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EXTENSIONS OF CONVEXITY–BASED INEQUALITIES VIA POSITIVE LINEAR FUNCTIONALS

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Abstract. This paper establishes refined extensions of fundamental inequalities such as Hardy, Hölder, Minkowski, AM-GM, and Ky Fan types. Our approach employs partition-based weights, positive linear functionals, and time scale calculus to obtain sharper estimates valid across continuous, discrete, and quantum domains. We derive new bounds for Csiszár divergences, Kullback-Leibler divergence, Shannon entropy, and related measures. These results link convexity-based refinements with modern information theory and provide analytic tools for optimization, error estimation, and hybrid dynamical systems.

1. Introduction

Functional inequalities play a central role in analysis, probability, and information theory. Classical results such as Hölder's inequality, Minkowski's inequality, Hardy's inequality, the arithmetic-geometric mean (AM-GM) inequality, and Ky Fan-type inequalities form the backbone of modern mathematical analysis [7, 12]. Refinements of Jensen's inequality [9, 13] have further provided a unifying principle that generates stronger forms of these results.

In recent years, convexity-based techniques have found deep connections with information theory and statistics. The framework of f-divergences and entropy-type measures links inequalities with discrepancy measures such as Csiszár divergence [5, 6], the Kullback-Leibler divergence [11], and Shannon entropy. These tools are central in data sciences, machine learning, and statistical inference [4].

At the same time, the theory of dynamic equations on time scales provides a unified approach to continuous, discrete, and quantum models [3]. Since time-scale integrals are positive linear functionals, they offer a natural platform for extending inequalities across hybrid domains. Several recent works have advanced refinements of inequalities in this direction [1, 2, 8, 10, 14, 15].

Motivated by these developments, our approach simultaneously refines Hardy, Hölder, Minkowski, AM-GM, and Ky Fan inequalities, while also establishing sharper bounds for divergence and entropy measures. These results provide a unified analytic

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toolkit with potential applications in optimization, error estimation, and hybrid dynamical systems.

Now we recall some fundamental concepts and results that will serve as the basis for our main developments. We begin with the notion of positive linear functionals, which provides a natural setting for extending convexity-based inequalities.

DEFINITION 1. Let $\mathscr{N}$ be a linear space of real-valued functions on a non-empty set S. A map $F : \mathscr{N} \to \mathbb{R}$ is called a *positive linear functional* if it is linear and satisfies $F(f) \geqslant 0$ whenever $f(x) \geqslant 0$ for all $x \in S$.

Jessen's extension of Jensen's inequality to positive linear functionals serves as a powerful unifying tool for deriving refinements of many classical inequalities. The following theorem, presented in [9], generalizes Jessen's inequality and forms a cornerstone for our subsequent developments.

THEOREM 1. (Jessen-type Inequality) *Suppose S, $\mathscr{N}$, and F are given as in Definition 1, and let $\psi \in C(I,\mathbb{R})$ be convex. Assume that $\zeta, \alpha, \beta \in \mathscr{N}$ are non-negative functions satisfying $\alpha(x)+\beta(x) = 1$ for all $x \in S$, and that $F(\zeta)$, $F(\alpha\zeta)$, and $F(\beta\zeta)$ are strictly positive. Then, for every $f \in \mathscr{N}$ with $f(S) \subset I$ and such that ζf, $\zeta\psi(f)$, $\alpha\zeta$, $\alpha\zeta f$, $\beta\zeta$, and $\beta\zeta f$ belong to $\mathscr{N}$, the following inequalities hold:*

$$\psi\left(\frac{F(\zeta f)}{F(\zeta)}\right) \leqslant \frac{F(\alpha\zeta)}{F(\zeta)}\,\psi\left(\frac{F(\alpha\zeta f)}{F(\alpha\zeta)}\right) + \frac{F(\beta\zeta)}{F(\zeta)}\,\psi\left(\frac{F(\beta\zeta f)}{F(\beta\zeta)}\right) \leqslant \frac{F(\zeta\psi(f))}{F(\zeta)}. \tag{1}$$

REMARK 1. If ψ is concave, i.e., $-\psi$ is convex, then all inequalities in Theorem 1 hold in the reverse order.

Let $\mathbb{T}_j$, $j = 1,\dots,m$, be time scales, and define their Cartesian product

$$\Lambda^m = \mathbb{T}_1 \times \cdots \times \mathbb{T}_m = \{(x_1,\dots,x_m) : x_j \in \mathbb{T}_j,\, 1 \leqslant j \leqslant m\},$$

an m-dimensional time scale. For a Δ-measurable function $f : X \to \mathbb{R}$, where $X \subseteq \Lambda^m$ is Δ-measurable, the corresponding Lebesgue Δ-integral may be expressed equivalently as

$$\int_X f(x_1,\dots,x_m)\,\Delta_1 x_1 \cdots \Delta_m x_m, \quad \int_X f(\mathbf{x})\,\Delta\mathbf{x}, \quad \int_X f\,d\mu_\Delta,$$

$$\text{or} \quad \int_X f(\mathbf{x})\,d\mu_\Delta(\mathbf{x}), \tag{2}$$

where μ_Δ denotes a σ-additive Lebesgue Δ-measure on Λ^m.

