1}{6}\left(\frac{1}{\cosh t}\right)^p\right).$$

Differentiation yields

$$\frac{\cosh^p t\left[\sinh^2 t\cosh t\left(5\cosh^p t+\frac{1}{\cosh^p t}\right)\right]}{5t\cosh t-5\sinh t}f'(t)$$
$$=\frac{2\sinh t\cosh^2 t-t\cosh t-\sinh t}{5t\cosh t-5\sinh t}-(\cosh t)^{2p}.$$

It is easy to see that

$$f'(t)\gtrless 0,\quad \text{according as}\quad g(t)\gtrless p,$$

where

$$g(t)=\frac{\ln\left(\frac{2\sinh t\cosh^2 t-t\cosh t-\sinh t}{5t\cosh t-5\sinh t}\right)}{2\ln(\cosh t)}.$$

For $t\geqslant 0$, let

$$g_1(t)=\begin{cases}\ln\left(\frac{2\sinh t\cosh^2 t-t\cosh t-\sinh t}{5t\cosh t-5\sinh t}\right), & t\neq 0\\ 0, & t=0,\end{cases}\qquad g_2(t)=2\ln(\cosh t).$$

Then, we have

$$g(t)=\frac{g_1(t)}{g_2(t)},\qquad t>0.$$

Elementary calculations reveal that

$$\frac{g_1'(t)}{g_2'(t)}=\frac{\sinh t\cosh t(2t\cosh^2 t-3\cosh t\sinh t+t)}{2t\cosh^3 t\sinh t-t^2\cosh^2 t-1+3\cosh^2 t-2\cosh^4 t}=:g_3(t).$$

Differentiation yields

$$g_3'(t)=\frac{g_4(t)}{g_5(t)},$$

where[1]

$$g_4(t)=2\cosh^7 t\sinh t-4t^3\cosh^6 t+(4t^2-8)\cosh^5 t\sinh t+(2t^3-8t)\cosh^4 t$$
$$+(5t^2+13)\cosh^3 t\sinh t-(t^3-7t)\cosh^2 t-7\cosh t\sinh t+t>0 \tag{20}$$

and

$$g_5(t)=(4t^2+4)\cosh^8 t-8t\cosh^7 t\sinh t-12\cosh^6 t-(4t^3-12t)\cosh^5 t\sinh t$$
$$+(t^4-6t^2+13)\cosh^4 t-4t\cosh^3 t\sinh t+(2t^2-6)\cosh^2 t+1>0. \tag{21}$$

[1]The inequality (20) is proved in the appendix. Following the same method as was used in the proof of (20), we can prove (21), we here omit it.

We then obtain

$$g_3'(t) > 0, \qquad t > 0.$$

Therefore, the functions $g_3(t)$ and $g_1'(t)/g_2'(t)$ are strictly increasing on $(0,\infty)$. By Lemma 1, the function

$$g(t) = \frac{g_1(t)}{g_2(t)} = \frac{g_1(t) - g_1(0)}{g_2(t) - g_2(0)}$$

is strictly increasing on $(0,\infty)$, and we have

$$\frac{12}{25} = \lim_{u\to 0^+} g(u) < g(t) < \lim_{u\to\infty} g(u) = 1 \quad \text{for} \quad t > 0.$$

For $p \geqslant 1$, we have $f'(t) < 0$. We then obtain

$$(t\coth t - 1) - \frac{1}{p}\ln\left(\frac{5}{6}(\cosh t)^p + \frac{1}{6}\left(\frac{1}{\cosh t}\right)^p\right) = f(t) < f(0) = 0,$$

which can be written, by Remark 1, as

$$I^p < \frac{5}{6}A^p + \frac{1}{6}H^p. \tag{22}$$

For $p \leqslant 12/25$, we have $f'(t) > 0$. We then obtain

$$(t\coth t - 1) - \frac{1}{p}\ln\left(\frac{5}{6}(\cosh t)^p + \frac{1}{6}\left(\frac{1}{\cosh t}\right)^p\right) = f(t) > f(0) = 0. \tag{23}$$

