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ONE–SIDED CONCENTRATION NEAR THE MEAN OF LOG–CONCAVE DISTRIBUTIONS

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Abstract. A lower bound on the probability $P(0<X<\delta)$ for all real $\delta>0$ and all random variables X with log-concave p.d.f.'s such that $EX=0$ and $EX^2=1$ is obtained.

1. Summary and discussion

Let L denote the set of all real random variables (r.v.'s) X with a log-concave p.d.f. f_X such that $EX=0$ and $EX^2=1$. A broad survey on log-concavity was given by Saumard and Wellner [5].

THEOREM 1.1. *For all $X\in L$ and all real $\delta>0$,*

$$P(0<X<\delta)\geqslant p(\delta):=\frac{\delta}{72(1+c\delta)}, \tag{1.1}$$

where $c:=\frac{419}{100}$.

Letting $\delta\downarrow 0$ in Theorem 1.1, we immediately obtain

COROLLARY 1.2. *If $X\in L$, then $f_X(0)\geqslant\frac{1}{72}$.*

Somewhat related to these results are the following ones by Barlow, Karlin, and Proschan: if X is a *nonnegative* r.v. with a log-concave p.d.f. f_X and $EX=1$, then

(i) $f_X(0)\leqslant 1$ ([1, formula (11)]);

(ii) $f_X(x)\leqslant e^{-x}$ for some (unspecified) real x_0 and all real $x>x_0$ ([1, Theorem 5]).

The latter result implies the inequality $P(0<X<\delta)\geqslant 1-e^{-\delta}$ for real $\delta\geqslant x_0$.

The lower bound $p(\delta)$ on $P(0<X<\delta)$ in (1.1) as well as the lower bound $\frac{1}{72}$ on $f_X(0)$ in Corollary 1.2 are likely rather far from optimality. In fact, as noted by the anonymous reviewer, [2, Proposition B.2] provides a better lower on $f_X(0)$ than the one in Corollary 1.2: $f_X(0)\geqslant\frac{1}{e\sqrt{3}}$. Further studies in this direction should be welcome.

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CONJECTURE 1.3. *Let Y be a r.v. with the standard exponential distribution, so that $Y-1 \in L$. Then for each real $\delta > 0$*

$$\min_{X \in L} P(0 < X < \delta) = P(0 < Y-1 < \delta) = e^{-1}(1-e^{-\delta}),$$

and hence

$$\min_{X \in L} f_X(0) = e^{-1}.$$

The target quantities $P(0 < X < \delta)$ and $f_X(0)$, as well as the quantities EX and EX^2 in the condition $X \in L$, are linear in the p.d.f. $f := f_X$. However, the condition that f be log concave is transcribed as the system of uncountably many inequalities $f((1-t)x+ty) \geqslant f(x)^{1-t} f(y)^t$ for all real x, y and all $t \in (0,1)$, and these inequalities are nonlinear in f. So, Conjecture 1.3 represents a problem of nonlinear infinite-dimensional optimization, which may be very nontrivial – cf. e.g. [3].

2. Proof of Theorem 1.1

LEMMA 2.1. *For all $X \in L$ we have $E|X| \geqslant 1/2$.*

Proof. Take any $X \in L$. Let m be the median of X, so that $P(X < m) = P(X > m) = 1/2$. Let $Z^{\pm}$ be r.v.'s whose respective distributions are the conditional distributions of $\pm(X-m)$ given $\pm(X-m) > 0$. Then $Z^{\pm}$ are positive r.v.'s with log-concave densities. So, by a well-known inequality (see e.g. [4, formula (0.3)]),