When the time scale interval $[a,b)_{\mathbb{T}}$ consists only of isolated points, the Δ-integral reduces to a weighted sum:

$$\int_a^b f(x)\,\Delta x = \sum_{x\in[a,b)_{\mathbb{T}}} \big(\sigma(x)-x\big)f(x),$$

where σ is the forward jump operator.

THEOREM 2. (Fubini Theorem on Time Scales) *Let* $(X,\mathscr{M},\mu_\Delta)$ *and* $(Y,\mathscr{L},\nu_\Delta)$ *be finite-dimensional time scale measure spaces, and let* $f:X\times Y\to\mathbb{R}$ *be* Δ*-integrable. Define*

$$\varphi(y)=\int_X f(x,y)\,\Delta x,\qquad \psi(x)=\int_Y f(x,y)\,\Delta y.$$

Then both φ *and* ψ *are* Δ*-integrable, and the following equality holds:*

$$\int_X\left(\int_Y f(x,y)\,\Delta y\right)\Delta x=\int_Y\left(\int_X f(x,y)\,\Delta x\right)\Delta y.$$

2. Extensions of classical inequalities

In this section, we present refinements of several well-known classical inequalities using the framework of positive linear functionals and time-scale calculus. Each subsection explores specific extensions with sharper bounds that are valid across discrete, continuous, and quantum settings.

2.1. Extensions of Hardy's inequality

We begin with a refinement of Jensen's inequality obtained through the Jessen-type inequality (Theorem 1) in the context of Δ-integrals.

THEOREM 3. *Let* $(X,\mathscr{M},\mu_\Delta)$ *be a finite-dimensional time scale measure space. If* $\psi\in C(I,\mathbb{R})$ *is convex, and* ζ,α,β *are non-negative,* μ_Δ*-integrable functions with* $\alpha(x)+\beta(x)=1$ *for all* $x\in X$ *and each*

$$\int_X\zeta(x)\Delta x,\quad \int_X\alpha(x)\zeta(x)\Delta x,\quad \int_X\beta(x)\zeta(x)\Delta x$$

is strictly positive, then for any μ_Δ*-integrable* $f:X\to\mathbb{R}$ *with* $f(X)\subset I$*, and for which* ζf *and related terms are* μ_Δ*-integrable, we have*

$$\begin{aligned}\psi\left(\frac{\int_X\zeta(x)f(x)\Delta x}{\int_X\zeta(x)\Delta x}\right)&\leqslant\frac{\int_X\alpha(x)\zeta(x)\Delta x}{\int_X\zeta(x)\Delta x}\psi\left(\frac{\int_X\alpha(x)\zeta(x)f(x)\Delta x}{\int_X\alpha(x)\zeta(x)\Delta x}\right)\\&\quad+\frac{\int_X\beta(x)\zeta(x)\Delta x}{\int_X\zeta(x)\Delta x}\psi\left(\frac{\int_X\beta(x)\zeta(x)f(x)\Delta x}{\int_X\beta(x)\zeta(x)\Delta x}\right)\\&\leqslant\frac{\int_X\zeta(x)\psi(f(x))\Delta x}{\int_X\zeta(x)\Delta x}.\end{aligned}$$

Moreover, if $\psi\in C(I,\mathbb{R})$ *is concave, the above inequalities hold in reverse order.*

Proof. The result follows directly from Theorem 1 together with Remark 1, since Δ-integrals on time scales satisfy the defining properties of positive linear functionals. □

We now state the foundational hypotheses required for establishing Hardy-type inequalities on arbitrary time scales, employing convexity assumptions and general measurable kernel functions.

(H_1) $(X, \mathscr{K}, \mu_\Delta)$ and $(Y, \mathscr{L}, \nu_\Delta)$ are Δ-measure spaces associated with the respective time scales.

(H_2) The kernel $K : X \times Y \to \mathbb{R}_+$ is Δ-measurable and satisfies

$$M_\Delta(x) := \int_Y K(x,y)\,\Delta y < \infty, \qquad \forall x \in X.$$

(H_3) The function $\xi : X \to \mathbb{R}_+$ is μ_Δ-integrable. Define the auxiliary weight $w : Y \to \mathbb{R}_+$ by

$$w(y) := \int_X \frac{K(x,y)\,\xi(x)}{M_\Delta(x)}\,\Delta x, \qquad \forall y \in Y.$$

(H_4) The functions $\alpha, \beta : Y \to [0,1]$ are nonnegative and ν_Δ-integrable, satisfying $\alpha(y) + \beta(y) = 1$ for all $y \in Y$, with $\beta(y) := 1 - \alpha(y)$.

