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THREE WEIGHTS HARDY–TYPE INEQUALITIES

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Abstract. In this paper we derive some new weighted iterated discrete Hardy-type inequalities with three independent weight sequences and various parameter ranges. In a special case, the operator reduces to the classical Hardy-type operator. Although integral analogues of such inequalities have been widely studied, the discrete setting has only recently attracted significant attention, including extensions to classes of matrix operators. The aim of the paper is to establish necessary and sufficient conditions for the validity of these inequalities in cases where the parameter p does not exceed one. The obtained results complement known criteria, extend them to new parameter ranges, and provide new insights into borderline cases. Some applications of Hardy-type inequalities are pointed out.

1. Introduction

This paper is within the very active research area concerning Hardy-type inequalities. G. H. Hardy proved his famous inequality aleardy in 1925 (see [7]). The dramatic prehistory of more than 10 years of research before is described in detail in [15]. After that it has now been 100 years of intensive research in this area, see e.g. the books [8], [13], [14] and especially the most complete and well cited book [16] in this area, where the most important developments up to 2017 are described. But the interest also after 2017 is great, see e.g. the papers [1] and [31] and the references therein. One reason for this great interest is the importance for various applications both in mathematics and engineering sciences. See e.g. the books mentioned above, the new related books [18] and [19] and also our final remark at the end of this paper. Most of this development is performed in the continuous case but there is also a parallel development in the discrete case but here there remains a number of open questions, where the answer many times is known in the continuous case. The aim of this paper is to fill in one important gap in this connection, which we describe and investigate in detail below.

Let $0 < p, q, \theta < \infty$ and $\varphi = \{\varphi_k\}_{k=1}^{\infty}$ be a non-negative sequence, and let $u = \{u_i\}_{i=1}^{\infty}$ and $\omega = \{\omega_i\}_{i=1}^{\infty}$ be positive sequences of real numbers, which will be referred to as weight sequences. We denote by $l_{p,u}$ the space of sequences $f = \{f_j\}_{j=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{j=1}^{\infty} |u_j f_j|^p\right)^{\frac{1}{p}} < +\infty, \ \ 0 < p < \infty.$$

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In this paper we consider the following weighted inequality:

$$\left(\sum_{n=1}^{\infty}\omega_n^{\theta}\left(\sum_{k=1}^{n}\left|\varphi_k\sum_{i=1}^{k}f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}\leqslant C\left(\sum_{j=1}^{\infty}|u_jf_j|^p\right)^{\frac{1}{p}},\ \forall f\in l_{p,u}, \tag{1.1}$$

for a quasilinear operator H_φ defined as follows for $\forall f\in l_1$:

$$(H_\varphi f)_n := \left(\sum_{k=1}^{n}\left|\varphi_k\sum_{i=1}^{k}f_i\right|^q\right)^{\frac{1}{q}}, \tag{1.2}$$

where $k\in\mathbb{N}$ and C is a positive constant of (1.1).

Note that, if we write $(H_\varphi f)_k := \varphi_k\sum\limits_{i=1}^{k}f_i$ separately in (1.2), then it will be a Hardy-type operator. So we can say that inequality (1.1) is a weighted quasilinear inequality and weighted iterated Hardy-type inequality.

The integral analogue of the iterated Hardy-type inequalities of the form (1.1), involving three independent general weights and relations between three different parameters, was first systematically studied in [20]. The results of this work were later applied by V. I. Burenkov and R. Oinarov to investigate the boundedness of the multidimensional Hardy-type operator from a weighted Lebesgue space to a Morrey-type space (see [3]). Most studies have focused on the continuous analogue of (1.1) (see for example [5], [6], [9], [11], [21], [28], [29] and [33]). While the integral analogues of iterated Hardy-type operators have been widely investigated, the discrete case has attracted increasing attention only in recent years (see, for example [4], [24], [27] and [37]). In particular, in [27] the criteria for the validity of inequality (1.1) were obtained for the following ranges of the parameters p,θ,q: $1<p\leqslant\min\{\theta,q\}<\infty$ and $0<q<\min\{p,\theta\}<\infty$, $p>1$. In [37], the validity of the inequality was studied for the following relations between the parameters $0<\theta<\min\{p,q\}<\infty$, $0<p\leqslant 1$ and $0<q<p\leqslant\theta<\infty$, $0<p\leqslant 1$. Moreover, weighted iterated discrete Hardy-type inequalities of the form (1.1) are also being investigated for the class of matrix operators (see [10], [12], [36] and [38]).

