

CARLSON TYPE INEQUALITIES AND OPTIMAL RECOVERY OF DIFFERENTIAL OPERATORS

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Abstract. The paper considers a multidimensional problem of optimal recovery of an operator whose action is represented by multiplying the original function by a weight function of a special type, based on inaccurately specified information about the values of operators of a similar type. An exact inequality for the norms of such operators is obtained. The problem under consideration is a generalization of the problem of optimal recovery of a derivative based on other inaccurately specified derivatives in the space \mathbb{R}^d and the problem of an exact inequality, which is an analogue of the Hardy–Littlewood–Polya inequality.

1. General setting

Let X be a linear space, Y_0, Y_1, \dots, Y_N be normed linear spaces, and $\Lambda_j: X \rightarrow Y_j$, $j = 0, 1, \dots, N$, be linear operators. Consider the problem of optimal recovery of Λ_0 on the set

$$W = \{x \in X : \|\Lambda_j x\|_{Y_j} \leq \delta_j, \delta_j > 0, j = m + 1, \dots, N\},$$

where $1 \leq m \leq N$ (if $m = N$, then $W = X$), by values of operators $\Lambda_1, \dots, \Lambda_m$ given with errors. More precisely, we will assume that for each $x \in W$ we know $y = (y_1, \dots, y_m) \in Y_1 \times \dots \times Y_m$ such that $\|\Lambda_j x - y_j\|_{Y_j} \leq \delta_j$, $\delta_j > 0$, $j = 1, \dots, m$.

Any recovery method by known information $y = (y_1, \dots, y_m)$ should give an element from Y_0 that is taken as an approximate value of $\Lambda_0 x$. Thus, every recovery method is a mapping $\Phi: Y_1 \times \dots \times Y_m \rightarrow Y_0$. The error of a method Φ is defined as

$$e(\Lambda_0, W, \delta, \Phi) = \sup_{\substack{x \in W, y = (y_1, \dots, y_m) \in Y_1 \times \dots \times Y_m \\ \|\Lambda_j x - y_j\|_{Y_j} \leq \delta_j, j = 1, \dots, m}} \|\Lambda_0 x - \Phi(y)\|_{Y_0},$$

where $\delta = (\delta_1, \dots, \delta_m)$.

We are interested in those methods for which the error is minimal, i.e. those methods $\widehat{\Phi}$ for which

$$e(\Lambda_0, W, \delta, \widehat{\Phi}) = \inf_{\Phi: Y_1 \times \dots \times Y_m \rightarrow Y_0} e(\Lambda_0, W, \delta, \Phi). \quad (1)$$

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We will call such methods *optimal recovery methods*. The quantity on the right-hand side of (1) will be called *the error of optimal recovery* and denoted by $E(\Lambda_0, W, \delta)$.

Let

$$\begin{aligned} \alpha &= (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k, \quad \varphi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_k(\cdot)), \\ \varphi^\alpha(\cdot) &= \varphi_1^{\alpha_1}(\cdot) \dots \varphi_k^{\alpha_k}(\cdot), \end{aligned} \tag{2}$$

where $\varphi_j(\cdot)$, $j = 1, \dots, k$ are continuous (generally speaking, complex-valued) functions on \mathbb{R}^d . Set

$$\mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d) = \left\{ x(\cdot) \in L_p(\mathbb{R}^d) : \varphi^{\alpha^j}(\cdot)x(\cdot) \in L_p(\mathbb{R}^d), j = 1, \dots, N \right\},$$

where $1 \leq p \leq \infty$, $\alpha^j \in \mathbb{R}_+^k$, $j = 1, \dots, N$, and $\mathcal{A} = \{\alpha^1, \dots, \alpha^N\}$. We define operators $\Lambda_j: \mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d) \rightarrow L_p(\mathbb{R}^d)$ as follows

$$\Lambda_j x(\cdot) = \varphi^{\alpha^j}(\cdot)x(\cdot), \quad j = 0, 1, \dots, N.$$

For these operators we consider problem (1), in which $X = \mathcal{W}_p^{\mathcal{A}}(\mathbb{R}^d)$, $Y_0 = Y_1 = \dots = Y_N = L_p(\mathbb{R}^d)$. The corresponding set of functions W is denoted by $W_p^{\mathcal{A}}(\mathbb{R}^d, \bar{\delta})$ where $\bar{\delta} = (\delta_{m+1}, \dots, \delta_N)$. The case when $\varphi(\xi) = i\xi$ was considered in [15].

The consideration of the problem posed is connected with the desire to generalize the recovery problem for functions of many variables from inaccurately given values of derivatives (see [14, p. 249]). In addition, as a consequence of the solution of the problem under study, a generalization of the exact inequality of the Hardy–Littlewood–Polya type is obtained (see [11]).

Note that the idea of considering recovery problems for a whole family of operators was used in [1, 2, 9, 10].

2. General result

Set

$$Q = \text{co}\{(\alpha^1, \ln 1/\delta_1), \dots, (\alpha^N, \ln 1/\delta_N)\},$$

where $\text{co}A$ is the convex hull of A . Define the function $S(\cdot)$ on \mathbb{R}^k by the equality

$$S(\alpha) = \max\{z \in \mathbb{R} : (\alpha, z) \in Q\} \tag{3}$$

($S(\alpha) = -\infty$, if $(\alpha, z) \notin Q$ for all z).

