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PATULA, HARTMAN—WINTNER AND LYAPUNOV TYPE INEQUALITIES FOR A CLASS OF STURM—LIOUVILLE PROBLEMS

MOHAMED JLELI AND BESSEM SAMET*

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Abstract. We derive Patula, Hartman–Wintner, and Lyapunov type inequalities for Sturm–Liouville problems of the form

$$-u''(x)+g(x)u(x)=f(x)u(x), \qquad a<x<b,$$

where $f,g\in C([a,b])$ and $g\geqslant 0$. We then apply these inequalities to several special cases. In particular, we obtain a refinement of Bargmann's inequality for the radial Schrödinger equation.

1. Introduction

One of the important results in the qualitative theory of differential equations is the Lyapunov result [15] that can be stated as follows: Let u be a nontrivial solution to the second-order differential equation

$$-u''(x)=f(x)u(x), \quad a<x<b, \tag{1}$$

satisfying $u(a)=u(b)=0$, where $a,b\in\mathbb{R}$, $a<b$, and $f\in C([a,b])$. Then,

$$\int_a^b |f(x)|\,dx>\frac{4}{b-a}.$$

The above inequality is known in the literature as: Lyapunov's inequality. The above result has several applications in the theory of differential equations. Typical applications include number of bound state solutions, stability criteria, and bounds of eigenvalues. For a discussion about several of these applications, we refer to the survey paper [7] and the recent monograph [1]. Numerous generalizations and extensions of Lyapunov's result have been proposed in the literature. In [21], using the Sturm-Picone comparison theorem, Wintner establihed the following result: Let u be a solution to (1) and a,b are two consecutive zeros of u. Then,

$$\int_a^b f^{+}(x)\,dx>\frac{4}{b-a}, \tag{2}$$

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* Corresponding author.

where

$$f^+(x) = \max\{0, f(x)\}, \quad a \leqslant x \leqslant b.$$

Hartman and Wintner [12] generalized the above result as follows: Let u be a solution to (1) and a, b are two consecutive zeros of u. Then,

$$\int_a^b (x-a)(b-x) f^+(x)\, dx > b-a. \tag{3}$$

Using that

$$\max_{a \leqslant x \leqslant b} (x-a)(b-x) = \left(\frac{b-a}{2}\right)^2, \tag{4}$$

it can be easily seen that (2) follows from (3). Patula [17] obtained another interesting generalization of (2). Namely, he proved that, if u is a solution to (1) and a, b are two consecutive zeros of u, then

$$\begin{aligned} \int_a^c f^+(x)\, dx &> \frac{1}{c-a}, \\ \int_c^b f^+(x)\, dx &> \frac{1}{b-c}, \\ \int_a^b f^+(x)\, dx &> \frac{b-a}{(c-a)(b-c)}, \end{aligned}$$

where $c \in (a,b)$ is a point where $|u|$ admits a maximum value. Using (4), it can be easily seen that (2) follows immediately from the last above inequality. Lyapunov and Hartman-Wintner type inequalities were also studied for various kinds of problems: Higher-order differential equations [3, 4, 8], nonlinear differential equations [3, 9, 11, 19, 22], fractional differential equations [10, 14, 18, 23], partial differential equations [2, 13, 16], and others.

In this paper, we consider Sturm-Liouville problems of the form

$$-L_g u(x) = f(x) u(x), \quad a < x < b, \tag{5}$$

where $a, b \in \mathbb{R}$, $a < b$, $f, g \in C([a,b])$, $g \geqslant 0$, and

$$L_g = \frac{d^2}{dx^2} - g(x). \tag{6}$$

We establish Patula and Hartman–Wintner type inequalities for the general problem (5). Next, we apply the obtained results to the special cases: $g \equiv k$ $(k > 0)$, $g(x) = \frac{\ell(\ell+1)}{x^2}$ $(\ell > 0)$, $g(x) = \frac{k}{x^4}$ $(k > 0)$, and $g(x) = x$. In particular, for the case $g(x) = \frac{\ell(\ell+1)}{x^2}$, we obtain a refinement of Bargmann's inequality [6] for the radial Schrödinger equation.

Our main results and their proofs are provided in Section 2. Some special cases of (5) are investigated in the same Section.

2. Main results

For two functions $\varphi_1, \varphi_2 \in C^1([a,b])$, we denote by $W[\varphi_1, \varphi_2]$ the Wronskian of $\{\varphi_1, \varphi_2\}$, that is,

$$W[\varphi_1, \varphi_2](x) = \varphi_1(x)\varphi_2'(x) - \varphi_1'(x)\varphi_2(x), \quad a \leqslant x \leqslant b.$$

Let $\varphi, \psi \in C^2([a,b])$ be two functions satisfying the following conditions:

(H_1): $\{\varphi, \psi\}$ is a fundamental set of solutions to the homogeneous differential equation

$$L_g v = 0, \quad a \leqslant x \leqslant b, \tag{7}$$

where L_g is the differential operator given by (6).

(H_2): $\varphi(a) = 0$, φ is strictly increasing.

(H_3): $\psi(b) = 0$, ψ is strictly decreasing.

By Abel's theorem and due to the above conditions, we know that

$$W[\psi, \varphi](x) = C_W[\psi, \varphi] > 0, \quad a \leqslant x \leqslant b,$$

where $C_W[\psi, \varphi]$ is a constant that depends only on ψ and φ. Furthermore, from (H_2) and (H_3), we have

$$\varphi(x) > 0, \quad a < x \leqslant b$$

and

$$\psi(x) > 0, \quad a \leqslant x < b.$$

Our first main result is the following theorem.