In the present paper we study the inequality (1.1) for the following cases:

a) $p\in(0,1]$ and $p<\min\{q,\theta\}$, (see Theorem 3.1);

b) $0<q<p=1\leqslant\theta$, (see Theorem 3.2);

c) $0<q<\theta<p=1$, (see Theorem 3.3).

Indeed, we establish necessary and sufficient conditions of the inequality (1.1) for each of the cases (a), (b) and (c). The problem we study in this paper is related to the research presented in the papers [10], [12], [36] and [38]. However, our focus is on a different situation so the results are new.

The paper is organized as follows: The main results (Theorems 3.1–3.3) are formulated in Section 3. The proofs are given is Sections 4, 5 and 6, respectively. In order not to disturb our discussion in these proofs the necessary preliminaries can be found in Section 2. In particular, a new Lemma of independent interest is proved in this section (see Lemma 2.1). Finally, in Section 7 we present some final remarks of general interest.

2. Preliminaries

In the proofs of our main results we will need the following well-known results on the discrete weighted Hardy inequality (see [2], [30]) and boundedness of matrix operators (see [23], [25] or [34]).

THEOREM A. ([30], see Theorem 1 (iv)). *Let* $0<q<p=1$ *and* u_i, v_i *and* f_i *be positive sequences. Then the inequality*

$$\Big(\sum_{i=1}^{\infty}\Big|\sum_{j=1}^{i} f_j\Big|^q v_i\Big)^{\frac{1}{q}} \leqslant C\sum_{i=1}^{\infty} u_i|f_i| \tag{2.1}$$

holds for some $C<\infty$ *if and only if*

$$H=\Big(\sum_{n=1}^{\infty} v_n\Big(\sum_{k=n}^{\infty} v_k\Big)^{\frac{q}{1-q}} \max_{1\leqslant k\leqslant n} u_k^{\frac{q}{1-q}}\Big)^{\frac{1-q}{q}}<\infty.$$

Moreover, $H\approx C$, *where* C *is the best constant in* (2.1)

DEFINITION. The matrix $\{a_{i,j}\}_{j=1}^{\infty}$, $i\geqslant j$ satisfies the (discrete) Oinarov condition ([25]), if there exist $d\geqslant 1$, a non-negative matrix $(a_{i,j})$, whose entries $a_{i,j}$ are almost non-decreasing in i and almost non-increasing in j such that the inequalities

$$\frac{1}{d}(a_{i,k}+a_{k,j})\leqslant a_{i,j}\leqslant d(a_{i,k}+a_{k,j}),$$

or $a_{i,j}\approx a_{i.k}+a_{k,j}$ hold for all $i\geqslant k\geqslant j\geqslant 1$.

THEOREM B. (see [25] or [34]). *Let* $1\leqslant p\leqslant q<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$ *and the entries of the matrix* $(a_{i,j})$ *satisfies the discrete Oinarov condition. Then the inequality*

$$\Big(\sum_{j=1}^{\infty}\Big|\sum_{i=j}^{\infty} a_{i,j}f_i\Big|^q u_i^q\Big)^{\frac{1}{q}} \leqslant C\Big(\sum_{i=1}^{\infty}|v_if_i|^p\Big)^{\frac{1}{p}} \tag{2.2}$$

holds for some $C<\infty$ *if and only if* $M=max\{M_1,M_2\}<\infty$, *where*

$$M_1=\sup_{k\geqslant 1}\Big(\sum_{j=1}^{k} u_j^q\Big)^{\frac{1}{q}}\Big(\sum_{i=k}^{\infty} a_{i,k}^{p'}v_i^{-p'}\Big)^{\frac{1}{p'}},$$

$$M_2=\sup_{k\geqslant 1}\Big(\sum_{j=1}^{k} a_{k,j}^q u_j^q\Big)^{\frac{1}{q}}\Big(\sum_{i=k}^{\infty} v_i^{-p'}\Big)^{\frac{1}{p'}}.$$

Moreover, $M\approx C$, *where* C *is the best constant in* (2.2).