For $0 < p \leqslant 12/25$, (23) can be written as (10). For $p < 0$, (23) can be written as (22). The proof of Theorem 1 is complete. □

*Proof of Theorem 2.* For $t > 0$, let

$$U(t) = \ln\left(\frac{\sinh t}{t}\right) - \frac{1}{p}\ln\left(\frac{2}{3} + \frac{1}{3}(\cosh t)^p\right).$$

Differentiation yields

$$\frac{t\sinh t\cosh t\left(2 + (\cosh t)^p\right)}{\cosh t\sinh t - t}U'(t) = \frac{2t\cosh^2 t - 2\cosh t\sinh t}{\cosh t\sinh t - t} - (\cosh t)^p.$$

It is easy to see that

$$U'(t) \gtrless 0, \quad \text{according as} \quad V(t) \gtrless p,$$

where

$$V(t) = \frac{\ln\left(\frac{2t\cosh^2 t - 2\cosh t\sinh t}{\cosh t\sinh t - t}\right)}{\ln(\cosh t)}.$$

For $t \geqslant 0$, let

$$V_1(t) = \begin{cases} \ln\left(\frac{2t\cosh^2 t - 2\cosh t\sinh t}{\cosh t\sinh t - t}\right), & t \neq 0 \\ 0, & t = 0, \end{cases} \qquad V_2(t) = \ln(\cosh t).$$

Then, we have

$$V(t) = \frac{V_1(t)}{V_2(t)}, \qquad t > 0.$$

Elementary calculations reveal that

$$\frac{V_1'(t)}{V_2'(t)} = \frac{\cosh^3 t - (2t^2+1)\cosh t + t\sinh t}{t\sinh t\cosh^2 t - t^2\cosh t + t\sinh t + \cosh t - \cosh^3 t} =: V_3(t).$$

Differentiation yields

$$V_3'(t) = -\frac{V_4(t)}{V_5(t)},$$

where

$$\begin{aligned} V_4(t) = {} & \cosh^5 t\sinh t - (4t^3+11t)\cosh^4 t + (10t^2+1)\cosh^3 t\sinh t \\ & + (2t^3+13t)\cosh^2 t - (t^2+2)\sinh t\cosh t - t^3 - 2t > 0 \end{aligned} \tag{24}$$

and

$$\begin{aligned} V_5(t) = {} & (t^2+1)\cosh^6 t - 2t\cosh^5 t\sinh t + (3t^2-2)\cosh^4 t - 2t^3\cosh^3 t\sinh t \\ & + (t^4-3t^2+1)\cosh^2 t - (2t^3-2t)\sinh t\cosh t - t^2 > 0. \end{aligned} \tag{25}$$

Following the same method as was used in the proof of (20), we can prove (24) and (25), we here omit them. We then obtain

$$V_3'(t) < 0, \qquad t > 0.$$

Therefore, the functions $V_3(t)$ and $V_1'(t)/V_2'(t)$ are strictly decreasing on $(0,\infty)$. By Lemma 1, the function

$$V(t) = \frac{V_1(t)}{V_2(t)} = \frac{V_1(t) - V_1(0)}{V_2(t) - V_2(0)}$$

is strictly decreasing on $(0,\infty)$, and we have

$$0 = \lim_{u\to\infty} V(u) < V(t) < \lim_{u\to 0^+} V(u) = \frac{4}{5} \quad \text{for} \quad t > 0.$$

For $p \geqslant 4/5$, we have $U'(t) < 0$. We then obtain

$$\ln\left(\frac{\sinh t}{t}\right) - \frac{1}{p}\ln\left(\frac{2}{3} + \frac{1}{3}(\cosh t)^p\right) = U(t) < U(0) = 0, \qquad t > 0,$$

which can be written, by Remark 1, as (12).