THEOREM 4. *Assume that (H_1)–(H_4) hold. Let $\Phi \in C(I, \mathbb{R})$ be convex on an interval $I \subseteq \mathbb{R}$. Then, for every ν_Δ-integrable function $f : Y \to \mathbb{R}$ with $f(Y) \subset I$, the following chain of inequalities holds:*

$$\begin{aligned}
&\int_X \xi(x)\,\Phi\left(\frac{1}{M_\Delta(x)}\int_Y K(x,y) f(y)\,\Delta y\right)\Delta x \\
&\leqslant \int_X \xi(x)\,\frac{\int_Y \alpha(y) K(x,y)\,\Delta y}{M_\Delta(x)}\,\Phi\left(\frac{\int_Y \alpha(y) K(x,y) f(y)\,\Delta y}{\int_Y \alpha(y) K(x,y)\,\Delta y}\right)\Delta x \\
&\quad + \int_X \xi(x)\,\frac{\int_Y \beta(y) K(x,y)\,\Delta y}{M_\Delta(x)}\,\Phi\left(\frac{\int_Y \beta(y) K(x,y) f(y)\,\Delta y}{\int_Y \beta(y) K(x,y)\,\Delta y}\right)\Delta x \\
&\leqslant \int_Y w(y)\,\Phi(f(y))\,\Delta y.
\end{aligned}$$

Moreover, if Φ is concave, then all inequalities above hold in the reverse order.

Proof. Applying Theorem 3 to the inner ν_Δ-integral for each $x \in X$ and using Fubini's theorem on time scales (Theorem 2), we obtain

$$\begin{aligned}
&\int_X \xi(x)\,\Phi\left(\frac{1}{M_\Delta(x)}\int_Y K(x,y) f(y)\,\Delta y\right)\Delta x \\
&= \int_X \xi(x)\,\Phi\left(\frac{\int_Y K(x,y) f(y)\,\Delta y}{\int_Y K(x,y)\,\Delta y}\right)\Delta x \\
&\leqslant \int_X \xi(x)\left[\frac{\int_Y \alpha(y) K(x,y)\,\Delta y}{\int_Y K(x,y)\,\Delta y}\,\Phi\left(\frac{\int_Y \alpha(y) K(x,y) f(y)\,\Delta y}{\int_Y \alpha(y) K(x,y)\,\Delta y}\right)\right. \\
&\quad \left. + \frac{\int_Y \beta(y) K(x,y)\,\Delta y}{\int_Y K(x,y)\,\Delta y}\,\Phi\left(\frac{\int_Y \beta(y) K(x,y) f(y)\,\Delta y}{\int_Y \beta(y) K(x,y)\,\Delta y}\right)\right]\Delta x \\
&\leqslant \int_X \frac{\xi(x)}{M_\Delta(x)}\left(\int_Y K(x,y)\,\Phi(f(y))\,\Delta y\right)\Delta x
\end{aligned}$$

$$= \int_Y \Phi(f(y)) \left(\int_X \frac{K(x,y)\xi(x)}{M_\Delta(x)} \Delta x \right) \Delta y$$
$$= \int_Y w(y)\, \Phi(f(y))\, \Delta y.$$

This proves the stated inequalities.

If Φ is concave, the reverse inequalities follow analogously by applying the reversed form of Theorem 3. □

REMARK 2. Choosing $\Phi(x) = x^p$ yields power-type inequalities, while taking $\Phi(x) = e^x$ produces exponential-type refinements. Classical Hardy, Hilbert, and Carleman inequalities are recovered as special cases on the continuous time scale $\mathbb{R}$ or the discrete time scale $\mathbb{Z}$.

In particular, for kernels of the form $K(x,y) = \frac{1}{x+y}$ one obtains Hilbert-type inequalities. On $\mathbb{R}$, the sharp constant $\pi/\sin(\pi/p)$ is recovered, thereby refining the classical Hilbert inequality.

Moreover, by choosing $\Phi(x) = e^x$ and restricting to discrete time scales, one obtains refinements of Carleman's inequality, which yield improved bounds of interest in summability problems.

2.2. Extensions of Hölder's Inequality

Hölder's inequality is one of the fundamental tools in functional analysis. We now present a refined version within the context of non-negative linear functionals. The refined inequality provides sharper bounds that hold across discrete, continuous, and quantum domains.

THEOREM 5. (Refined Hölder Inequality) *Let $p > 1$ and define q such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose S, $\mathcal{N}$ and F are given as in Definition 1, and w, g, h, α, β are non-negative functions belonging to the class $\mathcal{N}$, where $\alpha(x) + \beta(x) = 1$ for all $x \in S$. Assume the following functions also belong to $\mathcal{N}$:

$$wgh, whq, wgp, \alpha wgh, \beta wgh, \alpha whq, \beta whq.$$

Then, the refined Hölder-type inequality holds:

$$F(wgh) \leqslant F^{\frac{1}{q}}(wh^q) \left[\frac{F(\alpha wgh)}{F^{\frac{1}{q}}(\alpha wh^q)} + \frac{F(\beta wgh)}{F^{\frac{1}{q}}(\beta wh^q)} \right] \leqslant F^{\frac{1}{p}}(wg^p) F^{\frac{1}{q}}(wh^q), \tag{3}$$

where $F(wh^q)$, $F(\alpha wh^q) > 0$ and $F(\beta wh^q) > 0$. Further, if $0 < p < 1$, the above inequalities hold in reverse order. If $p < 0$ and $F(wg^p) > 0$, then inequalities also hold in reverse order.