$$2(EZ^{\pm})^2 \geqslant E(Z^{\pm})^2. \tag{45}$$

Since $EZ^{\pm} = 2E(X-m)_{\pm}$ and $E(Z^{\pm})^2 = 2E(X-m)_{\pm}^2$, where $u_{\pm} := \max(0, \pm u)$, we can rewrite (45) as $4(E(X-m)_{\pm})^2 \geqslant E(X-m)_{\pm}^2$. Therefore, and because (i) m is a minimizer of $E|X-a|$ in real a and (ii) $EX = 0$ is a minimizer of $E(X-a)^2$ in real a, we get

$$\begin{aligned}
4(E|X|)^2 &\geqslant 4(E|X-m|)^2 \\
&= 4(E(X-m)_+ + E(X-m)_-)^2 \\
&\geqslant 4(E(X-m)_+)^2 + 4(E(X-m)_-)^2 \\
&\geqslant E(X-m)_+^2 + E(X-m)_-^2 \\
&= E(X-m)^2 \geqslant EX^2 = 1.
\end{aligned}$$

So, $2E|X| \geqslant 1$. □

LEMMA 2.2. *For all real $\delta > 0$, $b \geqslant 0$, and $x \geqslant 0$,*

$$\min(\delta, x) \geqslant r_{\delta,b}(x) := a(\delta, b)\Big(\frac{x^2}{2} - b\frac{x^3}{3}\Big), \tag{2.1}$$

where

$$a(\delta,b) := \min\Big(\frac{16b}{3}, 6b^2\delta\Big) = \begin{cases} \frac{16b}{3} & \textit{if } b \geqslant \frac{8}{9\delta}, \\ 6b^2\delta & \textit{if } b \leqslant \frac{8}{9\delta}. \end{cases} \tag{2.2}$$

Moreover, for each pair $(\delta,b) \in (0,\infty)\times[0,\infty)$, $a(b,\delta)$ *is the best (that is, the largest) constant factor in* (2.1).

Proof. First, let us verify (2.1).

The case $b=0$ is obvious. So, without loss of generality (wlog) $b>0$. Then the maximum of $\frac{x^2}{2} - b\frac{x^3}{3}$ in $x \geqslant 0$ is $\frac{1}{6b^2}$, so that for all $x \geqslant 0$ we have

$$r_{\delta,b}(x) \leqslant a(\delta,b)\frac{1}{6b^2} \leqslant 6b^2\delta\frac{1}{6b^2} = \delta.$$

The case $x=0$ is obvious. So, wlog $x>0$. Then the maximum of $\frac{1}{x}\big(\frac{x^2}{2} - b\frac{x^3}{3}\big)$ in $x>0$ is $\frac{3}{16b}$, so that

$$r_{\delta,b}(x) \leqslant a(\delta,b)\frac{3}{16b}x \leqslant \frac{16b}{3}\frac{3}{16b}x = x.$$

Thus, for all real $\delta>0$, $b>0$, and $x>0$ we have $r_{\delta,b}(x) \leqslant \delta$ and $r_{\delta,b}(x) \leqslant x$, so that (2.1) follows.

As for the optimality of the constant factor $a(\delta,b)$, note that (2.1) turns into an equality if (i) $b \leqslant \frac{8}{9\delta}$ and $x=\frac{1}{b}$ or (ii) $b \geqslant \frac{8}{9\delta}$ and $x=\frac{3}{4b}$. □

LEMMA 2.3. *Take any real* $\delta>0$ *and any* $X \in L$ *such that the p.d.f.* f *of* X *is nonincreasing on* $[0,\infty)$. *Then*

$$P(0<X<\delta) \geqslant p_1(\delta) := \begin{cases} \frac{\delta}{72} & \textit{if } 0<\delta \leqslant \frac{16}{3}, \\ \frac{32(9\delta-32)}{243\delta^2} & \textit{if } \frac{16}{3} \leqslant \delta \leqslant \frac{64}{9}, \\ \frac{1}{12} & \textit{if } \delta \geqslant \frac{64}{9}. \end{cases}$$