Let $\alpha^0 \in \text{co}\mathcal{A}$ and let $z = \langle \alpha, \hat{\eta} \rangle + \hat{a}$, where $\hat{\eta} = (\hat{\eta}_1, \dots, \hat{\eta}_k) \in \mathbb{R}^k$, be a support hyperplane to the graph of $S(\cdot)$ at α^0 . By Caratheodory’s theorem there exist points in this hyperplane $(\alpha^{j_s}, \ln 1/\delta_{j_s})$, $s = 1, \dots, l$, $l \leq k + 1$, such that

$$\alpha^0 = \sum_{s=1}^l \theta_{j_s} \alpha^{j_s}, \quad \theta_{j_s} > 0, \quad s = 1, \dots, l, \quad \sum_{s=1}^l \theta_{j_s} = 1. \tag{4}$$

Set

$$M = \{j_1, \dots, j_l\} \cap \{1, \dots, m\}.$$

THEOREM 1. *Let $\alpha^0 \in \text{co}\mathcal{A}$. Assume that for any $a_1, \dots, a_k > 0$ there exists $\widehat{\xi} \in \mathbb{R}^d$ such that $|\varphi_j(\widehat{\xi})| = a_j$, $j = 1, \dots, k$. Then*

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) = e^{-S(\alpha^0)}. \tag{5}$$

If $M \neq \emptyset$, then all methods

$$\Phi(y(\cdot))(\cdot) = \sum_{j \in M} a_j(\cdot)y_j(\cdot), \tag{6}$$

where functions $a_{j_s}(\cdot)$, $s = 1, \dots, l$, satisfy the conditions

$$\begin{cases} \sum_{s=1}^l \varphi^{\alpha^{j_s}}(\xi) a_{j_s}(\xi) = \varphi^{\alpha^0}(\xi), & (7) \\ \left\{ \begin{array}{l} \sum_{s=1}^l \frac{\delta_{j_s}^{p'}}{\theta_{j_s}^{p'/p}} |a_{j_s}(\xi)|^{p'} \leq e^{-p'S(\alpha^0)}, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \\ \max_{1 \leq s \leq l} \frac{\delta_{j_s} |a_{j_s}(\xi)|}{\theta_{j_s}} \leq e^{-S(\alpha^0)}, \quad p = 1, \\ \sum_{s=1}^l |a_{j_s}(\xi)| \delta_{j_s} \leq e^{-S(\alpha^0)}, \quad p = \infty, \end{array} \right. & (8) \end{cases}$$

for almost all $\xi \in \mathbb{R}^d$, are optimal.

If $M = \emptyset$ then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.

Proof. We prove that

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) \geq \sup_{\substack{x(\cdot) \in W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}) \\ \|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j, j=1, \dots, m}} \|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)}. \tag{9}$$

Indeed, for any method $\Phi: (L_2(\mathbb{R}^d))^m \rightarrow L_2(\mathbb{R}^d)$ and any function $x(\cdot) \in W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta})$ such that $\|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j$, $j = 1, \dots, m$, we have

$$\begin{aligned} 2\|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} &= \|\Lambda_0 x(\cdot) - \Phi(0)(\cdot) - (\Lambda_0(-x(\cdot)) - \Phi(0)(\cdot))\|_{L_p(\mathbb{R}^d)} \\ &\leq \|\Lambda_0 x(\cdot) - \Phi(0)(\cdot)\|_{L_p(\mathbb{R}^d)} + \|\Lambda_0(-x(\cdot)) - \Phi(0)(\cdot)\|_{L_p(\mathbb{R}^d)} \\ &\leq 2e(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta, \Phi). \end{aligned}$$

Due to the arbitrariness of Φ it follows that

$$\|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta).$$

Taking the upper bound over all functions $x(\cdot)$ satisfying the given conditions, we obtain inequality (9).

The extremal problem in the right-hand side of (9) may be written in the form

$$\| |\varphi(\cdot)|^{\alpha^0} x(\cdot) \|_{L_p(\mathbb{R}^d)} \rightarrow \max, \quad \| |\varphi(\cdot)|^{\alpha^j} x(\cdot) \|_{L_p(\mathbb{R}^d)} \leq \delta_j, \quad j = 1, \dots, N, \quad (10)$$

where $|\varphi(\xi)|^\alpha = |\varphi_1(\xi)|^{\alpha_1} \dots |\varphi_k(\xi)|^{\alpha_k}$ for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$.

Let $\widehat{\xi}_{\widehat{\eta}} \in \mathbb{R}^d$ be such that $|\varphi_j(\widehat{\xi}_{\widehat{\eta}})| = e^{-\widehat{\eta}_j}$, $j = 1, \dots, k$. Consider the case when $1 \leq p < \infty$. Due to the continuity of the functions $\varphi_s(\cdot)$, $s = 1, \dots, k$, for any $\varepsilon > 0$ and any $j \in \{0, 1, \dots, N\}$ there exist $\widetilde{\delta}_j$ such that

$$\left| |\varphi(\xi)|^{p\alpha^j} - |\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^j} \right| \leq \varepsilon \quad (11)$$

for all $\xi \in B_{\widetilde{\delta}_j}(\widehat{\xi}_{\widehat{\eta}})$ where

$$B_\delta(\xi_0) = \{ \xi \in \mathbb{R}^d : |\xi - \xi_0| < \delta \}.$$

Set $\widetilde{\delta} = \min\{\widetilde{\delta}_1, \dots, \widetilde{\delta}_N\}$. Then for all $\xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})$ and all $j \in \{0, 1, \dots, N\}$ inequalities (11) hold.