Proof. Let $\psi(x) = x^p$ for $x > 0$, set $\zeta = wh^q$, and define $f = gh^{-q/p}$. Substituting these choices into inequality (1) yields

$$\left(\frac{F(wgh)}{F(wh^q)}\right)^p \leqslant \frac{F(\alpha wh^q)}{F(wh^q)}\left(\frac{F(\alpha wgh)}{F(\alpha wh^q)}\right)^p + \frac{F(\beta wh^q)}{F(wh^q)}\left(\frac{F(\beta wgh)}{F(\beta wh^q)}\right)^p \leqslant \frac{F(wg^p)}{F(wh^q)}.$$

Multiplying through by $F^p(wh^q)$ and taking the p-th root gives (3).

For $0 < p < 1$ and for $p < 0$ (with $F(wg^p) > 0$), the corresponding reversed inequalities follow analogously by applying the concavity case of Theorem 1. □

COROLLARY 1. *Let $(X, \mathscr{K}, \mu_\Delta)$ be a Δ-measure space. Under the assumptions of Theorem 5, if α, β, w, g, h are positive Lebesgue Δ-integrable functions on time scale intervals $X \subseteq \mathbb{T}$, then*

$$\begin{aligned} &\int_X w(x)g(x)h(x)\,\Delta x \qquad (4)\\ &\leqslant \left(\int_X w(x)h^q(x)\,\Delta x\right)^{\frac{1}{q}} \left[\frac{\int_X \alpha(x)w(x)g(x)h(x)\,\Delta x}{\left(\int_X \alpha(x)w(x)h^q(x)\,\Delta x\right)^{\frac{1}{q}}} + \frac{\int_X \beta(x)w(x)g(x)h(x)\,\Delta x}{\left(\int_X \beta(x)w(x)h^q(x)\,\Delta x\right)^{\frac{1}{q}}}\right]\\ &\leqslant \left(\int_X w(x)g^p(x)\,\Delta x\right)^{\frac{1}{p}} \left(\int_X w(x)h^q(x)\,\Delta x\right)^{\frac{1}{q}}, \end{aligned}$$

where $\int_X w(x)h^q(x)\,\Delta x$, $\int_X \alpha(x)w(x)h^q(x)\,\Delta x$, $\int_X \beta(x)w(x)h^q(x)\,\Delta x > 0$. Further, if $0 < p < 1$, the above inequalities hold in reverse order. If $p < 0$ and $\int_X w(x)g^p(x)\,\Delta x > 0$, then inequalities also hold in reverse order.

Proof. Immediate from Theorem 5 with F taken as the normalized Δ-integral. □

COROLLARY 2. *If the functions in Theorem 5 are Lebesgue integrable on $X \subseteq \mathbb{R}$, then inequality (3) holds with F the Lebesgue integral.*

Proof. Direct consequence of Corollary 1 by setting $X \subseteq \mathbb{R}$. □

COROLLARY 3. *If $\mathbb{T}$ is a time scale consisting only of isolated points, then inequality (3) holds with F the weighted sum*

$$\int_a^b f(x)\,\Delta x = \sum_{x \in [a,b)_{\mathbb{T}}} (\sigma(x) - x) f(x). \qquad (5)$$

Proof. Follows directly from Corollary 1 since the Δ-integral reduces to a discrete sum. □

EXAMPLE 1. The discrete version specializes further on standard time scales:

(1) If $\mathbb{T} = \mathbb{N}$, $[a,b) = [1, m+1)$, then inequality (5) reduces to the refinement of classical discrete Hölder form.

(2) If $\mathbb{T} = h\mathbb{N}$, $[a,b) = [h,(m+1)h)$, $h > 0$, the inequality becomes a weighted discrete refinement.

(3) If $\mathbb{T} = \mathbb{N}^2$, $[a,b) = [1,(m+1)^2)$, then an analogous refinement holds with weights $(2l+1)$.

(4) If $\mathbb{T} = q^{\mathbb{N}}$, $[a,b) = [q,q^{m+1})$, $q > 0$, the refinement takes the form with weights q^l.

2.3. Extensions of integral Minkowski's inequality

Minkowski's inequality is a cornerstone of L^p-spaces. Here, we provide a refinement using time-scale calculus.

THEOREM 6. *Assume that* (H_1) *holds and let* $\Phi \in \mathrm{C}(I,\mathbb{R})$ *be a convex function defined on an interval* $I \subseteq \mathbb{R}$. *If* α,β,ξ *are Lebesgue* Δ*-integrable functions on* X, f *is Lebesgue* Δ*-integrable on* Y, *and* F *is Lebesgue* Δ*-integrable on* $X \times Y$, *then, for* $p > 1$, *the following chain of inequalities is satisfied:*

$$\begin{aligned}
&\left(\int_X \xi(x)\left(\int_Y F(x,y)f(y)\,\Delta y\right)^p \Delta x\right)^{\frac{1}{p}} \qquad (6)\\
&\leqslant \int_Y \left[\frac{\int_X \alpha(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \alpha(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}} + \frac{\int_X \beta(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \beta(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}}\right] f(y)\,\Delta y\\
&\leqslant \int_Y \left(\int_X \xi(x)F^p(x,y)\Delta x\right)^{\frac{1}{p}} f(y)\,\Delta y,
\end{aligned}$$

where $H(x) = \int_Y F(x,y)f(y)\,\Delta y$ *such that*

$$\int_X \xi(x)\left(\int_Y F(x,y)f(y)\,\Delta y\right)^p \Delta x > 0, \quad \int_X \alpha(x)\xi(x)H^p(x)\Delta x > 0,$$
$$\int_X \beta(x)\xi(x)H^p(x)\Delta x > 0.$$