Proof. Wlog the p.d.f. f is left continuous. Let ν be the (nonnegative) Lebesgue–Stieltjes measure over the interval $[0,\infty)$ defined by the formula

$$\nu([x,\infty)) = f(x)$$

for real $x \geqslant 0$. Then

$$\begin{aligned} P(0<X<\delta) &= \int_0^\delta dx\, f(x) \\ &= \int_0^\delta dx \int_{[x,\infty)} \nu(dy) \\ &= \int_{[0,\infty)} \nu(dy) \int_0^{\min(\delta,y)} dx \\ &= \int_{[0,\infty)} \nu(dy)\, \min(\delta,y), \end{aligned} \tag{2.3}$$

$$\begin{aligned} EX_+ &= \int_0^\infty dx\, x f(x) \\ &= \int_0^\infty dx\, x \int_{[x,\infty)} \nu(dy) \\ &= \int_{[0,\infty)} \nu(dy) \int_0^y dx\, x \\ &= \int_{[0,\infty)} \nu(dy)\, \frac{y^2}{2}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} EX_+^2 &= \int_0^\infty dx\, x^2 f(x) \\ &= \int_0^\infty dx\, x^2 \int_{[x,\infty)} \nu(dy) \\ &= \int_{[0,\infty)} \nu(dy) \int_0^y dx\, x^2 \\ &= \int_{[0,\infty)} \nu(dy)\, \frac{y^3}{3}. \end{aligned} \tag{2.5}$$

It follows from (2.3), (2.4), (2.5), and Lemma 2.2 that

$$P(0<X<\delta) \geqslant a(\delta,b)(EX_+ - b\,EX_+^2). \tag{2.6}$$

By Lemma 2.1 and the condition $EX = 0$,

$$1/2 \leqslant E|X| = EX_+ + EX_- = 2EX_+,$$

whence $EX_+ \geqslant 1/4$. Also, $EX_+^2 \leqslant EX^2 = 1$. So, in view of (2.6), for all real $\delta \geqslant 0$ and $b \geqslant 0$,

$$P(0<X<\delta) \geqslant p_2(\delta,b) := a(\delta,b)\Big(\frac{1}{4}-b\Big) = \begin{cases} \frac{16}{3} b\Big(\frac{1}{4}-b\Big) & \text{if } b \geqslant \frac{8}{9\delta} \\ 6\Big(\frac{1}{4}-b\Big) b^2 \delta & \text{if } b \leqslant \frac{8}{9\delta}. \end{cases} \tag{2.7}$$

So, $p_2(\delta,b)$ is piecewise polynomial in b, with the two polynomial pieces of degrees 2 and 3. Therefore, it is straightforward but somewhat tedious to maximize $p_2(\delta,b)$ in $b \geqslant 0$, to get

$$\max_{b \geqslant 0} p_2(\delta,b) = p_1(\delta).$$

However, to complete the proof of Lemma 2.3, it is enough to note that

$$p_1(\delta) = p_2(\delta, b_\delta),$$

where

$$b_\delta := \begin{cases} \dfrac{1}{6} & \text{if } \delta \leqslant \dfrac{16}{3}, \\ \dfrac{8}{9\delta} & \text{if } \dfrac{16}{3} \leqslant \delta \leqslant \dfrac{64}{9}, \\ \dfrac{1}{8} & \text{if } \delta \geqslant \dfrac{64}{9}; \end{cases}$$

concerning the piecewise expression of $p_2(\delta,b)$ in (2.7), note also that $b_\delta \leqslant \frac{8}{9\delta}$ if $\delta \leqslant \frac{16}{3}$, and $b_\delta \geqslant \frac{8}{9\delta}$ if $\delta \geqslant \frac{64}{9}$. $\square$