Put $\widehat{A} = e^{-2\widehat{a}}$ and

$$x_{\varepsilon,p}(\xi) = \begin{cases} \left(\frac{\widehat{A}}{|B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})|\gamma_\varepsilon} \right)^{1/p}, & \xi \in B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \\ 0, & \xi \notin B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}}), \end{cases}$$

where

$$\gamma_\varepsilon = 1 + \varepsilon \frac{\widehat{A}}{\min_{1 \leq j \leq N} \delta_j^p}$$

($|B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})|$ denotes the volume of the ball $B_{\widetilde{\delta}}(\widehat{\xi}_{\widehat{\eta}})$).

We have

$$\int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^j} |x_{\varepsilon,p}(\xi)|^p d\xi \leq \frac{\widehat{A}}{\gamma_\varepsilon} \left(|\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^j} + \varepsilon \right) = \frac{e^{-p(\langle \alpha^j, \widehat{\eta} \rangle + \widehat{a})} + \widehat{A}\varepsilon}{\gamma_\varepsilon}.$$

Since $z = \langle \alpha, \widehat{\eta} \rangle + \widehat{a}$ is the equation of support hyperplane of Q , we get

$$\langle \alpha^j, \widehat{\eta} \rangle + \widehat{a} \geq \ln 1 / \delta_j.$$

Consequently,

$$\int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^j} |x_{\varepsilon,p}(\xi)|^p d\xi \leq \frac{\delta_j^p + \widehat{A}\varepsilon}{\gamma_\varepsilon} \leq \delta_j^p.$$

Thus, $x_{\varepsilon,p}(\cdot)$ is an admissible function for extremal problem (10). Therefore, taking into account (9) and (11), we obtain

$$\begin{aligned} E^p(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta) &\geq \int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^0} |x_\varepsilon(\xi)|^p d\xi \geq \frac{\widehat{A}}{\gamma_\varepsilon} \left(|\varphi(\widehat{\xi}_{\widehat{\eta}})|^{p\alpha^0} - \varepsilon \right) \\ &= \frac{e^{-p(\langle \alpha^0, \widehat{\eta} \rangle + \widehat{a})} - \widehat{A}\varepsilon}{\gamma_\varepsilon}. \end{aligned}$$

Making ε tends to zero, we have

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \bar{\delta}), \delta) \geq e^{-((\alpha^0, \hat{\eta}) + \hat{a})} = e^{-S(\alpha^0)}.$$

Now let $p = \infty$. Similar to the previous reasoning, for any $\varepsilon > 0$ there exists $\tilde{\delta} > 0$ such that for all $\xi \in B_{\tilde{\delta}}(\hat{\xi}_{\hat{\eta}})$ and all $j \in \{0, 1, \dots, N\}$ inequalities

$$\left| |\varphi(\xi)|^{\alpha^j} - |\varphi(\hat{\xi}_{\hat{\eta}})|^{\alpha^j} \right| \leq \varepsilon$$

hold.

Put

$$x_{\varepsilon, \infty}(\xi) = \begin{cases} \widehat{A} \tilde{\gamma}_{\varepsilon}^{-1}, & \xi \in B_{\tilde{\delta}}(\hat{\xi}_{\hat{\eta}}), \\ 0, & \xi \notin B_{\tilde{\delta}}(\hat{\xi}_{\hat{\eta}}), \end{cases}$$

where

$$\tilde{\gamma}_{\varepsilon} = 1 + \varepsilon \frac{\widehat{A}}{\min_{1 \leq j \leq N} \delta_j}$$

We have

$$\| |\varphi(\cdot)|^{\alpha^j} x_{\varepsilon, \infty}(\cdot) \|_{L_{\infty}(\mathbb{R}^d)} \leq \frac{\widehat{A}}{\tilde{\gamma}_{\varepsilon}} \left(|\varphi(\hat{\xi}_{\hat{\eta}})|^{p\alpha^j} + \varepsilon \right) = \frac{e^{-((\alpha^j, \hat{\eta}) + \hat{a})} + \widehat{A}\varepsilon}{\tilde{\gamma}_{\varepsilon}} \leq \delta_j.$$

Consequently, $x_{\varepsilon, \infty}(\cdot)$ is an admissible function for extremal problem (10). Thus,

$$\begin{aligned} E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \bar{\delta}), \delta) &\geq \| |\varphi(\cdot)|^{\alpha^0} x_{\varepsilon, \infty}(\cdot) \|_{L_{\infty}(\mathbb{R}^d)} \geq \frac{\widehat{A}}{\tilde{\gamma}_{\varepsilon}} \left(|\varphi(\hat{\xi}_{\hat{\eta}})|^{\alpha^0} - \varepsilon \right) \\ &= \frac{e^{-((\alpha^0, \hat{\eta}) + \hat{a})} - \widehat{A}\varepsilon}{\tilde{\gamma}_{\varepsilon}}. \end{aligned}$$

Making ε tends to zero, we have

$$E(\Lambda_0, W_p^{\mathcal{A}}(\mathbb{R}^d, \bar{\delta}), \delta) \geq e^{-((\alpha^0, \hat{\eta}) + \hat{a})} = e^{-S(\alpha^0)}. \tag{12}$$

Now we prove the optimality of the recovery methods (6). To estimate the error of methods (6) we consider the following extremal problem

$$\begin{aligned} \left\| \varphi^{\alpha^0}(\cdot)x(\cdot) - \sum_{j \in M} a_j(\cdot)y_j(\cdot) \right\|_{L_p(\mathbb{R}^d)} &\rightarrow \max, \\ \|\varphi^{\alpha^j}(\cdot)x(\cdot) - y_j(\cdot)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = 1, \dots, m, \\ \|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = m + 1, \dots, N. \end{aligned} \tag{13}$$

For $M = \emptyset$, the corresponding sums are considered equal to zero.