Further, if $0 < p < 1$, *the sequence of inequalities in* (6) *holds in reverse order. If* $p < 0$ *and* $\int_X \xi(x)F^p(x,y)\Delta x > 0$, *then inequalities also hold in reverse order.*

Proof. Applying the refined Hölder inequality (4) and using Fubini's theorem for time scales (Theorem 2) yields

$$\begin{aligned}
&\int_X \xi(x)H^p(x)\Delta x\\
&= \int_X \xi(x)H(x)H^{p-1}(x)\Delta x\\
&= \int_X \xi(x)\int_Y F(x,y)f(y)\,\Delta y H^{p-1}(x)\Delta x
\end{aligned}$$

$$
\begin{aligned}
&= \int_Y \int_X \xi(x)F(x,y)H^{p-1}(x)\Delta x f(y)\Delta y \\
&\leqslant \int_Y \left(\int_X \xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}} \\
&\quad \times \left[\frac{\int_X \alpha(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \alpha(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}} + \frac{\int_X \beta(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \beta(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}}\right] f(y)\,\Delta y \\
&\leqslant \int_Y \left(\int_X \xi(x)F^p(x,y)\Delta x\right)^{\frac{1}{p}} \left(\int_X \xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}} f(y)\,\Delta y.
\end{aligned}
$$

Dividing through by $\left(\int_X \xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}$, we get

$$
\begin{aligned}
&\left(\int_X \xi(x)H^p(x)\Delta x\right)^{\frac{1}{p}} \\
&\leqslant \int_Y \left[\frac{\int_X \alpha(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \alpha(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}} + \frac{\int_X \beta(x)\xi(x)F(x,y)H^{p-1}(x)\Delta x}{\left(\int_X \beta(x)\xi(x)H^p(x)\Delta x\right)^{\frac{p-1}{p}}}\right] f(y)\,\Delta y \\
&\leqslant \int_Y \left(\int_X \xi(x)F^p(x,y)\Delta x\right)^{\frac{1}{p}} f(y)\,\Delta y.
\end{aligned}
$$

This establishes the desired inequalities. For $0 < p < 1$ and $p < 0$, the corresponding result can be obtained similarly. □

REMARK 3. The refinement in Theorem 6 extends Minkowski's inequality to arbitrary time scales, thereby unifying continuous, discrete, and quantum cases. In particular, when $T = \mathbb{R}$, the Δ-integral reduces to the Lebesgue integral, yielding a refinement of classical Minkowski inequality. For $T = \mathbb{Z}$, the result becomes a sum inequality for sequences, useful in summability problems.

2.4. Extensions of AM-GM inequality

The arithmetic-geometric mean (AM-GM) inequality is fundamental in convexity theory. We derive multivariable refinements in the setting of positive linear functionals and Δ-integrals on arbitrary time scales, thereby unifying continuous, discrete, and quantum forms.

THEOREM 7. *Suppose S, $\mathcal{N}$ and F are given as in Definition 1, and $\zeta, \alpha, \beta \in \mathcal{N}$ are positive functions such that $\alpha(x) + \beta(x) = 1$ for all $x \in S$, and $F(\zeta)$, $F(\alpha\zeta)$, $F(\beta\zeta) > 0$. For all functions $f \in \mathcal{N}$ with $f(S) \subset I \subseteq \mathbb{R}^+$ such that ζf, $\zeta \ln(f)$, $\alpha\zeta$, $\alpha\zeta f$, $\beta\zeta$, $\beta\zeta f \in \mathcal{N}$, the following inequalities hold:*

$$
A(\zeta, f) \geqslant (A(\alpha\zeta, f))^{A(\zeta,\alpha)} (A(\beta\zeta, f))^{A(\zeta,\beta)} \geqslant G(\zeta, f),
$$

where

$$A(\zeta,f) := \frac{F(\zeta f)}{F(\zeta)} \quad \textit{and} \quad G(\zeta,f) := \exp\left(\frac{F(\zeta \ln f)}{F(\zeta)}\right). \tag{7}$$

Proof. Let $\psi(x) = -\ln x$, convex on $(0,\infty)$. Applying the Jessen-type inequality for the positive linear functional F, with weights $\alpha,\beta \geqslant 0$, $\alpha+\beta=1$, yields

$$\ln\left(\frac{F(\zeta f)}{F(\zeta)}\right) \geqslant \frac{F(\alpha\zeta)}{F(\zeta)}\ln\left(\frac{F(\alpha\zeta f)}{F(\alpha\zeta)}\right) + \frac{F(\beta\zeta)}{F(\zeta)}\ln\left(\frac{F(\beta\zeta f)}{F(\beta\zeta)}\right) \geqslant \frac{F(\zeta\ln f)}{F(\zeta)}.$$