By the $x \leftrightarrow -x$ reflection, from Lemma 2.3 we get

COROLLARY 2.4. *Take any real* $\delta > 0$ *and any* $X \in L$ *such that the p.d.f.* f *of* X *is nondecreasing on* $(-\infty, 0]$. *Then*

$$P(-\delta < X < 0) \geqslant p_1(\delta).$$

We can now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that for all real $\delta > 0$

$$r_1(\delta) := \frac{p_1(\delta)}{p(\delta)} = \begin{cases} 1 + c\delta & \text{if } \delta \leqslant \dfrac{16}{3}, \\ \dfrac{256(9\delta - 32)(1 + c\delta)}{27\delta^3} & \text{if } \dfrac{16}{3} \leqslant \delta \leqslant \dfrac{64}{9}, \\ 6(c + 1/\delta) & \text{if } \delta \geqslant \dfrac{64}{9}. \end{cases}$$

Clearly, $r_1(\delta) \geqslant 1$ if $\delta \leqslant \frac{16}{3}$ or $\delta \geqslant \frac{64}{9}$. In the remaining case, when $\frac{16}{3} \leqslant \delta \leqslant \frac{64}{9}$, the second derivative in δ of the rational expression for $r_1(\delta)$ is

$$\frac{512\left(3c\delta^2 + (9 - 32c)\delta - 64\right)}{9\delta^5} \leqslant 0.$$

So, the minimum of the rational expression for $r_1(\delta)$ in δ in the interval $[\frac{16}{3}, \frac{64}{9}]$ is attained at an endpoint of this interval. So, $r_1(\delta) \geqslant 1$ for all real $\delta > 0$.

This proves Theorem 1.1 in the case when the p.d.f. f of X is nonincreasing on $[0, \infty)$.

It remains to consider the case when f is not nonincreasing on $[0,\infty)$. Then f is nondecreasing on $(-\infty,0]$. So, by Corollary 2.4,

$$f(0)=\lim_{\delta\downarrow 0}\frac{P(-\delta<X<0)}{\delta}\geqslant\lim_{\delta\downarrow 0}\frac{p_1(\delta)}{\delta}=\frac{1}{72}.$$

Also, since f is not nonincreasing on $[0,\infty)$, there is some real $u>0$ such that f is nondecreasing on the interval $[0,u]$ and nonincreasing on the interval $[u,\infty)$. In particular, $f\geqslant f(0)\geqslant\frac{1}{72}$ on the interval $[0,u]$.

So, if $u\geqslant\delta$, then $f\geqslant f(0)\geqslant\frac{1}{72}$ on the interval $[0,\delta]$ and hence

$$P(0<X<\delta)=\int_0^\delta dx\, f(x)\geqslant\frac{\delta}{72}\geqslant p(\delta).$$

So, wlog

$$0<u<\delta. \tag{2.8}$$

Note that, to get (2.6), we did not use any conditions on the r.v. X except that its p.d.f. f_X be nonincreasing on $[0,\infty)$. Moreover, to get (2.7) from (2.6), we only used the conditions $EX_+\geqslant 1/4$ and $EX_+^2\leqslant 1$. So, for all real $b\geqslant 0$, noting that the p.d.f. of the r.v. $X-u$ is nonincreasing on $[0,\infty)$, $E(X-u)_+\geqslant EX_+-u\geqslant 1/4-u$, and $E(X-u)_+^2\leqslant EX_+^2\leqslant 1$, we similarly get

$$\begin{aligned}P(u<X<\delta)&=P(0<X-u<\delta-u)\\&\geqslant p_{2;u}(\delta,b):=a(\delta-u,b)(1/4-u-b)\\&=\begin{cases}q_1(b)\text{ if }b\geqslant b_u,\\ q_2(b)\text{ if }b\leqslant b_u,\end{cases}\end{aligned}$$

where

$$q_1(b):=\frac{16}{3}b\Big(\frac{1}{4}-u-b\Big),\quad q_2(b):=6b^2\Big(\frac{1}{4}-u-b\Big)(\delta-u),$$

$$b_u:=\frac{8}{9(\delta-u)}.$$

It follows that

$$\begin{aligned}P(0<X<\delta)&=P(0<X<u)+P(u<X<\delta)\\&\geqslant\frac{u}{72}+p_{2;u}(\delta,b).\end{aligned}$$