THEOREM 1. *Let $a, b \in \mathbb{R}$, $a < b$, $f, g \in C([a,b])$, and $g \geqslant 0$. Assume that $u \in C^2([a,b])$ is a solution to* (5) *and a, b are two consecutive zeros of u. Let $\varphi, \psi \in C^2([a,b])$ be two functions satisfying* (H_1)–(H_3). *Then,*

$$\int_a^c \varphi(x) f^+(x)\, dx \geqslant \varphi'(c), \tag{8}$$

$$\int_a^c f^+(x)\, dx \geqslant \frac{\varphi'(c)}{\varphi(c)}, \tag{9}$$

$$\int_c^b \psi(x) f^+(x)\, dx \geqslant -\psi'(c), \tag{10}$$

$$\int_c^b f^+(x)\, dx \geqslant \frac{-\psi'(c)}{\psi(c)}, \tag{11}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

Proof. Let $c \in (a,b)$ be such that $|u(c)| = \max_{x\in[a,b]} |u(x)|$. Since a and b are two consecutive zeros of u, we have $u(x) \neq 0$ on (a,b), hence u has a constant sign on (a,b). Replacing u by $-u$ if necessary, we may assume that

$$u(x) > 0, \quad a < x < b. \tag{12}$$

Decompose

$$f = f^+ - f^-,$$

with $f^-(x) = \max\{0, -f(x)\}$. Then, by (5),

$$-L_g u(x) + f^-(x)u(x) = f^+(x)u(x), \qquad x \in (a,c).$$

Multiplying the above equation by φ and integrating over (a,c), we obtain

$$-\int_a^c L_g u(x)\,\varphi(x)\,dx + \int_a^c f^-(x)u(x)\varphi(x)\,dx = \int_a^c f^+(x)u(x)\varphi(x)\,dx. \tag{13}$$

Since $f^- \geqslant 0$, $\varphi > 0$ on (a,c), and $u > 0$ by (12), we have

$$-\int_a^c L_g u(x)\,\varphi(x)\,dx \leqslant -\int_a^c L_g u(x)\,\varphi(x)\,dx + \int_a^c f^-(x)u(x)\varphi(x)\,dx. \tag{14}$$

On the other hand, integrating by parts, we obtain

$$\begin{aligned}
-\int_a^c L_g u(x)\,\varphi(x)\,dx &= -\int_a^c u''(x)\,\varphi(x)\,dx + \int_a^c g(x)u(x)\varphi(x)\,dx \\
&= \Big[-u'(x)\varphi(x)\Big]_a^c + \int_a^c u'(x)\varphi'(x)\,dx + \int_a^c g(x)u(x)\varphi(x)\,dx \\
&= u'(a)\varphi(a) - u'(c)\varphi(c) + \int_a^c u'(x)\varphi'(x)\,dx + \int_a^c g(x)u(x)\varphi(x)\,dx \\
&= u'(a)\varphi(a) - u'(c)\varphi(c) \\
&\quad + \Big[u(x)\varphi'(x)\Big]_a^c - \int_a^c u(x)\big(\varphi''(x) - g(x)\varphi(x)\big)\,dx \\
&= u'(a)\varphi(a) - u'(c)\varphi(c) + u(c)\varphi'(c) - u(a)\varphi'(a) \\
&\quad - \int_a^c u(x)L_g\varphi(x)\,dx.
\end{aligned}$$

Since $\varphi(a) = 0$ (by (H$_2$)), $u'(c) = 0$, $u(a) = 0$, and $L_g\varphi(x) = 0$ (by (H$_1$)), we obtain

$$-\int_a^c L_g u(x)\,\varphi(x)\,dx = u(c)\varphi'(c). \tag{15}$$

Furthermore, since $f^+(x)\varphi(x) \geqslant 0$, by the definition of $u(c)$, we have

$$\int_a^c f^+(x)u(x)\varphi(x)\,dx \leqslant u(c)\int_a^c f^+(x)\varphi(x)\,dx. \tag{16}$$

Hence, combining (13)–(16) and using that $u(c) > 0$, we obtain (8).

Since φ is increasing and $f^+ \geqslant 0$, by (8), we have

$$\varphi'(c) \leqslant \int_a^c \varphi(x) f^+(x)\,dx \leqslant \varphi(c) \int_a^c f^+(x)\,dx,$$

that is,

$$\varphi'(c) \leqslant \varphi(c) \int_a^c f^+(x)\,dx.$$

Dividing by $\varphi(c)$ (note that $\varphi(c) > 0$), we obtain (9).

Similarly, by (5), we have

$$-L_g u(x) + f^-(x)u(x) = f^+(x)u(x), \quad x \in (c,b).$$

Multiplying the above differential equation by ψ and integrating over (c,b), we obtain

$$-\int_c^b L_g u(x)\,\psi(x)\,dx + \int_c^b f^-(x)u(x)\psi(x)\,dx = \int_c^b f^+(x)u(x)\psi(x)\,dx.$$

Arguing as above, we deduce

$$-\int_c^b L_g u(x)\,\psi(x)\,dx \leqslant u(c) \int_c^b f^+(x)\psi(x)\,dx. \tag{17}$$

Integrating by parts over (c,b) and using $u(b) = 0$, $u'(c) = 0$, $\psi(b) = 0$ (by (H$_3$)), and $L_g \psi = 0$ (by (H$_1$)), we obtain

$$-\int_c^b L_g u(x)\,\psi(x)\,dx = -u(c)\psi'(c). \tag{18}$$

Then, by (17) and (18), we obtain (10).

Since ψ is decreasing and $f^+ \geqslant 0$, by (10), we have

$$-\psi'(c) \leqslant \int_c^b \psi(x) f^+(x)\,dx \leqslant \psi(c) \int_c^b f^+(x)\,dx,$$

that is,

$$-\psi'(c) \leqslant \psi(c) \int_c^b f^+(x)\,dx.$$

Dividing by $\psi(c)$ (note that $\psi(c) > 0$), we obtain (11). This completes the proof. □

Our second main result is the following Hartman–Wintner-type inequality.