Put $z_j(\xi) = \varphi^{\alpha^j}(\xi)x(\xi) - y_j(\xi)$, $j = 1, \dots, m$, and

$$\omega(\xi) = \varphi^{\alpha^0}(\xi) - \sum_{j \in M} \varphi^{\alpha^j}(\xi)a_j(\xi).$$

Then (13) may be written in the form

$$\begin{aligned} \left\| \omega(\xi)x(\xi) + \sum_{j \in M} a_j(\xi)z_j(\xi) \right\|_{L_p(\mathbb{R}^d)} &\rightarrow \max, \\ \|z_j(\xi)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = 1, \dots, m, \\ \|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = m + 1, \dots, N. \end{aligned}$$

It follows from (7) that

$$\omega(\xi) = \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi).$$

Thus, we have to estimate the value of the following problem

$$\begin{aligned} \left\| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi) + \sum_{j \in M} a_j(\xi)z_j(\xi) \right\|_{L_p(\mathbb{R}^d)} &\rightarrow \max, \\ \|z_j(\xi)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = 1, \dots, m, \\ \|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} &\leq \delta_j, \quad j = m + 1, \dots, N. \end{aligned}$$

Let $1 \leq p < \infty$. Set

$$\widehat{\lambda}_{j_s} = \theta_{j_s} e^{p(\alpha^{j_s} - \alpha^0, \widehat{\eta})}, \quad s = 1, \dots, l,$$

Consider the case when $1 < p < \infty$. Then by Hölder's inequality we have

$$\begin{aligned} &\left| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi)a_j(\xi)x(\xi) + \sum_{j \in M} a_j(\xi)z_j(\xi) \right| \\ &= \left| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \frac{a_j(\xi)}{\widehat{\lambda}_j^{1/p}} \widehat{\lambda}_j^{1/p} \varphi^{\alpha^j}(\xi)x(\xi) + \sum_{j \in M} \frac{a_j(\xi)}{\widehat{\lambda}_j^{1/p}} \widehat{\lambda}_j^{1/p} z_j(\xi) \right| \\ &\leq Q_p(\xi) \left(\sum_{j \in \{j_1, \dots, j_l\} \setminus M} \widehat{\lambda}_j |\varphi^{\alpha^j}(\xi)|^p |x(\xi)|^p + \sum_{j \in M} \widehat{\lambda}_j |z_j(\xi)|^p \right)^{1/p}, \end{aligned}$$

where

$$Q_p(\xi) = \left(\sum_{s=1}^l \frac{|a_{j_s}(\xi)|^{p'}}{\widehat{\lambda}_{j_s}^{p'/p}} \right)^{1/p'}.$$

In view of equalities

$$\ln 1/\delta_{j_s} = \langle \alpha^{j_s}, \widehat{\eta} \rangle + \widehat{a}, \quad s = 1, \dots, l, \quad S(\alpha^0) = \langle \alpha^0, \widehat{\eta} \rangle + \widehat{a}$$

we obtain

$$\widehat{\lambda}_{j_s} = \theta_{j_s} e^{p(\alpha^{j_s} - \alpha^0, \widehat{\eta})} = \theta_{j_s} e^{p(\ln 1/\delta_{j_s} - S(\alpha^0))} = \frac{\theta_{j_s}}{\delta_{j_s}^p} e^{-pS(\alpha^0)}.$$

Hence,

$$Q_p(\xi) = \left(e^{pS(\alpha^0)} \sum_{s=1}^l \frac{\delta_{j_s}^{p'} |a_{j_s}(\xi)|^{p'}}{\theta_{j_s}^{p'/p}} \right)^{1/p'}. \tag{14}$$

It follows from (8) that $Q_p(\xi) \leq 1$ almost for all $\xi \in \mathbb{R}^d$. Therefore, we get

$$\begin{aligned} & \left\| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi) a_j(\xi) x(\xi) + \sum_{j \in M} a_j(\xi) z_j(\xi) \right\|_{L_p(\mathbb{R}^d)}^p \\ & \leq \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \widehat{\lambda}_j \int_{\mathbb{R}^d} |\varphi(\xi)|^{p\alpha^j} |x(\xi)|^p d\xi + \sum_{j \in M} \widehat{\lambda}_j \int_{\mathbb{R}^d} |z_j(\xi)|^p d\xi \\ & \leq \sum_{s=1}^l \widehat{\lambda}_{j_s} \delta_{j_s}^p = \sum_{s=1}^l \theta_{j_s} e^{-pS(\alpha^0)} = e^{-pS(\alpha^0)}. \end{aligned}$$

Consequently,

$$e(\Lambda_0, W_p^{\omega}(\mathbb{R}^d, \overline{\delta}), \delta, \Phi) \leq e^{-S(\alpha^0)}.$$

Taking into account (12), it means that methods (6) are optimal and equality (5) holds.