Let us now maximize $p_{2;u}(\delta,b)$ in $b\geqslant 0$; the latter condition on b will be henceforth assumed by default. In view of (2.8), $q_1(b)\leqslant 0$ and $q_2(b)\leqslant 0$ if $u\geqslant\frac{1}{4}$, and $q_1(b)=q_2(b)=0$ if $b=0$. So,

$$\max_{b\geqslant 0}p_{2;u}(\delta,b)=0\quad\text{if }u\geqslant\frac{1}{4}.$$

It remains to maximize $p_{2;u}(\delta,b)$ in $b \geqslant 0$ assuming that $0 < u < \frac{1}{4}$. Then

$$\max_{b\geqslant 0} q_1(b) = q_1(b_{1;u}), \quad \text{where } b_{1;u} := \frac{1}{2}\Big(\frac{1}{4} - u\Big),$$

$$\max_{b\geqslant 0} q_2(b) = q_2(b_{2;u}), \quad \text{where } b_{2;u} := \frac{2}{3}\Big(\frac{1}{4} - u\Big),$$

$$b_{1;u} \geqslant b_u \iff h \geqslant r_1(u) := \frac{64}{9}\frac{1}{1-4u},$$

$$b_{2;u} \geqslant b_u \iff h \geqslant r_2(u) := \frac{16}{3}\frac{1}{1-4u},$$

where

$$h := \delta - u > 0$$

(and the latter two displayed logical equivalences hold if each of the entries of $\geqslant$ there is replaced by $\leqslant$). Note also that $r_1 \geqslant r_2$.

Collecting the pieces, we see that

$$\begin{aligned}
&\max_{b\geqslant 0} p_{2;u}(\delta,b) \\
&= p_{1;u}(\delta) \\
&:= \begin{cases} q_2(b_{2;u}) = \dfrac{1}{72}(1-4u)^3 h & \text{if } u < \dfrac{1}{4} \ \& \ h \leqslant r_2(u), \\ q_1(b_u) = q_2(b_u) = \dfrac{128}{27h}\Big(\dfrac{1}{4} - u - \dfrac{8}{9h}\Big) & \text{if } u < \dfrac{1}{4} \ \& \ r_2(u) \leqslant h \leqslant r_1(u), \\ q_1(b_{1;u}) = \dfrac{1}{12}(1-4u)^2 & \text{if } u < \dfrac{1}{4} \ \& \ h \geqslant r_1(u), \\ 0 & \text{if } u \geqslant \dfrac{1}{4}. \end{cases}
\end{aligned} \tag{2.9}$$

So,

$$P(0 < X < \delta) \geqslant p_u(\delta) := \frac{u}{72} + p_{1;u}(\delta). \tag{2.10}$$

It remains to show that

$$p_u(\delta) \geqslant p(\delta) \tag{2.11}$$

if $0 < u < \delta < \infty$.

This follows immediately from the four lemmas below. □

LEMMA 2.5. *If* $0 < u < \frac{1}{4}$, $0 < h \leqslant r_2(u)$, *and* $\delta = u + h$, *then (cf.* (2.10)*, the first of the four lines in* (2.9)*, and* (1.1)*)*

$$d_1 := \frac{u}{72} + \frac{1}{72}(1-4u)^3 h - \frac{\delta}{72(1+c\delta)} \geqslant 0.$$