THEOREM 2. *Let $a,b \in \mathbb{R}$, $a < b$, $f,g \in C([a,b])$, and $g \geqslant 0$. Assume that $u \in C^2([a,b])$ is a solution to* (5) *and that a,b are two consecutive zeros of u. Let $\varphi,\psi \in C^2([a,b])$ satisfy* (H$_1$)–(H$_3$)*. Then,*

$$\int_a^b \varphi(x)\psi(x) f^+(x)\,dx \geqslant C_W[\psi,\varphi]. \tag{19}$$

Proof. Let $c \in (a,b)$ be such that $|u(c)| = \max_{x\in[a,b]} |u(x)|$. Writing

$$\int_a^b \varphi(x)\psi(x)f^+(x)\,dx = \int_a^c \varphi(x)\psi(x)f^+(x)\,dx + \int_c^b \varphi(x)\psi(x)f^+(x)\,dx,$$

and using that φ (resp. ψ) is increasing (resp. decreasing), we obtain

$$\int_a^b \varphi(x)\psi(x)f^+(x)\,dx \geqslant \psi(c)\int_a^c f^+(x)\varphi(x)\,dx + \varphi(c)\int_c^b f^+(x)\psi(x)\,dx.$$

Hence, by (8) and (10) (and noting that $\varphi(c) > 0$ and $\psi(c) > 0$), we get

$$\begin{aligned}\int_a^b \varphi(x)\psi(x)f^+(x)\,dx &\geqslant \psi(c)\varphi'(c) - \psi'(c)\varphi(c)\\ &= W[\psi,\varphi](c) = C_W[\psi,\varphi],\end{aligned}$$

which proves (19). □

REMARK 1. We point out that (8), (9), (10), (11), and (19) are independent of the choice of the fundamental set of solutions $\{\varphi,\psi\}$. Indeed, suppose that $\{\overline{\varphi},\overline{\psi}\}$ is another fundamental set of solutions to (7), and that $\overline{\varphi}$ (resp. $\overline{\psi}$) satisfies (H_2) (resp. (H_3)). Then

$$\overline{\varphi}(x) = \lambda_1\varphi(x) + \lambda_2\psi(x), \quad a \leqslant x \leqslant b$$

for some $\lambda_1,\lambda_2 \in \mathbb{R}$. In particular,

$$0 = \overline{\varphi}(a) = \lambda_1\varphi(a) + \lambda_2\psi(a) = \lambda_2\psi(a),$$

which implies (since $\psi(a) > 0$) that $\lambda_2 = 0$. Consequently,

$$\overline{\varphi}(x) = \lambda_1\varphi(x), \quad a \leqslant x \leqslant b,$$

where $\lambda_1 > 0$. Similarly, one shows that

$$\overline{\psi}(x) = \mu_1\psi(x), \quad a \leqslant x \leqslant b$$

for some $\mu_1 > 0$. Then

$$\int_a^c \overline{\varphi}(x)f^+(x)\,dx \geqslant \overline{\varphi}'(c)$$

is equivalent to

$$\lambda_1\int_a^c \varphi(x)f^+(x)\,dx \geqslant \lambda_1\varphi'(c),$$

that is, (8). The same argument applies to (9), (10), (11), and (19).

By Theorem 2, we obtain the following Lyapunov type inequality.

THEOREM 3. *Let $a,b \in \mathbb{R}$, $a < b$, $f,g \in C([a,b])$, and $g \geqslant 0$. Assume that $u \in C^2([a,b])$ is a solution to (5) and that a,b are two consecutive zeros of u. Let $\varphi,\psi \in C^2([a,b])$ satisfy (H_1)–(H_3). Then*

$$\int_a^b f^+(x)\,dx \geqslant \frac{C_W[\psi,\varphi]}{\max_{x\in[a,b]}\varphi(x)\psi(x)}. \tag{20}$$

2.1. The case $g \equiv 0$

In this case, (5) reduces to (1). Consider the functions

$$\varphi(x) = x - a, \qquad \psi(x) = b - x, \qquad a \leqslant x \leqslant b.$$

Then φ and ψ satisfy (H_1)–(H_3) (with $g \equiv 0$). Moreover, for all $x \in [a,b]$,

$$\begin{aligned} W[\psi,\varphi](x) &= \psi(x)\varphi'(x) - \psi'(x)\varphi(x) \\ &= (b-x)\cdot 1 - (-1)\cdot(x-a) \\ &= b-a \\ &= C_W[\psi,\varphi]. \end{aligned}$$

Applying Theorem 1, we obtain the following result of Patula [17] with non-strict inequalities).

COROLLARY 1. *Let $a,b \in \mathbb{R}$, $a < b$ and $f \in C([a,b])$ Assume that $u \in C^2([a,b])$ is a solution to (1) and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned} \int_a^c (x-a)f^+(x)\,dx &\geqslant 1, \\ \int_a^c f^+(x)\,dx &\geqslant \frac{1}{c-a}, \\ \int_c^b (b-x)f^+(x)\,dx &\geqslant 1, \\ \int_c^b f^+(x)\,dx &\geqslant \frac{1}{b-c}, \end{aligned}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

Applying Theorem 2, we obtain the following Hartman-Wintner result [12].

COROLLARY 2. *Let $a,b \in \mathbb{R}$, $a < b$ and $f \in C([a,b])$ Assume that $u \in C^2([a,b])$ is a solution to (1) and a,b are two consecutive zeros of u. Then, (3) holds with a non-strict inequality).*

Applying Theorem 3, we obtain the following Lyapunov type inequality.