If $p = 1$ we use the inequality

$$\begin{aligned} & \left| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi) a_j(\xi) x(\xi) + \sum_{j \in M} a_j(\xi) z_j(\xi) \right| \\ & \leq Q_1(\xi) \left(\sum_{j \in \{j_1, \dots, j_l\} \setminus M} \widehat{\lambda}_j |\varphi(\xi)|^{\alpha^j} |x(\xi)| + \sum_{j \in M} \widehat{\lambda}_j |z_j(\xi)| \right), \end{aligned}$$

where

$$Q_1(\xi) = \max_{1 \leq s \leq l} \frac{|a_{j_s}(\xi)|}{\widehat{\lambda}_{j_s}}.$$

By analogy with (14) we get

$$Q_1(\xi) = e^{S(\alpha^0)} \max_{1 \leq s \leq l} \frac{\delta_{j_s} |a_{j_s}(\xi)|}{\theta_{j_s}}.$$

Using the same arguments as for the case $1 < p < \infty$, we obtain the assertion of the theorem in the case under consideration.

For $p = \infty$ we use the inequality

$$\left\| \sum_{j \in \{j_1, \dots, j_l\} \setminus M} \varphi^{\alpha^j}(\xi) a_j(\xi) x(\xi) + \sum_{j \in M} a_j(\xi) z_j(\xi) \right\|_{L_\infty(\mathbb{R}^d)} \leq \left\| \sum_{s=1}^l |a_{j_s}(\xi)| \delta_{j_s} \right\|_{L_\infty(\mathbb{R}^d)}.$$

It follows from (8) that

$$e(\Lambda_0, W_\infty^{\mathcal{A}}(\mathbb{R}^d, \overline{\delta}), \delta, \Phi) \leq e^{-S(\alpha^0)}.$$

Consequently, using (12), we obtain that methods (6) are optimal and (5) holds.

It remains to show that the set of functions $\alpha_{j_s}(\cdot)$, $s = 1, \dots, l$, satisfying conditions (7) and (8) is nonempty. Let $1 \leq p < \infty$. Consider the function

$$f(\eta) = -1 + \sum_{s=1}^l \widehat{\lambda}_{j_s} e^{-p(\alpha_{j_s} - \alpha^0, \eta)}, \quad \eta \in \mathbb{R}^k.$$

This is obviously a convex function, and it is easy to verify that $f(\widehat{\eta}) = 0$ and the derivative of this function at the point $\widehat{\eta}$ is also zero. It follows that $f(\eta) \geq 0$ for all $\eta \in \mathbb{R}^k$. Consequently,

$$f_1(\eta) = e^{-p(\alpha^0, \eta)} f(\eta) \geq 0$$

for all $\eta \in \mathbb{R}^k$. Putting $e^{-\eta_j} = |\varphi_j(\xi)|$, $j = 1, \dots, k$, we obtain that

$$g(\xi) = -|\varphi(\xi)|^{p\alpha^0} + \sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{p\alpha_{j_s}} \geq 0$$

for all $\xi \in \mathbb{R}^d$. Thus,

$$\frac{|\varphi(\xi)|^{p\alpha^0}}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{p\alpha_{j_s}}} \leq 1.$$

Set

$$a_{j_s}(\xi) = \varphi^{\alpha^0}(\xi) \frac{\widehat{\lambda}_{j_s} \varphi^{(p/2-1)\alpha_{j_s}}(\xi) \overline{\varphi}^{p/2\alpha_{j_s}}(\xi)}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{p\alpha_{j_s}}}, \quad s = 1, \dots, l.$$

It is easy to check that condition (7) is valid. If $1 < p < \infty$, then

$$\begin{aligned} \sum_{s=1}^l \frac{\delta_{j_s}^{p'} |a_{j_s}(\xi)|^{p'}}{\theta_{j_s}^{p'/p}} &= e^{-p'S(\alpha^0)} \sum_{s=1}^l \frac{|a_{j_s}(\xi)|^{p'}}{\lambda_{j_s}^{p'/p}} \\ &= e^{-p'S(\alpha^0)} \left(\frac{|\varphi(\xi)|^{p\alpha^0}}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{p\alpha_{j_s}}} \right)^{p'-1} \leq e^{-p'S(\alpha^0)}. \end{aligned}$$

If $p = 1$, then

$$\frac{\delta_{j_s} |a_{j_s}(\xi)|}{\theta_{j_s}} = e^{-S(\alpha^0)} \frac{|\varphi(\xi)|^{\alpha^0}}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{\alpha_{j_s}}} \leq e^{-S(\alpha^0)}, \quad s = 1, \dots, l.$$

If $p = \infty$, we set

$$a_{j_s}(\xi) = \varphi^{\alpha^0}(\xi) \frac{\widehat{\lambda}_{j_s} e^{-i \arg\{\varphi^{\alpha_{j_s}}(\xi)\}}}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{\alpha_{j_s}}}, \quad s = 1, \dots, l,$$

where

$$\widehat{\lambda}_{j_s} = \frac{\theta_{j_s}}{\delta_{j_s}} e^{-S(\alpha^0)}, \quad s = 1, \dots, l.$$