Exponentiating both sides yields the desired inequality. □

COROLLARY 4. *Let $(X,\mathscr{K},\mu_\Delta)$ be a Δ-measure space. Suppose $f,\zeta,\alpha,\beta : X \to \mathbb{R}^+$ are positive Δ-integrable functions, with $\alpha(x)+\beta(x)=1$ for all $x \in X$. Then the following inequality holds:*

$$A_\Delta(\zeta,f) \geqslant (A_\Delta(\alpha\zeta,f))^{A_\Delta(\zeta,\alpha)}\,(A_\Delta(\beta\zeta,f))^{A_\Delta(\zeta,\beta)} \geqslant G_\Delta(\zeta,f),$$

where

$$A_\Delta(\alpha,f) := \frac{\int_X \alpha(x)f(x)\Delta x}{\int_X \alpha(x)\Delta x}, \quad G_\Delta(\zeta,f) := \exp\left(\frac{\int_X \zeta(x)\ln f(x)\Delta x}{\int_X \zeta(x)\Delta x}\right). \tag{8}$$

Proof. Immediate from Theorem 7 by taking F as the normalized Δ-integral on X. □

REMARK 4. The corollary establishes refined arithmetic mean-geometric mean (AM-GM) inequalities across various important time scales by specializing the general Δ-integral framework. Specifically, on the continuous time scale $T=\mathbb{R}$, the Δ-integral reduces to the classical Lebesgue integral, yielding a continuous refined AM-GM inequality. For the discrete time scale $T=\mathbb{N}$, the Δ-integral becomes a sum, resulting in a discrete refined AM-GM inequality suitable for sequences. Similarly, on the discrete time scale including zero $T=\mathbb{N}_0=\{0,1,2,\ldots\}$, a sum-based inequality holds incorporating the zero index. Moreover, for the quantum time scale $T=q^{\mathbb{N}_0}$ with $q>1$, the Δ-integral corresponds to the q-integral, producing a quantum version of the refined AM-GM inequality relevant in quantum calculus and related fields.

REMARK 5. Analogous refinements can be derived for the arithmetic-harmonic mean (AM-HM) inequality by applying the same framework with $\psi(x)=1/x$.

2.5. Extensions of Ky Fan-type inequalities

We now establish refined Ky Fan-type inequalities within the framework of positive linear functionals and time scale calculus. The following results generalize the classical Ky Fan inequality by introducing partition-based weights and extending validity across discrete, continuous, and quantum domains.

Let $\mathbf{1} \in \mathscr{N}$ denote the constant function defined by $\mathbf{1}(t)=1$ for all $t \in S$.

THEOREM 8. *Suppose $r > 0$ and S, $\mathscr{N}$, F are given as in Definition 1. If $f, \zeta, \alpha, \beta \in \mathscr{N}$ are positive functions such that $f(x) \in (0, \frac{r}{2}]$, $\alpha(x) + \beta(x) = 1$ for all $x \in S$, and ζf, $\zeta \ln(f)$, $\zeta(r\mathbf{1} - f)$, $\zeta \ln(r\mathbf{1} - f)$, $\alpha\zeta$, $\alpha\zeta f$, $\alpha\zeta(r\mathbf{1} - f)$, $\beta\zeta$, $\beta\zeta f$, $\beta\zeta(r\mathbf{1} - f) \in \mathscr{N}$, then the following inequalities hold.*

$$\frac{A(\zeta, r\mathbf{1} - f)}{G(\zeta, r\mathbf{1} - f)} \leqslant \frac{A(\zeta, f)}{G(\zeta, r\mathbf{1} - f)} \left(\frac{A(\alpha\zeta, r\mathbf{1} - f)}{A(\alpha\zeta, f)}\right)^{A(\zeta,\alpha)} \left(\frac{A(\beta\zeta, r\mathbf{1} - f)}{A(\beta\zeta, f)}\right)^{A(\zeta,\beta)} \leqslant \frac{A(\zeta, f)}{G(\zeta, f)},$$

where $A(\zeta, f)$ and $G(\zeta, f)$ are defined in (7).