COROLLARY 3. *Let $a,b \in \mathbb{R}$, $a < b$ and $f \in C([a,b])$ Assume that $u \in C^2([a,b])$ is a solution to (1) and a,b are two consecutive zeros of u. Then, (2) holds with a non-strict inequality).*

2.2. The case $g \equiv k$

We consider the Sturm-Liouville problem

$$-u''(x) + ku(x) = f(x)u(x), \quad a < x < b, \tag{21}$$

where $k > 0$ is a constant and $f \in C([a,b])$. Clearly, (21) is a special case of (5) with $g(x) = k$. Let us consider the functions

$$\varphi(x) = \sinh\left[\sqrt{k}(x-a)\right], \quad a \leqslant x \leqslant b$$

and

$$\psi(x) = \sinh\left[\sqrt{k}(b-x)\right], \quad a \leqslant x \leqslant b.$$

It can be easily seen that the functions φ and ψ satisfy (H_1)–(H_3) (with $g \equiv k$). Furthermore, for all $a \leqslant x \leqslant b$, we have

$$\begin{aligned} W[\psi,\varphi](x) &= \psi(x)\varphi'(x) - \psi'(x)\varphi(x) \\ &= \sqrt{k}\sinh\left[\sqrt{k}(b-x)\right]\cosh\left[\sqrt{k}(x-a)\right] \\ &\quad + \sqrt{k}\cosh\left[\sqrt{k}(b-x)\right]\sinh\left[\sqrt{k}(x-a)\right] \\ &= \sqrt{k}\sinh\left[\sqrt{k}(b-x)+\sqrt{k}(x-a)\right] \\ &= \sqrt{k}\sinh\left[\sqrt{k}(b-a)\right] \\ &= C_W[\psi,\varphi]. \end{aligned}$$

Then, by Theorem 1, we obtain the following Patula type inequalities.

COROLLARY 4. *Let $a,b \in \mathbb{R}$, $a < b$, $k > 0$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to (21) and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned} \int_a^c \sinh\left[\sqrt{k}(x-a)\right] f^+(x)\,dx &\geqslant \sqrt{k}\cosh\left[\sqrt{k}(c-a)\right], \\ \int_a^c f^+(x)\,dx &\geqslant \sqrt{k}\coth\left[\sqrt{k}(c-a)\right], \\ \int_c^b \sinh\left[\sqrt{k}(b-x)\right] f^+(x)\,dx &\geqslant \sqrt{k}\cosh\left[\sqrt{k}(b-c)\right], \\ \int_c^b f^+(x)\,dx &\geqslant \sqrt{k}\coth\left[\sqrt{k}(b-c)\right], \end{aligned}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

By Theorem 2, we obtain the following Hartman-Wintner type inequality.

COROLLARY 5. *Let $a,b \in \mathbb{R}$, $a < b$, $k > 0$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to (21) and a,b are two consecutive zeros of u. Then,*

$$\int_a^b \sinh\left[\sqrt{k}(x-a)\right]\sinh\left[\sqrt{k}(b-x)\right] f^+(x)\,dx \geqslant \sqrt{k}\sinh\left[\sqrt{k}(b-a)\right]. \qquad (22)$$

REMARK 2. Inequality (22) is equivalent to

$$\int_a^b \frac{\sinh\left[\sqrt{k}(x-a)\right]\sinh\left[\sqrt{k}(b-x)\right]}{k} f^+(x)\,dx \geqslant \frac{\sinh\left[\sqrt{k}(b-a)\right]}{\sqrt{k}}.$$

Passing to the limit as $k \to 0^+$, the above inequality reduces to the standard Hartman-Wintner inequality (3) (with a non-strict inequality).

Now, let us consider the function

$$\xi(x) = \varphi(x)\psi(x), \quad a \leqslant x \leqslant b.$$

An elementary calculation shows that

$$\xi'(x) = \sqrt{k}\sinh\left[\sqrt{k}(a+b-2x)\right], \quad a \leqslant x \leqslant b,$$

which implies that ξ admits a maximum value at

$$x^* = \frac{a+b}{2}.$$

Consequently, we have

$$\begin{aligned}\max_{a\leqslant x\leqslant b} \xi(x) &= \varphi\left(\frac{a+b}{2}\right)\psi\left(\frac{a+b}{2}\right)\\ &= \sinh^2\left[\sqrt{k}\left(\frac{b-a}{2}\right)\right].\end{aligned}$$

Then, by Theorem 3, we obtain the following Lyapunov type inequality.

COROLLARY 6. *Let* $a,b \in \mathbb{R}$, $a < b$, $k > 0$, *and* $f \in C([a,b])$. *Assume that* $u \in C^2([a,b])$ *is a solution to* (21) *and* a,b *are two consecutive zeros of* u. *Then,*

$$\int_a^b f^+(x)\,dx \geqslant 2\sqrt{k}\coth\left[\sqrt{k}\left(\frac{b-a}{2}\right)\right].$$

2.3. The radial Schrödinger equation

We consider the radial Schrödinger equation

$$-u''(x) + \frac{\ell(\ell+1)}{x^2}u(x) = f(x)u(x), \quad a < x < b, \tag{23}$$

where $a > 0$, $\ell > 0$ is the angular momentum, and $f \in C([a,b])$. Note that (23) is a particular case of (5) with

$$g(x) = \frac{\ell(\ell+1)}{x^2}, \quad a \leqslant x \leqslant b.$$

A result due to Bargmann (see [6], Inequality (6)) states that, if u is a solution to (23), and a,b are two consecutive zeros of u, then

$$\int_a^b x|f(x)|\,dx > \ell(\ell+1). \tag{24}$$

Using our main results, we shall provide some refinements of (24).