For $p<0$, we have $U'(t)>0$. We then obtain

$$\ln\left(\frac{\sinh t}{t}\right)-\frac{1}{p}\ln\left(\frac{2}{3}+\frac{1}{3}(\cosh t)^p\right)=U(t)>U(0)=0, \qquad t>0,$$

which can be written as (12). The proof of Theorem 2 is complete. □

*Proof of Theorem 3.* By Proposition 2, (14) may be written as

$$\frac{4x}{\ln\frac{1+x}{1-x}}<\sqrt{1-x^2}+\frac{x}{\arcsin x}, \qquad 0<x<1. \tag{26}$$

By an elementary change of variable $x=\sin t$ $(0<t<\pi/2)$, (26) becomes

$$\frac{4\tan t}{1+\frac{\tan t}{t}}<\ln\left(\frac{1+\sin t}{1-\sin t}\right), \qquad 0<t<\frac{\pi}{2}. \tag{27}$$

The inequality (27) is obtained by considering the function $\lambda(t)$ defined by

$$\lambda(t)=\ln\left(\frac{1+\sin t}{1-\sin t}\right)-\frac{4\tan t}{1+\frac{\tan t}{t}}, \qquad 0<t<\frac{\pi}{2}.$$

Differentiation yields

$$\lambda'(t)=\frac{2\mu(t)}{\cos t\left(t\sin(2t)+t^2\cos^2 t+\sin^2 t\right)},$$

where[2]

$$\mu(t)=t\sin(2t)+t^2\cos^2 t-(2t^2+2)\cos t+\sin^2 t+2\cos^3 t>0. \tag{28}$$

Hence, $\lambda'(t)>0$ for $0<t<\pi/2$. So, $\lambda(t)$ is strictly increasing for $0<t<\pi/2$, and we have

$$\lambda(t)>\lim_{x\to 0^+}\lambda(x)=0, \quad 0<t<\frac{\pi}{2},$$

which means that (27) holds. The proof of Theorem 3 is complete. □

[2]The inequality (28) is proved in the appendix.

**Appendix A: A proof of (20)**

Elementary calculations reveal that

$$\begin{aligned}
g_4(t) &= \frac{1}{32}\Big(\cosh(7t)+7\cosh(5t)+21\cosh(3t)+35\cosh t\Big)\sinh t\\
&\quad -\frac{1}{8}t^3\Big(\cosh(6t)+6\cosh(4t)+15\cosh(2t)+10\Big)\\
&\quad +\left(\frac{1}{4}t^2-\frac{1}{2}\right)\Big(\cosh(5t)+5\cosh(3t)+10\cosh t\Big)\sinh t\\
&\quad +\left(\frac{1}{4}t^3-t\right)\Big(\cosh(4t)+4\cosh(2t)+3\Big)\\
&\quad +\left(\frac{5}{4}t^2+\frac{13}{4}\right)\Big(\cosh(3t)+3\cosh t\Big)\sinh t\\
&\quad -\left(\frac{1}{2}t^3-\frac{7}{2}t\right)\Big(\cosh(2t)+1\Big)-\frac{7}{2}\sinh(2t)+t\\
&= \frac{1}{64}\sinh(8t)+\left(\frac{1}{8}t^2-\frac{5}{32}\right)\sinh(6t)+\left(\frac{9}{8}t^2+\frac{27}{32}\right)\sinh(4t)\\
&\quad +\left(\frac{15}{8}t^2-\frac{41}{32}\right)\sinh(2t)-\frac{1}{8}t^3\cosh(6t)-\left(\frac{1}{2}t^3+t\right)\cosh(4t)\\
&\quad -\left(\frac{11}{8}t^3+\frac{1}{2}t\right)\cosh(2t)-t^3+\frac{3}{2}t\\
&= \sum_{n=6}^{\infty}\frac{a_n}{144\cdot(2n+1)!}t^{2n+1},
\end{aligned}$$

where

$$\begin{aligned}
a_n &= 18\cdot 64^n-(4n^3-12n^2-7n+135)36^n-(36n^3-162n^2+198n-342)16^n\\
&\quad -(396n^3-540n^2-225n+441)4^n.
\end{aligned}$$