Proof of Lemma 2.5. Note that

$$\begin{aligned} D_1(h) &:= 72(100+419(h+u))d_1 \\ &= 419(1-4u)^3h^2 - 2u(13408u^3 - 6856u^2 + 114u + 181)h + 419u^2 \end{aligned}$$

is a strictly convex polynomial in h, whose discriminant

$$16u^2\left(16u^2 - 12u + 3\right)\left(2808976u^4 - 765932u^3 - 318917u^2 + 131400u - 11900\right)$$

is $\leqslant 0$ and hence $D_1(h) \geqslant 0$ for all real h if $u \leqslant 24/100$. On the other hand, if $u \geqslant 24/100$, then the critical value of h for this polynomial $D_1(h)$ is

$$\frac{u\left(13408u^3 - 6856u^2 + 114u + 181\right)}{419(1-4u)^3} \leqslant 0,$$

so that $D_1(h) \geqslant D_1(0) = 419u^2 \geqslant 0$. So, $D_1(h) \geqslant 0$ and hence $d_1 \geqslant 0$ for all $u \in (0, \frac{1}{4})$ and all real $h > 0$. (The condition $h \leqslant r_2(u)$ was not needed or used in this proof.) □

REMARK 1. The sign pattern of a polynomial in $\mathbb{R}[x]$ can be determined using Sturm's theorem – see e.g. [6, p. 244]. Especially when the degree of the polynomial is rather small, one can also use calculus to determine the convexity and monotonicity patterns. In principle, all this can be done by hand. However, this is much more easy to do using any computer algebra system.

LEMMA 2.6. *If* $0 < u < \frac{1}{4}$, $r_2(u) \leqslant h \leqslant r_1(u)$, *and* $\delta = u + h$, *then (cf.* (2.10)*, the second of the four lines in* (2.9)*, and* (1.1)*)*

$$d_2 := \frac{u}{72} + \frac{128}{27h}\left(\frac{1}{4} - u - \frac{8}{9h}\right) - \frac{\delta}{72(1+c\delta)} \geqslant 0.$$

Proof of Lemma 2.6. Note that

$$\begin{aligned} D_2 &:= 2169h^2(100+419(h+u))\,d_2 \\ &= 27h^3(419u-100) + 3771h^2\left(3u^2 - 1024u + 256\right) \\ &\quad - 256h\left(15084u^2 - 171u + 12508\right) - 8192(419u+100) \end{aligned}$$

is a polynomial in u, h. Consider the partial derivatives of D_2 in u and in h:

$$\begin{aligned} D_{21} &:= \partial_u D_2 \\ &= 11313h^3 + 22626h^2u - 3861504h^2 - 7723008hu + 43776h - 3432448, \\ D_{22} &:= \partial_h D_2 \\ &= 33939h^2u - 8100h^2 + 22626hu^2 - 7723008hu + 1930752h - 3861504u^2 \\ &\qquad + 43776u - 3202048. \end{aligned}$$

The resultants of the polynomials D_{21} and D_{22} in u,h with respect to u and h are

$$\begin{aligned}R_1(h) := &-383951907h^7+360127950804h^6-111582387392256h^5\\&+11376013014678528h^4+7495868032745472h^3-5874950492651520h^2\\&+17672548909056h-3016115013812224,\end{aligned}$$

$$\begin{aligned}R_2(u) := &-1807585595653280832u^7-847701988664287064715u^6\\&-785233657767535818607 62u^5+448923109288432998756 96u^4\\&+432521111274031740646 08u^3-353983573223105052591 36u^2\\&+291657451375181150330 88u-519226651413957931 8272.\end{aligned}$$

Note that $r_2(u) > \frac{16}{3}$, so that the condition $r_2(u) \leqslant h \leqslant r_1(u)$ in Lemma 2.6 implies $h > \frac{16}{3}$.

There are three real roots of $R_1(h)$ that are $> \frac{16}{3}$:

$$(h_1,h_2,h_3) \approx (255.934, 341.325, 341.342).$$

There is only one root of $R_2(u)$ in the interval $(0,\frac{1}{4})$, $u_1 \approx 0.218271$. Therefore (see e.g. [6, top of p. 104]) $D_{21} = 0 = D_{22}$ only if $(u,h) = (u_1,h_j)$ for some $j = 1,2,3$.