Let us consider the functions

$$\varphi(x) = \frac{x^{2\ell+1}-a^{2\ell+1}}{x^\ell}, \quad a \leqslant x \leqslant b$$

and

$$\psi(x) = \frac{b^{2\ell+1}-x^{2\ell+1}}{x^\ell}, \quad a \leqslant x \leqslant b.$$

It can be easily seen that the functions φ and ψ satisfy (H_1)–(H_3) (with $g(x) = \frac{\ell(\ell+1)}{x^2}$). On the other hand, for all $a \leqslant x \leqslant b$, we have

$$\begin{aligned} W[\psi,\varphi](x) &= \psi(x)\varphi'(x) - \psi'(x)\varphi(x) \\ &= \frac{b^{2\ell+1}-x^{2\ell+1}}{x^\ell}\left(\ell a^{2\ell+1}x^{-\ell-1} + (\ell+1)x^\ell\right) \\ &\quad + \frac{x^{2\ell+1}-a^{2\ell+1}}{x^\ell}\left(\ell b^{2\ell+1}x^{-\ell-1} + (\ell+1)x^\ell\right) \\ &= (2\ell+1)\left(b^{2\ell+1}-a^{2\ell+1}\right) \\ &= C_W[\psi,\varphi]. \end{aligned}$$

Then, by Theorem 1, we obtain the following Patula type inequalities.

COROLLARY 7. *Let $a,b \in \mathbb{R}$, $0 < a < b$, $\ell > 0$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to (23) and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned} \int_a^c \frac{x^{2\ell+1}-a^{2\ell+1}}{x^\ell} f^+(x)\,dx &\geqslant \frac{(\ell+1)c^{2\ell+1}+\ell a^{2\ell+1}}{c^{\ell+1}}, \\ \int_a^c f^+(x)\,dx &\geqslant \frac{(\ell+1)c^{2\ell+1}+\ell a^{2\ell+1}}{c\,(c^{2\ell+1}-a^{2\ell+1})}, \\ \int_c^b \frac{b^{2\ell+1}-x^{2\ell+1}}{x^\ell} f^+(x)\,dx &\geqslant \frac{(\ell+1)c^{2\ell+1}+\ell b^{2\ell+1}}{c^{\ell+1}}, \\ \int_c^b f^+(x)\,dx &\geqslant \frac{(\ell+1)c^{2\ell+1}+\ell b^{2\ell+1}}{c\,(b^{2\ell+1}-c^{2\ell+1})}, \end{aligned}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

By Theorem 2, we obtain the following Hartman-Wintner type inequality.

COROLLARY 8. *Let $a,b\in\mathbb{R}$, $0<a<b$, $\ell>0$, and $f\in C([a,b])$. Assume that $u\in C^2([a,b])$ is a solution to* (23) *and a,b are two consecutive zeros of u. Then,*

$$\int_a^b \frac{\left(x^{2\ell+1}-a^{2\ell+1}\right)\left(b^{2\ell+1}-x^{2\ell+1}\right)}{x^{2\ell}} f^+(x)\,dx \geqslant (2\ell+1)\left(b^{2\ell+1}-a^{2\ell+1}\right). \qquad (25)$$

REMARK 3. Passing to the limit as $\ell\to 0^+$, (25) reduces to the standard Hartman-Wintner inequality (3) (with a non-strict inequality).

The following lemma will be used in the proof of our next result.

LEMMA 1. *Let $a,b\in\mathbb{R}$, $0<a<b$, and $\ell>0$. Then,*

$$\max_{a\leqslant x\leqslant b}\varphi(x)\psi(x)=\varphi(x^*)\psi(x^*),$$

where

$$x^*=\left[\frac{a^{2\ell+1}+b^{2\ell+1}+\sqrt{\left(a^{2\ell+1}+b^{2\ell+1}\right)^2+16\ell(\ell+1)\,(ab)^{2\ell+1}}}{4(\ell+1)}\right]^{\frac{1}{2\ell+1}}. \qquad (26)$$

Proof. Let

$$\xi(x)=\varphi(x)\psi(x),\quad a\leqslant x\leqslant b,$$

that is,

$$\xi(x)=\frac{\left(x^{2\ell+1}-a^{2\ell+1}\right)\left(b^{2\ell+1}-x^{2\ell+1}\right)}{x^{2\ell}},\quad a\leqslant x\leqslant b.$$

We set

$$X=x^{2\ell+1},\quad A=a^{2\ell+1},\quad B=b^{2\ell+1},\quad p=\frac{2\ell}{2\ell+1}.$$

Then,

$$\xi(x)=H(X)=X^{-p}(X-A)(B-X),\quad A\leqslant X\leqslant B.$$

Differentiating the function H, we obtain

$$H'(X)=X^{-(p+1)}P(X),\quad A\leqslant X\leqslant B,$$

where

$$P(X)=-(2-p)X^2+(1-p)(A+B)X+pAB.$$

The discriminant of the polynomial function P is given by

$$\Delta=(1-p)^2(A+B)^2+4(2-p)pAB>0,$$

which shows that P admits two distinct roots

$$X_1=\frac{(1-p)(A+B)-\sqrt{\Delta}}{2(2-p)},\quad X_2=\frac{(1-p)(A+B)+\sqrt{\Delta}}{2(2-p)}.$$

Clearly, we have

$$X_1 < 0 < X_2.$$

On the other hand, we have

$$P_{|(-\infty,X_1)} < 0, \quad P_{|[X_1,X_2]} \geqslant 0, \quad P_{|(X_2,\infty)} < 0.$$

Calculating $P(A)$, we obtain

$$P(A) = A(B-A) > 0.$$

Calculating $P(B)$, we obtain

$$P(B) = -B(B-A) < 0.$$

Consequently, we obtain

$$0 < A < X_2 < B, \quad H'_{|[A,X_2]} \geqslant 0, \quad H'_{|(X_2,B)} < 0,$$

which shows that the function H admits a maximum value at

$$X^* = X_2.$$

Then,

$$\max_{a \leqslant x \leqslant b} \xi(x) = H(X^*) = \xi(x^*),$$

where $x^* = X^{*\frac{1}{2\ell+1}}$, that is, x^* is given by (26). □

By Theorem 3 and Lemma 1, we obtain the following Lyapunov type inequality.