Then condition (7) is obviously satisfied, and

$$\sum_{s=1}^l |a_{j_s}(\xi)| \delta_{j_s} = e^{-S(\alpha^0)} \frac{|\varphi(\xi)|^{\alpha^0}}{\sum_{s=1}^l \widehat{\lambda}_{j_s} |\varphi(\xi)|^{\alpha_{j_s}}} \leq e^{-S(\alpha^0)}. \quad \square$$

Note that it follows from the proof of Theorem 1 that

$$\begin{aligned} \sup_{\substack{x(\cdot) \in W_p^{s,d}(\mathbb{R}^d, \bar{\delta}) \\ \|\Lambda_j x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \delta_j, \quad j=1, \dots, m}} \|\Lambda_0 x(\cdot)\|_{L_p(\mathbb{R}^d)} &= e^{-S(\alpha^0)} = \prod_{s=1}^l \delta_{j_s}^{\theta_{j_s}} \\ &= \min \left\{ \prod_{j=1}^N \delta_j^{\theta_j} : \theta_j \geq 0, \quad j = 1, \dots, N, \quad \sum_{j=1}^N \theta_j = 1, \quad \alpha^0 = \sum_{j=1}^N \theta_j \alpha^j \right\}. \end{aligned} \quad (15)$$

3. Exact Carlson type inequalities

The Carlson inequality [6]

$$\|x(t)\|_{L_1(\mathbb{R}_+)} \leq \sqrt{\pi} \|x(t)\|_{L_2(\mathbb{R}_+)}^{1/2} \|tx(t)\|_{L_2(\mathbb{R}_+)}^{1/2}, \quad \mathbb{R}_+ = [0, +\infty),$$

was generalized by many authors (see [3–5, 7, 8, 12, 13]). In [12] we found exact constants in inequalities of the form

$$\|w(\cdot)x(\cdot)\|_{L_q(T,\mu)} \leq K \|w_0(\cdot)x(\cdot)\|_{L_p(T,\mu)}^{\gamma} \|w_1(\cdot)x(\cdot)\|_{L_r(T,\mu)}^{1-\gamma}, \quad (16)$$

where T is a cone in a linear space, $w(\cdot)$, $w_0(\cdot)$, and $w_1(\cdot)$ are homogenous functions, μ is a homogenous measure, and $1 \leq q < p, r < \infty$ (for $T = \mathbb{R}^d$ the exact inequality was obtained in [5]). Recall that a constant K is called exact if it cannot be replaced by a smaller value. The inequality in this case is called exact.

Weighted inequalities and inequalities for derivatives do not always have a multiplicative form. For example, for analytic and bounded in the strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ functions $x(t) \not\equiv 0$ the inequality

$$\|x'(\cdot)\|_{C(\mathbb{R})} \leq \frac{1}{2\beta} \|x(\cdot)\|_{C(\mathbb{R})} \|x(\cdot)\|_{C(S_\beta)}^2 \int_0^{\pi/2} \frac{dt}{\sqrt{\|x(\cdot)\|_{C(S_\beta)}^4 \cos^2 t + \|x(\cdot)\|_{C(\mathbb{R})}^4 \sin^2 t}}$$

holds (see [14, c. 177]).

In this regard, it becomes necessary to use a more general definition of exact inequalities. Let X be a linear space, Y_0, Y_1, \dots, Y_N be normed linear spaces, and $\Lambda_j: X \rightarrow Y_j, j = 0, 1, \dots, N$, be linear operators. We say that the inequality

$$\|\Lambda_0 x\|_{Y_0} \leq \kappa(\Lambda x), \quad \Lambda x = (\|\Lambda_1 x\|_{Y_1}, \dots, \|\Lambda_N x\|_{Y_N}), \quad \kappa: \mathbb{R}_+^N \rightarrow \mathbb{R}_+, \quad (17)$$

is exact, if it is fulfilled for all $x \in X$ and there is no such $x_0 \in X$ for which

$$\sup_{\substack{x \in X \\ \|\Lambda_j x\|_{Y_j} \leq \|\Lambda_j x_0\|_{Y_j}, j=1, \dots, N}} \|\Lambda_0 x\|_{Y_0} < \kappa(\Lambda x_0).$$

PROPOSITION 1. Let $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}_+^N$. Set

$$\kappa(\delta) = \sup_{\substack{x \in X \\ \|\Lambda_j x\|_{Y_j} \leq \delta_j, j=1, \dots, N}} \|\Lambda_0 x\|_{Y_0}.$$

Then inequality (17) is exact.

Proof. For $x \in X$ we put

$$\delta = (\|\Lambda_1 x\|_{Y_1}, \dots, \|\Lambda_N x\|_{Y_N}).$$

From the definition of $\kappa(\cdot)$ inequality (17) holds. Assume that it is not exact. Then there is an element $x_0 \in X$ for which

$$\sup_{\substack{x \in X \\ \|\Lambda_j x\|_{Y_j} \leq \delta_j^0, j=1, \dots, N}} \|\Lambda_0 x\|_{Y_0} < \kappa(\delta^0),$$

where

$$\delta^0 = (\delta_1^0, \dots, \delta_N^0) = (\|\Lambda_1 x_0\|_{Y_1}, \dots, \|\Lambda_N x_0\|_{Y_N}).$$

This contradicts the definition of $\kappa(\cdot)$. \square

The solution of extremal problem (15) allows us to obtain new exact inequalities of Carlson type.