On the other hand, for all $j = 1,2,3$,

$$D_2|_{u=u_1,\, h=h_j} > 3\times 10^9 > 0.$$

So, $D_2 > 0$ at all critical points point (u,h) of D_2 such that $0 < u < \frac{1}{4}$ and $r_2(u) < h < r_1(u)$.

Also, $r_1(u) > r_2(u) \to \infty$ as $u \uparrow \frac{1}{4}$ and hence

$$\lim_{u\uparrow\frac{1}{4}} \min_{h\in[r_2(u),r_1(u)]} \frac{D_2}{h^3} = 27(419\times\tfrac{1}{4}-100) > 0,$$

so that $D_2 \to \infty > 0$ uniformly in $h \in [r_2(u), r_1(u)]$ as $u \uparrow \frac{1}{4}$, which shows that $D_2 > 0$ for some $u_0 \in (0,\frac{1}{4})$, all $u \in (u_0,\frac{1}{4})$, and all $h \in [r_2(u), r_1(u)]$.

Therefore, to complete the proof of Lemma 2.6, it remains to show that $D_2 \geqslant 0$ on the boundary of the (unbounded) set

$$G := \{(u,h)\colon 0 < u < \tfrac{1}{4},\ r_2(u) \leqslant h \leqslant r_1(u)\}.$$

To do this, note first that $r_2(0) = \frac{16}{3}$, $r_1(0) = \frac{64}{9}$, and

$$D_2|_{u=0} = 4(-675h^3+241344h^2-800512h-204800) > 0$$

if $r_2(0) \leqslant h \leqslant r_1(0)$. It remains to note that

$$\begin{aligned}&D_2|_{h=r_1(u)}\\&= -\frac{4096}{27(1-4u)^3}\left(1448064u^4-725364u^3-2565795u^2+1302526u-159896\right) \geqslant 0,\end{aligned}$$

$$D_2|_{h=r_2(u)} = -\frac{256}{3(1-4u)^3}\left(1287168u^4 - 643092u^3 - 1709051u^2 + 875488u - 107264\right) \geqslant 0$$

if $0 < u < \frac{1}{4}$. □

LEMMA 2.7. *If* $0 < u < \frac{1}{4}$, $h \geqslant r_1(u)$, *and* $\delta = u + h$, *then (cf.* (2.10)*, the third of the four lines in* (2.9)*, and* (1.1)*)*

$$d_3 := \frac{u}{72} + \frac{1}{12}(1-4u)^2 - \frac{\delta}{72(1+c\delta)} \geqslant 0.$$

Proof of Lemma 2.7. Note that

$$\begin{aligned} D_3(\delta) &:= 72(100+419\delta)d_3 \\ &= (40224u^2 - 19693u + 2414)\delta + 100\left(96u^2 - 47u + 6\right) \end{aligned}$$

is a polynomial of degree 1 in δ with coefficients that are positive for all real u. So, for all $\delta > 0$ and all real u, we have $D_3(\delta) \geqslant 0$ and hence $d_3 \geqslant 0$. (The conditions $0 < u < \frac{1}{4}$ and $h \geqslant r_1(u)$ were not needed or used in this proof.) □

LEMMA 2.8. *If* $u \geqslant \frac{1}{4}$, *then (cf.* (2.10)*, the fourth of the four lines in* (2.9)*, and* (1.1)*)*

$$d_4 := \frac{u}{72} - \frac{\delta}{72(1+c\delta)} \geqslant 0.$$

Proof of Lemma 2.8. Note that, for $u \geqslant \frac{1}{4}$ and $\delta > 0$,

$$d_4 \geqslant \frac{1/4}{72} - \frac{\delta}{72(0+c\delta)} = \frac{1}{72\times 4} - \frac{1}{72c} \geqslant 0.$$

Lemma 2.8 is proved. □

This completes the proof of Theorem 1.1. □

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