Proof. The proof follows by applying Theorem 1 with the convex function

$$\psi(x) = \ln\left(\frac{r - x}{x}\right).$$

The inequality emerges naturally from the convexity property combined with the weighting scheme. By substituting f and the weights into the theorem, the stated inequalities are derived, refining the classical Ky Fan inequality. □

COROLLARY 5. *Let $r > 0$ and let $(X, \mathscr{K}, \mu_\Delta)$ be a Δ-measure space. Suppose $f, \zeta, \alpha, \beta : X \to \mathbb{R}^+$ are positive Δ-integrable functions, with $\alpha(x) + \beta(x) = 1$ and $f(x) \in (0, \frac{r}{2}]$ for all $x \in X$. Then the following inequalities hold.*

$$\frac{A_\Delta(\zeta, r\mathbf{1} - f)}{G_\Delta(\zeta, r\mathbf{1} - f)} \leqslant \frac{A_\Delta(\zeta, f)}{G_\Delta(\zeta, r\mathbf{1} - f)} \left(\frac{A_\Delta(\alpha\zeta, r\mathbf{1} - f)}{A_\Delta(\alpha\zeta, f)}\right)^{A_\Delta(\zeta,\alpha)} \left(\frac{A_\Delta(\beta\zeta, r\mathbf{1} - f)}{A_\Delta(\beta\zeta, f)}\right)^{A_\Delta(\zeta,\beta)} \leqslant \frac{A_\Delta(\zeta, f)}{G_\Delta(\zeta, f)},$$

where A_Δ and G_Δ are the time scale means defined in (8) and $\mathbf{1}(x) = 1$ for all $x \in X$.

Proof. Immediate from Theorem 8 by taking F as the normalized Δ-integral on X. □

REMARK 6. Corollary 5 establishes a unified refined Ky Fan inequality that is valid on arbitrary time scales. By specializing the general Δ-integral framework, this result recovers the classical continuous, discrete, and quantum cases as particular instances. For example, when the time scale $T = \mathbb{R}$, the Δ-integral reduces to the classical Lebesgue integral, yielding a refinement of the continuous Ky Fan inequality. When $T = \mathbb{N}$, the Δ-integral becomes a sum, providing a discrete inequality suitable for sequences and difference equations. For the quantum time scale $T = q^{\mathbb{N}_0}$ with $q > 1$, the Δ-integral corresponds to the q-integral, resulting in a quantum version of the inequality that is relevant in quantum calculus and related fields. This unification highlights the versatility of the time scale approach, allowing the refined Ky Fan inequality to be applied seamlessly across continuous and discrete settings, as well as in quantum systems.

3. Extensions of inequalities for divergence measures

Divergence and entropy measures are central in information theory and statistics. Here we establish refined Csiszár-type inequalities within the framework of positive linear functionals and time scale calculus.

In this section, we focus on the Csiszár divergence, a fundamental measure of discrepancy between probability distributions widely used in information theory and statistics. Originally formulated in continuous settings, the Csiszár divergence generalizes several well-known divergences such as the Kullback-Leibler divergence and total variation distance. Within the time scale calculus framework, the Csiszár divergence has been naturally extended by defining it via the Δ-integral, thereby unifying its treatment across continuous, discrete, and hybrid time domains. Specifically, on continuous time scales ($T=\mathbb{R}$), the divergence is expressed as an integral with respect to the Lebesgue measure. On discrete time scales ($T=\mathbb{N}$ or $T=\mathbb{N}_0$), the Δ-integral reduces to sums over the relevant indices, preserving the classical discrete form of Csiszár divergence. Moreover, in quantum calculus settings ($T=q^{\mathbb{N}_0}$), the definition adapts to a q-integral, enabling the analysis of divergences in quantum discrete models. This versatile formulation allows the Csiszár divergence to serve as a unifying tool for measuring distributional differences across diverse temporal structures encountered in modern applications.

THEOREM 9. *Suppose S, $\mathscr{N}$ and F are given as in Definition 1, and $\psi \in$ $\mathrm{C}((0,\infty),\mathbb{R})$ is convex. Assume that $t,s,\alpha,\beta \in \mathscr{N}$ are positive functions such that $\alpha(x)+\beta(x)=1$ for all $x\in S$. Assume also that αs, αt, βs and βt belong to $\mathscr{N}$. Then the the following inequalities hold:*

$$C_{\psi}(s,t) \geqslant F(\alpha s)\,\psi\left(\frac{F(\alpha t)}{F(\alpha s)}\right)+F(\beta s)\,\psi\left(\frac{F(\beta t)}{F(\beta s)}\right) \geqslant F(s)\,\psi\left(\frac{F(t)}{F(s)}\right), \tag{9}$$

where

$$C_{\psi}(s,t)=F\left(s\cdot\psi\left(\frac{t}{s}\right)\right)$$

Proof. Set $f=\frac{t}{s}$ and $\zeta=s$ in Theorem 1. Applying the theorem with these substitutions yields the stated inequalities. □

COROLLARY 6. *Let $(X,\mathscr{K},\mu_{\Delta})$ be a Δ-measure space. Suppose $\psi:(0,\infty)\to\mathbb{R}$ is a convex function, and t,s,μ,ν are positive Δ-integrable functions on X. Then the following inequalities hold:*

$$\begin{aligned} C_{\Delta\psi}(s,t) &\geqslant \left(\int_X \mu(x)s(x)\,\Delta x\right)\psi\left(\frac{\int_X \mu(x)t(x)\,\Delta x}{\int_X \mu(x)s(x)\,\Delta x}\right) \\ &\quad+\left(\int_X \nu(x)s(x)\,\Delta x\right)\psi\left(\frac{\int_X \nu(x)t(x)\,\Delta x}{\int_X \nu(x)s(x)\,\Delta x}\right) \\ &\geqslant \left(\int_X s(x)\,\Delta x\right)\psi\left(\frac{\int_X t(x)\,\Delta x}{\int_X s(x)\,\Delta x}\right), \end{aligned}$$

where

$$C_{\Delta\psi}(s,t) = \int_X s(x)\,\psi\left(\frac{t(x)}{s(x)}\right)\Delta x.$$

Proof. Immediate from Theorem 9 taking F as the normalized Δ-integral. □

REMARK 7. (Special cases) By suitable choices of ψ, Theorem 9 recovers refined inequalities for many well-known divergence measures.