COROLLARY 9. *Let* $a,b \in \mathbb{R}$, $0 < a < b$, $\ell > 0$, *and* $f \in C([a,b])$. *Assume that* $u \in C^2([a,b])$ *is a solution to* (23) *and* a,b *are two consecutive zeros of* u. *Then,*

$$\int_a^b f^+(x)\,dx \geqslant \frac{(2\ell+1)\left(b^{2\ell+1} - a^{2\ell+1}\right) x^{*2\ell}}{\left(x^{*2\ell+1} - a^{2\ell+1}\right)\left(b^{2\ell+1} - x^{*2\ell+1}\right)}, \tag{27}$$

where x^* *is given by* (26).

REMARK 4. For $\ell = 0$, by (26), we have

$$x^* = \frac{a+b}{2}$$

and

$$\begin{aligned}\frac{(2\ell+1)\left(b^{2\ell+1} - a^{2\ell+1}\right) x^{*2\ell}}{\left(x^{*2\ell+1} - a^{2\ell+1}\right)\left(b^{2\ell+1} - x^{*2\ell+1}\right)} &= \frac{b-a}{\left(\frac{a+b}{2} - a\right)\left(b - \frac{a+b}{2}\right)} \\ &= \frac{4}{b-a}.\end{aligned}$$

Then, passing to the limit as $\ell \to 0^+$, (27) reduces to the standard Lyapunov inequality (2) (with a non-strict inequality).

From Corollary 8, we deduce the following refinement of Bargmann's inequality (24).

COROLLARY 10. *Let* $a,b\in\mathbb{R}$, $0<a<b$, $\ell>0$, *and* $f\in C([a,b])$. *Assume that* $u\in C^2([a,b])$ *is a solution to* (23) *and* a,b *are two consecutive zeros of* u. *Then,*

$$\int_a^b x|f(x)|\,dx \geqslant \int_a^b xf^+(x)\,dx \geqslant (2\ell+1)\frac{b^{\ell+\frac{1}{2}}+a^{\ell+\frac{1}{2}}}{b^{\ell+\frac{1}{2}}-a^{\ell+\frac{1}{2}}} > 2\ell+1. \tag{28}$$

Proof. Let us consider the function

$$\theta(x)=\frac{\left(x^{2\ell+1}-a^{2\ell+1}\right)\left(b^{2\ell+1}-x^{2\ell+1}\right)}{x^{2\ell+1}},\quad a\leqslant x\leqslant b.$$

Setting

$$X=x^{2\ell+1},\quad A=a^{2\ell+1},\quad B=b^{2\ell+1},$$

we obtain

$$\theta(x)=\rho(X)=\frac{(X-A)(B-X)}{X},\quad A\leqslant X\leqslant B.$$

Differentiating the function ρ, we obtain

$$\rho'(X)=\frac{AB-X^2}{X^2},\quad A\leqslant X\leqslant B,$$

which shows that ρ admits a maximum value at

$$X^*=\sqrt{AB}.$$

Consequently, we have

$$\begin{aligned}\max_{a\leqslant x\leqslant b}\theta(x)&=\rho\left(\sqrt{AB}\right)\\&=\frac{\left(\sqrt{AB}-A\right)\left(B-\sqrt{AB}\right)}{\sqrt{AB}}\\&=\left(\sqrt{B}-\sqrt{A}\right)^2\\&=\left(b^{\ell+\frac{1}{2}}-a^{\ell+\frac{1}{2}}\right)^2.\end{aligned}$$

Then,

$$\begin{aligned}\int_a^b\frac{\left(x^{2\ell+1}-a^{2\ell+1}\right)\left(b^{2\ell+1}-x^{2\ell+1}\right)}{x^{2\ell}}f^+(x)\,dx&=\int_a^b\theta(x)xf^+(x)\,dx\\&\leqslant\left(b^{\ell+\frac{1}{2}}-a^{\ell+\frac{1}{2}}\right)^2\int_a^b xf^+(x)\,dx.\end{aligned}$$

Then, (28) follows from (25) and the above inequality. □

The following result follows also from Corollary 8.

COROLLARY 11. *Let* $a,b \in \mathbb{R}$, $0<a<b$, $\ell>0$, *and* $f \in C([a,b])$. *Assume that* $u \in C^2([a,b])$ *is a solution to* (23) *and* a,b *are two consecutive zeros of* u. *Then,*

$$\int_a^b \frac{f^+(x)}{x^{2\ell}}\,dx \geqslant \frac{4(2\ell+1)}{b^{2\ell+1}-a^{2\ell+1}}. \tag{29}$$

Proof. Let us introduce the function

$$\eta(x) = \left(x^{2\ell+1}-a^{2\ell+1}\right)\left(b^{2\ell+1}-x^{2\ell+1}\right), \quad a \leqslant x \leqslant b.$$

It can be easily seen that η admits a maximum value at

$$x^* = \left(\frac{a^{2\ell+1}+b^{2\ell+1}}{2}\right)^{\frac{1}{2\ell+1}},$$

which implies that for all $a \leqslant x \leqslant b$,

$$\begin{aligned}\eta(x) &\leqslant \eta(x^*)\\ &= \left(\frac{b^{2\ell+1}-a^{2\ell+1}}{2}\right)^2.\end{aligned}$$

Then, using (25), we obtain

$$\begin{aligned}\eta(x^*)\int_a^b \frac{f^+(x)}{x^{2\ell}}\,dx &\geqslant \int_a^b \eta(x)\frac{f^+(x)}{x^{2\ell}}\,dx\\ &\geqslant (2\ell+1)\left(b^{2\ell+1}-a^{2\ell+1}\right),\end{aligned}$$

which yields (29). □

2.4. The case $g(x) = \frac{k}{x^4}$

We consider the Sturm-Liouville problem

$$-u''(x) + \frac{k}{x^4}u(x) = f(x)u(x), \quad a<x<b, \tag{30}$$

where $a>0$, $k>0$, and $f \in C([a,b])$. Clearly, (30) is a special case of (5) with

$$g(x) = \frac{k}{x^4}, \quad a \leqslant x \leqslant b.$$

We consider the functions

$$\varphi(x) = x\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{x}\right)\right], \quad a \leqslant x \leqslant b$$

and

$$\psi(x) = x\sinh\left[\sqrt{k}\left(\frac{1}{x}-\frac{1}{b}\right)\right], \quad a \leqslant x \leqslant b.$$