Let $\varphi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_k(\cdot))$. Assume that $\varphi_1(\cdot), \dots, \varphi_k(\cdot)$ are continuous on \mathbb{R}^d and for any $a_1, \dots, a_k > 0$ there exists $\hat{\xi} \in \mathbb{R}^d$ for which $|\varphi_j(\hat{\xi})| = a_j, j = 1, \dots, k$. It follows from Proposition 1 and (15) that for $\alpha^0 \in \text{co}\{\alpha^1, \dots, \alpha^N\}$ we have the exact inequality

$$\|\varphi^{\alpha^0}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)} \leq \min \left\{ \prod_{j=1}^N \|\varphi^{\alpha^j}(\cdot)x(\cdot)\|_{L_p(\mathbb{R}^d)}^{\theta_j} : \theta_j \geq 0, j = 1, \dots, N, \sum_{j=1}^N \theta_j = 1, \alpha^0 = \sum_{j=1}^N \theta_j \alpha^j \right\}.$$

In particular, for $\varphi(\xi) = i\xi$ and $\alpha^0 \in \text{co}\{\alpha^1, \dots, \alpha^N\}$ we obtain the exact inequality

$$\| |\xi|^{\alpha^0} x(\xi) \|_{L_p(\mathbb{R}^d)} \leq \min \left\{ \prod_{j=1}^N \| |\xi|^{\alpha^j} x(\xi) \|_{L_p(\mathbb{R}^d)}^{\theta_j} : \theta_j \geq 0, j = 1, \dots, N, \sum_{j=1}^N \theta_j = 1, \alpha^0 = \sum_{j=1}^N \theta_j \alpha^j \right\}.$$

Consider one more example. Let

$$\varphi(\xi) = \psi_\theta(\xi) = \left(|\xi_1|^\theta + \dots + |\xi_d|^\theta \right)^{2/\theta}, \quad \theta > 0. \tag{18}$$

In this case $k = 1$, Q is a convex set on \mathbb{R}^2 , and $S(\cdot)$ is a broken line. For $\alpha^0 \in \text{co}\{\alpha^1, \dots, \alpha^N\}$ we have the exact inequality

$$\| \psi_\theta^{\alpha^0}(\cdot)x(\cdot) \|_{L_p(\mathbb{R}^d)} \leq \min \left\{ \| \psi_\theta^{\alpha^{j_1}}(\cdot)x(\cdot) \|_{L_p(\mathbb{R}^d)}^\lambda \| \psi_\theta^{\alpha^{j_2}}(\cdot)x(\cdot) \|_{L_p(\mathbb{R}^d)}^{1-\lambda} : \right. \\ \left. 0 \leq \lambda \leq 1, \quad \alpha^0 = \lambda \alpha^{j_1} + (1-\lambda) \alpha^{j_2} \right\}.$$

4. Recovery of differential operators and exact inequalities

Using notation (2), we set

$$\mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d) = \left\{ x(\cdot) \in L_2(\mathbb{R}^d) : \varphi^{\alpha^j}(\cdot)Fx(\cdot) \in L_2(\mathbb{R}^d), j = 1, \dots, N \right\},$$

where

$$Fx(t) = \int_{\mathbb{R}^d} x(\tau) e^{-i\langle \tau, t \rangle} d\tau, \quad \langle \tau, t \rangle = \tau_1 t_1 + \dots + \tau_d t_d,$$

is the Fourier transform of $x(\cdot)$. Define operators $\Lambda_j: \mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)$, $j = 0, 1, \dots, N$, as follows

$$\Lambda_j x(\cdot) = F^{-1}(\varphi^{\alpha^j}(\cdot)Fx(\cdot))(\cdot), \quad j = 0, 1, \dots, N. \tag{19}$$

Consider problem (1) of optimal recovery of Λ_0 on the set

$$W_F^{\mathcal{A}}(\mathbb{R}^d) = \{ x(\cdot) \in \mathcal{W}_F^{\mathcal{A}}(\mathbb{R}^d) : \| \Lambda_j x(\cdot) \|_{L_2(\mathbb{R}^d)} \leq \delta_j, \delta_j > 0, j = m + 1, \dots, N \}$$

by values of $\Lambda_1, \dots, \Lambda_m$ given with errors.

Passing to Fourier transforms, we have

$$e(\Lambda_0, W_F^{\mathcal{A}}(\mathbb{R}^d), \delta, \Phi) \\ = \frac{1}{(2\pi)^d} \sup_{\substack{x(\cdot) \in W_F^{\mathcal{A}}(\mathbb{R}^d), y=(y_1, \dots, y_m) \in (L_2(\mathbb{R}^d))^m \\ \frac{1}{(2\pi)^d} \| \varphi^{\alpha^j}(\cdot)Fx(\cdot) - Fy_j(\cdot) \|_{Y_j} \leq \delta_j, j=1, \dots, m}} \| \varphi^{\alpha^0}(\cdot)Fx(\cdot) \\ - F(\Phi(y))(\cdot) \|_{L_2(\mathbb{R}^d)}.$$

Putting

$$z(\cdot) = \frac{1}{(2\pi)^d} Fx(\cdot), \quad z_j(\cdot) = \frac{1}{(2\pi)^d} Fy_j(\cdot) \quad j = 1, \dots, m,$$

it is easy to verify that the problem under consideration is reduced to the one considered earlier in Theorem 1 for $p = 2$.