We now prove, for $n \geqslant 6$,

$$a_n > 0.$$

Direct computations show that $a_n > 0$ is valid for $n = 6, 7, \ldots, 21$. For $n \geqslant 22$, we have

$$\begin{aligned}
a_n &> 18\cdot 64^n-(4n^3-12n^2-7n+135)36^n-(36n^3-162n^2+198n-342)36^n\\
&\quad -(396n^3-540n^2-225n+441)36^n\\
&= 18\cdot 64^n-2(218n^3-357n^2-17n+117)36^n\\
&= 18\cdot 36^n\left\{\left(\frac{64}{36}\right)^n-\frac{218n^3-357n^2-17n+117}{9}\right\}>0.
\end{aligned}$$

The last inequality can be proved by induction on $n$, we here omit it. Hence, $a_n > 0$ holds for all $n \geqslant 6$.

We then obtain $g_4(t) > 0$ for $t > 0$.

## Appendix B: A proof of (28)

Elementary calculations reveal that

$$\begin{aligned}\mu(t) &= t\sin(2t) + t^2\cos^2 t - (2t^2+2)\cos t + \sin^2 t + 2\cos^3 t\\ &= t\sin(2t) + t^2\left(\frac{1+\cos(2t)}{2}\right) - (2t^2+2)\cos t + \frac{1-\cos(2t)}{2}\\ &\quad + 2\left(\frac{\cos(3t)+3\cos t}{4}\right)\\ &= t\sin(2t) + \frac{1}{2}(t^2+1) + \frac{1}{2}(t^2-1)\cos(2t) - \left(2t^2+\frac{1}{2}\right)\cos t + \frac{1}{2}\cos(3t)\\ &= \frac{1}{18}t^6 + \frac{1}{90}t^8 - \frac{523}{151200}t^{10} + \sum_{n=6}^{\infty}(-1)^n v_n(t),\end{aligned}$$

where

$$v_n(t) = \left\{9^n - \left(n^2+\frac{3}{2}n+1\right)4^n + 16n^2 - 8n - 1\right\}\frac{1}{2\cdot(2n)!}t^{2n}.$$

If we define $P(n) = n^2 + 3n/2 + 1$ and $Q(n) = (4n-1)^2$, then, for $0 < t < \pi/2$ and $n \geqslant 6$,

$$\begin{aligned}\frac{v_{n+1}(t)}{v_n(t)} &= \frac{9t^2}{(2n+1)(2n+2)}\,\frac{1 - P(n+1)(\frac{4}{9})^{n+1} + \frac{Q(n+1)}{9^{n+1}}}{1 - P(n)(\frac{4}{9})^n + \frac{Q(n)}{9^n}}\\ &< \frac{27}{(2n+1)(2n+2)}\,\frac{1+\frac{Q(n+1)}{9^{n+1}}}{1-P(n)(\frac{4}{9})^n}.\end{aligned}$$

Noting that the sequence $x_n = 1 + \frac{Q(n+1)}{9^{n+1}}$ is strictly decreasing, and the sequence $y_n = 1 - P(n)(\frac{4}{9})^n$ is strictly increasing for $n \geqslant 6$, we find that the sequence $x_n/y_n$ is strictly decreasing for $n \geqslant 6$, and we have, for $n \geqslant 6$,

$$0 < \frac{1+\frac{Q(n+1)}{9^{n+1}}}{1-P(n)(\frac{4}{9})^n} = \frac{x_n}{y_n} < \frac{x_6}{y_6} = \frac{531522}{343025}.$$

We then obtain, for $0 < t < \pi/2$ and $n \geqslant 6$,

$$\frac{v_{n+1}(t)}{v_n(t)} < \frac{27}{13\cdot 14}\,\frac{531522}{343025} = \frac{7175547}{31215275} < 1.$$

Hence, for every $t \in (0, \pi/2)$, the sequence $n \longmapsto v_n(t)$ is strictly decreasing for $n \geqslant 6$. We then obtain

$$\mu(t) > t^6\left(\frac{1}{18} + \frac{1}{90}t^2 - \frac{523}{151200}t^4\right) > 0, \qquad 0 < t < \frac{\pi}{2}.$$

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