It can be easily seen that the functions φ and ψ satisfy (H_1)–(H_3) (with $g(x)=\frac{k}{x^4}$). On the other hand, an elementary calculation shows that for all $a \leqslant x \leqslant b$,

$$\begin{aligned} W[\psi,\varphi](x) &= \sqrt{k}\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{b}\right)\right] \\ &= C_W[\psi,\varphi]. \end{aligned}$$

Using Theorem 1, we obtain the following Patula type inequalities.

COROLLARY 12. *Let $a,b \in \mathbb{R}$, $0<a<b$, $k>0$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to (30) and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned} \int_a^c x\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{x}\right)\right] f^+(x)\,dx &\geqslant \frac{c\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{c}\right)\right]+\sqrt{k}\cosh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{c}\right)\right]}{c}, \\ \int_a^c f^+(x)\,dx &\geqslant \frac{c+\sqrt{k}\coth\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{c}\right)\right]}{c^2}, \\ \int_c^b x\sinh\left[\sqrt{k}\left(\frac{1}{x}-\frac{1}{b}\right)\right] &\geqslant \frac{\sqrt{k}\cosh\left[\sqrt{k}\left(\frac{1}{c}-\frac{1}{b}\right)\right]-c\sinh\left[\sqrt{k}\left(\frac{1}{c}-\frac{1}{b}\right)\right]}{c}, \\ \int_c^b f^+(x)\,dx &\geqslant \frac{\sqrt{k}\coth\left[\sqrt{k}\left(\frac{1}{c}-\frac{1}{b}\right)\right]-c}{c^2}, \end{aligned}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

Using Theorem 2, we obtain the following Hartman-Wintner type inequality.

COROLLARY 13. *Let $a,b \in \mathbb{R}$, $0<a<b$, $k>0$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to (30) and a,b are two consecutive zeros of u. Then,*

$$\int_a^b x^2\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{x}\right)\right]\sinh\left[\sqrt{k}\left(\frac{1}{x}-\frac{1}{b}\right)\right] f^+(x)\,dx \geqslant \sqrt{k}\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{b}\right)\right]. \tag{31}$$

REMARK 5. Inequality (31) is equivalent to

$$\int_a^b x^2\frac{\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{x}\right)\right]\sinh\left[\sqrt{k}\left(\frac{1}{x}-\frac{1}{b}\right)\right]}{k} f^+(x)\,dx \geqslant \frac{\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{b}\right)\right]}{\sqrt{k}}.$$

Passing to the limit as $k \to 0^+$, the above inequality reduces to the standard Hartman-Wintner inequality (3) (with a non-strict inequality).

Using Corollary 13, we obtain the following result

COROLLARY 14. *Let* $a,b\in\mathbb{R}$, $0<a<b$, $k>0$, *and* $f\in C([a,b])$. *Assume that* $u\in C^2([a,b])$ *is a solution to* (30) *and* a,b *are two consecutive zeros of* u. *Then,*

$$\int_a^b x^2 f^+(x)\,dx \geqslant 2\coth\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right]. \tag{32}$$

Proof. Let us consider the function

$$G(x)=\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{x}\right)\right]\sinh\left[\sqrt{k}\left(\frac{1}{x}-\frac{1}{b}\right)\right],\quad a\leqslant x\leqslant b.$$

Differentiating the function G, we obtain

$$G'(x)=\frac{\sqrt{k}}{x^2}\sinh\left[\sqrt{k}\left(\frac{2}{x}-\frac{1}{a}-\frac{1}{b}\right)\right],\quad a\leqslant x\leqslant b,$$

which shows that G admits its maximum value at

$$x^*=\frac{2ab}{a+b}.$$

and

$$G(x)\leqslant G(x^*)=\sinh^2\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right].$$

Then, by (31), we obtain

$$\begin{aligned}\int_a^b x^2 f^+(x)\,dx &\geqslant \frac{\sqrt{k}\sinh\left[\sqrt{k}\left(\frac{1}{a}-\frac{1}{b}\right)\right]}{\sinh^2\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right]}\\ &=\frac{2\sinh\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right]\cosh\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right]}{\sinh^2\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right]}\\ &=2\coth\left[\frac{\sqrt{k}}{2}\left(\frac{1}{a}-\frac{1}{b}\right)\right],\end{aligned}$$

which proves (32). ☐

2.5. The Airy differential equation

We consider the Airy differential equation

$$-u''(x)+xu(x)=f(x)u(x),\quad a<x<b, \tag{33}$$

where $a>0$. This differential equation is a special case of (5) with $g(x)=x$. Before studying (33), let us recall some properties related to the Airy functions Ai and Bi. For more details about these functions, we refer to the monograph [20].

On the half-interval $(0,\infty)$, the Airy function Ai has the following integral representation:

$$\mathrm{Ai}(x)=\frac{\sqrt{3}}{2\pi}\int_0^\infty e^{-\frac{1}{3}\left(x^3t^3+\frac{1}{t^3}\right)}\,dt,\quad x>0.$$

The function Ai (for $x>0$) is positive, convex, and decreasing exponentially to zero.

The Airy function Bi has the following integral representation:

$$\mathrm{Bi}(x)=\frac{1}{\pi}\int_0^\infty\left[e^{-\frac{t^3}{3}+tx}+\sin\left(\frac{t^3}{3}+xt\right)\right]dt,\quad x>0.$$

The function Bi (for $x>0$) is positive, convex, and increasing exponentially.