THEOREM 2. *Let the conditions of Theorem 1 be satisfied with respect to the functions $\varphi_j(\cdot)$, $j = 1, \dots, k$, and $\alpha^0 \in \text{co } \mathcal{A}$. Then*

$$E(\Lambda_0, W_{F^{\mathcal{A}}}(\mathbb{R}^d), \delta) = e^{-S(\alpha^0)}.$$

If $M \neq \emptyset$, then all methods of the form

$$\Phi(y(\cdot))(\cdot) = F^{-1} \left(\sum_{j \in M} a_j(\cdot) F y_j(\cdot) \right),$$

where functions $a_{j_s}(\cdot)$, $s = 1, \dots, l$, satisfy the conditions

$$\begin{aligned} \sum_{s=1}^l \varphi^{\alpha_{j_s}}(\xi) a_{j_s}(\xi) &= \varphi^{\alpha^0}(\xi), \\ \sum_{s=1}^l \frac{\delta_{j_s}^2 |a_{j_s}(\xi)|^2}{\theta_{j_s}} &\leq e^{-2S(\alpha^0)} \end{aligned}$$

for almost all $\xi \in \mathbb{R}^d$, are optimal.

If $M = \emptyset$, then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.

The exact inequality

$$\|\Lambda_0(\cdot)x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \prod_{s=1}^l \|\Lambda_{j_s}(\cdot)x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_s}}$$

holds.

For $\varphi(\xi) = i\xi$ the operators Λ_j defined by (19) are the Weyl derivatives of orders α^j which are denoted by D^{α^j} . Thus, it follows from Theorem 2 the following result.

COROLLARY 1. ([14], Theorem 5.19) *Let $\alpha^0 \in \text{co } \mathcal{A}$. Then*

$$E(D^{\alpha^0}, W_{F^{\mathcal{A}}}(\mathbb{R}^d), \delta) = e^{-S(\alpha^0)}.$$

If $M \neq \emptyset$, then all methods of the form

$$\Phi(y(\cdot))(\cdot) = F^{-1} \left(\sum_{j \in M} a_j(\cdot) F y_j(\cdot) \right),$$

where functions $a_{j_s}(\cdot)$, $s = 1, \dots, l$, satisfy the conditions

$$\begin{aligned} \sum_{s=1}^l (i\xi)^{\alpha_{j_s}} a_{j_s}(\xi) &= (i\xi)^{\alpha^0}, \\ \sum_{s=1}^l \frac{\delta_{j_s}^2 |a_{j_s}(\xi)|^2}{\theta_{j_s}} &\leq e^{-2S(\alpha^0)} \end{aligned}$$

for almost all $\xi \in \mathbb{R}^d$, are optimal.

If $M = \emptyset$, then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.

The exact inequality

$$\|D^{\alpha_0} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \prod_{s=1}^l \|D^{\alpha_{js}} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{js}}$$

holds.

Set

$$\Lambda_\theta^{\alpha_j} = F^{-1}(\psi_\theta^{\alpha_j}(\cdot)Fx(\cdot))(\cdot), \quad j = 0, 1, \dots, N,$$

where the function $\psi_\theta(\cdot)$ is defined by (18). Note that $\Lambda_2 = -\Delta$, where Δ is the Laplace operator. In this case Q is the set on \mathbb{R}^2 because $k = 1$. Consequently, $1 \leq l \leq 2$. Assume that $\alpha_0 \notin \mathcal{A}$ (otherwise the answer is written out in a trivial way). Then $l = 2$.

COROLLARY 2. Let $\alpha^0 \in \text{co } \mathcal{A}$ and $\alpha_0 \notin \mathcal{A}$. Then

$$E(\Lambda_\theta^{\alpha^0}, W_F^{\mathcal{A}}(\mathbb{R}^d), \delta) = e^{-S(\alpha^0)}.$$

If $M \neq \emptyset$, then all methods of the form

$$\Phi(y(\cdot))(\cdot) = F^{-1}\left(\sum_{j \in M} a_j(\cdot)Fy_j(\cdot)\right),$$

where functions $a_{j_1}(\cdot)$, $a_{j_2}(\cdot)$ satisfy the conditions

$$\begin{aligned} \psi_\theta^{\alpha^{j_1}}(\xi)a_{j_1}(\xi) + \psi_\theta^{\alpha^{j_2}}(\xi)a_{j_2}(\xi) &= \psi_\theta^{\alpha^0}(\xi), \\ \frac{\delta_{j_1}^2 |a_{j_1}(\xi)|^2}{\theta_1} + \frac{\delta_{j_2}^2 |a_{j_2}(\xi)|^2}{1 - \theta_1} &\leq e^{-2S(\alpha^0)} \end{aligned}$$

for almost all $\xi \in \mathbb{R}^d$, are optimal.

If $M = \emptyset$, then the method $\Phi(y(\cdot))(\cdot) = 0$ is optimal.

The exact inequality

$$\|\Lambda_\theta^{\alpha^0} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \|\Lambda_\theta^{\alpha^{j_1}} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_1}} \|\Lambda_\theta^{\alpha^{j_2}} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\theta_{j_1}}$$

holds.

In particular, for $\theta = 2$ we obtain the exact inequality

$$\|(-\Delta)^{\alpha^0} x(\cdot)\|_{L_2(\mathbb{R}^d)} \leq \|(-\Delta)^{\alpha^{j_1}} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{\theta_{j_1}} \|(-\Delta)^{\alpha^{j_2}} x(\cdot)\|_{L_2(\mathbb{R}^d)}^{1-\theta_{j_1}}.$$

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