(1) By substituting the convex function $\psi(x) = -\ln x$ into (9), we obtain a refined entropy type inequality:

$$F(s\ln(t)) + S(t) \leqslant F(\mu s)\ln\left(\frac{F(\mu t)}{F(\mu s)}\right) + F(\nu s)\ln\left(\frac{F(\nu t)}{F(\nu s)}\right) \leqslant F(s)\ln\left(\frac{F(t)}{F(s)}\right),$$

where

$$S(s) = -F(s\ln(s))$$

(2) By taking $\psi(x) = x\ln x$ in (9), we obtain a refined Kullback-Leibler divergence type inequality:

$$D_{KL}(s\|t) \geqslant F(\mu t)\ln\left(\frac{F(\mu t)}{F(\mu s)}\right) + F(\nu t)\ln\left(\frac{F(\nu t)}{F(\nu s)}\right) \geqslant F(t)\ln\left(\frac{F(t)}{F(s)}\right).$$

where

$$D_{KL}(s\|t) = F\left(t\ln\left(\frac{t}{s}\right)\right).$$

(3) By taking $\psi(x) = |x-1|$ in (9), we obtain a refined Variational distance type inequality:

$$V(s,t) \geqslant |F(\mu t) - F(\mu s)| + |F(\nu t) - F(\nu s)| \geqslant |F(t) - F(s)|,$$

where

$$V(s,t) = F(|t-s|).$$

(4) By taking $\psi(x) = (x-1)\ln x$ in (9), we obtain a refined Jeffrey divergence type inequality:

$$J(t,s) \geqslant (F(\mu t) - F(\mu s))\ln\left(\frac{F(\mu t)}{F(\mu s)}\right) + (F(\nu t) - F(\nu s))\ln\left(\frac{F(\nu t)}{F(\nu s)}\right)$$
$$\geqslant (F(t) - F(s))\ln\left(\frac{F(t)}{F(s)}\right),$$

where

$$J(t,s) = F\left((t-s)\ln\left(\frac{t}{s}\right)\right).$$

(5) By taking $\psi(x) = -\sqrt{x}$ in (9), we obtain a refined Bhattacharya coefficient type inequality:

$$B(s,t) \leqslant \sqrt{F(\mu s)F(\mu t)} + \sqrt{F(\nu s)F(\nu t)} \leqslant \sqrt{F(s)F(t)},$$

where

$$B(s,t) = F\left(\sqrt{st}\right).$$

(6) By taking $\psi(x) = (\sqrt{x}-1)^2$ in (9), we obtain a refined Hellinger distance type inequality:

$$\begin{aligned} H(t,s) &\geqslant \left(\sqrt{F(\mu t)} - \sqrt{F(\mu s)}\right)^2 + \left(\sqrt{F(\nu t)} - \sqrt{F(\nu s)}\right)^2 \\ &\geqslant \left(\sqrt{F(t)} - \sqrt{F(s)}\right)^2, \end{aligned}$$

where

$$H(t,s) = F\left(\left(\sqrt{t} - \sqrt{s}\right)^2\right).$$

(7) By taking $\psi(x) = \frac{(x-1)^2}{x+1}$ in (9), we obtain a refined Triangular discrimination type inequality:

$$\Delta(t,s) \geqslant \frac{(F(\mu t) - F(\mu s))^2}{F(\mu s) + F(\mu t)} + \frac{(F(\nu t) - F(\nu s))^2}{F(\nu s) + F(\nu t)} \geqslant \frac{(F(t) - F(s))^2}{F(s) + F(t)},$$

where

$$\Delta(t,s) = F\left(\frac{(t(x) - s(x))^2}{s(x) + t(x)}\right).$$

Thus a broad family of divergence inequalities-classical and modern-follows as direct consequences of the general Csiszár refinement, valid uniformly across continuous, discrete, and q-discrete time scales.

4. Conclusion

In this work, we developed a unified framework for refining classical inequalities-such as Hardy, Hölder, Minkowski, AM-GM, and Ky Fan-type inequalities – within the setting of positive linear functionals and time scale calculus. This approach provided sharper bounds that remain valid simultaneously in discrete, continuous, and quantum settings. Furthermore, by extending the same methodology to Csiszár divergences and related entropy measures, we established new refinements of Kullback-Leibler divergence, Shannon entropy, Hellinger distance, and other statistical distances.

The results highlight the versatility of convexity-based techniques and their capacity to unify inequalities across analytic and information-theoretic domains. Future directions include extending these refinements to non-commutative frameworks such as operator algebras, quantum information theory, and stochastic processes on hybrid domains.

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