The functions Ai and Bi are two linearly independent solutions to the homogeneous differential equation

$$v''(x)-xv(x)=0,\quad x>0.$$

Furthermore, we have

$$W[\mathrm{Ai},\mathrm{Bi}](x)=\frac{1}{\pi},\quad x>0. \tag{34}$$

Now, let us introduce the functions

$$\varphi(x)=\mathrm{Ai}(a)\,\mathrm{Bi}(x)-\mathrm{Ai}(x)\,\mathrm{Bi}(a),\quad a\leqslant x\leqslant b$$

and

$$\psi(x)=\mathrm{Ai}(x)\,\mathrm{Bi}(b)-\mathrm{Ai}(b)\,\mathrm{Bi}(x),\quad a\leqslant x\leqslant b$$

The functions φ and ψ satisfy (H_1)–(H_3) (with $g(x)=x$). Furthermore, for all $a\leqslant x\leqslant b$, we have

$$\begin{aligned}
W[\psi,\varphi](x)&=\psi(x)\varphi'(x)-\psi'(x)\varphi(x)\\
&=(\mathrm{Ai}(x)\,\mathrm{Bi}(b)-\mathrm{Ai}(b)\,\mathrm{Bi}(x))\left(\mathrm{Ai}(a)\,\mathrm{Bi}'(x)-\mathrm{Ai}'(x)\,\mathrm{Bi}(a)\right)\\
&\quad-\left(\mathrm{Ai}'(x)\,\mathrm{Bi}(b)-\mathrm{Ai}(b)\,\mathrm{Bi}'(x)\right)(\mathrm{Ai}(a)\,\mathrm{Bi}(x)-\mathrm{Ai}(x)\,\mathrm{Bi}(a))\\
&=\mathrm{Ai}(a)\,\mathrm{Bi}(b)\left(\mathrm{Ai}(x)\,\mathrm{Bi}'(x)-\mathrm{Ai}'(x)\,\mathrm{Bi}(x)\right)\\
&\quad-\mathrm{Ai}(b)\,\mathrm{Bi}(a)\left(\mathrm{Ai}(x)\,\mathrm{Bi}'(x)-\mathrm{Ai}'(x)\,\mathrm{Bi}(x)\right)\\
&=W[\mathrm{Ai},\mathrm{Bi}](x)\,(\mathrm{Ai}(a)\,\mathrm{Bi}(b)-\mathrm{Ai}(b)\,\mathrm{Bi}(a)),
\end{aligned}$$

which implies by (34) that

$$\begin{aligned}
W[\psi,\varphi](x)&=\frac{\mathrm{Ai}(a)\,\mathrm{Bi}(b)-\mathrm{Ai}(b)\,\mathrm{Bi}(a)}{\pi}\\
&=C_W[\psi,\varphi].
\end{aligned}$$

Then, by Theorem 1, we obtain the following Patula type inequalities.

COROLLARY 15. *Let $a,b \in \mathbb{R}$, $0 < a < b$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to* (33) *and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned}\int_a^c [\mathrm{Ai}(a)\,\mathrm{Bi}(x) - \mathrm{Ai}(x)\,\mathrm{Bi}(a)]\,f^+(x)\,dx &\geqslant \mathrm{Ai}(a)\,\mathrm{Bi}'(c) - \mathrm{Ai}'(c)\,\mathrm{Bi}(a),\\ \int_a^c f^+(x)\,dx &\geqslant \frac{\mathrm{Ai}(a)\,\mathrm{Bi}'(c) - \mathrm{Ai}'(c)\,\mathrm{Bi}(a)}{\mathrm{Ai}(a)\,\mathrm{Bi}(c) - \mathrm{Ai}(c)\,\mathrm{Bi}(a)},\\ \int_c^b [\mathrm{Ai}(x)\,\mathrm{Bi}(b) - \mathrm{Ai}(b)\,\mathrm{Bi}(x)]\,f^+(x)\,dx &\geqslant \mathrm{Ai}(b)\,\mathrm{Bi}'(c) - \mathrm{Ai}'(c)\,\mathrm{Bi}(b),\\ \int_c^b f^+(x)\,dx &\geqslant \frac{\mathrm{Ai}(b)\,\mathrm{Bi}'(c) - \mathrm{Ai}'(c)\,\mathrm{Bi}(b)}{\mathrm{Ai}(c)\,\mathrm{Bi}(b) - \mathrm{Ai}(b)\,\mathrm{Bi}(c)},\end{aligned}$$

where $c \in (a,b)$ is a point at which $|u|$ attains its maximum value.

Using Theorem 2, we obtain the following Hartman-Wintner type inequality.

COROLLARY 16. *Let $a,b \in \mathbb{R}$, $0 < a < b$, and $f \in C([a,b])$. Assume that $u \in C^2([a,b])$ is a solution to* (33) *and a,b are two consecutive zeros of u. Then,*

$$\begin{aligned}&\int_a^b [\mathrm{Ai}(a)\,\mathrm{Bi}(x) - \mathrm{Ai}(x)\,\mathrm{Bi}(a)]\,[\mathrm{Ai}(x)\,\mathrm{Bi}(b) - \mathrm{Ai}(b)\,\mathrm{Bi}(x)]\,f^+(x)\,dx\\ &\geqslant \frac{\mathrm{Ai}(a)\,\mathrm{Bi}(b) - \mathrm{Ai}(b)\,\mathrm{Bi}(a)}{\pi}.\end{aligned}$$

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Mohamed Jleli
Department of Mathematics
College of Science, King Saud University
Riyadh 11451, Saudi Arabia
e-mail: `jleli@ksu.edu.sa`

Bessem Samet
Department of Mathematics
College of Science, King Saud University
Riyadh 11451, Saudi Arabia
e-mail: `bsamet@ksu.edu.sa`