THEOREM C. (see [23]). *Let* $1 \leqslant q < p < \infty$ *and the entries of the matrix* $(a_{i,j})$ *satisfies the discrete Oinarov condition. Then the inequality* (2.2) *holds for some* $C < \infty$ *if and only if* $M^* = max\{M_1^*, M_2^*\} < \infty$, *where*

$$M_1^* = \Big(\sum_{k=1}^{\infty} \Big(\sum_{i=k}^{\infty} a_{i,k}^{p'} v_i^{-p'} \Big)^{\frac{q(p-1)}{p-q}} \Big(\sum_{j=1}^{k} u_j^q \Big)^{\frac{q}{p-q}} u_k^q \Big)^{\frac{p-q}{pq}},$$

$$M_2^* = \Big(\sum_{k=1}^{\infty} \Big(\sum_{j=1}^{k} a_{k,j}^q u_j^q \Big)^{\frac{p}{p-q}} \Big(\sum_{i=k}^{\infty} v_i^{-p'} \Big)^{\frac{p(q-1)}{p-q}} v_k^{-p'} \Big)^{\frac{p-q}{pq}}.$$

Moreover, $M^* \approx C$, *where* C *is the best constant in* (2.2).

We also need the following well-known version of the discrete Minkowski inequality:

LEMMA A. *Let* $\{a_{i,j}\}$, $i = 1, 2, \dots, n \leqslant +\infty$, $j = 1, 2, \dots, m$, *be a positive matrix. Then the inequalities*

$$\Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} a_{i,j} \Big|^{\sigma} \Big)^{\frac{1}{\sigma}} \leqslant \sum_{j=1}^{m} \Big(\sum_{i=1}^{n} |a_{i,j}|^{\sigma} \Big)^{\frac{1}{\sigma}}, \tag{2.3}$$

and

$$\Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{i} a_{i,j} \Big|^{\sigma} \Big)^{\frac{1}{\sigma}} \leqslant \sum_{j=1}^{n} \Big(\sum_{i=j}^{n} |a_{i,j}|^{\sigma} \Big)^{\frac{1}{\sigma}}, \tag{2.4}$$

holds, where $\sigma \geqslant 1$.

We also need the following elementary inequalities: If $a_i > 0$, $i = 1, 2, \dots, k$, then

$$\Big(\sum_{i=1}^{k} a_i \Big)^{\alpha} \leqslant \sum_{i=1}^{k} a_i^{\alpha}, \;\; 0 < \alpha \leqslant 1, \tag{2.5}$$

and

$$\Big(\sum_{i=1}^{k} a_i \Big)^{\alpha} \geqslant \sum_{i=1}^{k} a_i^{\alpha}, \;\; \alpha \geqslant 1. \tag{2.6}$$

Finally, for the proof of Theorem 3.2 we need the following new Lemma of independent interest:

LEMMA 2.1. *Let* $\alpha > 0$ *and* $\{u_i\}_{i=1}^{\infty}$ *be a non-negative and non-decreasing sequence, let* $\{v_i\}_{i=1}^{\infty}$ *- non-negative sequence, assume that* $\Delta u_n = u_n - u_{n-1}$, $n \geqslant 1$ *and* $u_0 \equiv 0$. *Then*

$$\sum_{n=1}^{\infty} v_n \Big(\sum_{k=n}^{\infty} v_k \Big)^{\alpha} u_n^{\alpha} \approx \sum_{n=1}^{\infty} \Big(\sum_{k=n}^{\infty} v_k \Big)^{\alpha+1} \Delta u_n^{\alpha}. \tag{2.7}$$

Proof. Since $u_0 = 0$, we have $u_n^\alpha = \sum\limits_{i=1}^{n} \Delta u_i^\alpha$. By changing the order of summation and applying Lemma 2.1 in [37], we obtain that

$$\sum_{n=1}^{\infty} v_n \left(\sum_{k=n}^{\infty} v_k\right)^{\alpha} u_n^\alpha = \sum_{n=1}^{\infty} v_n \left(\sum_{k=n}^{\infty} v_k\right)^{\alpha} \sum_{i=1}^{n} \Delta u_i^\alpha$$

$$= \sum_{i=1}^{\infty} \Delta u_i^\alpha \sum_{n=i}^{\infty} v_n \left(\sum_{k=n}^{\infty} v_k\right)^{\alpha} \approx \sum_{i=1}^{\infty} \left(\sum_{n=i}^{\infty} v_n\right)^{\alpha+1} \Delta u_i^\alpha. \quad \square$$

The proof is complete.

REMARK 2.1. This Lemma was formulated, but not proved in the Proceedings [26].

3. Main results

THEOREM 3.1. *Let* $p \in (0,1]$ *and* $p \leqslant \min\{q,\theta\}$. *Then the inequality* (1.1) *holds for some* $C < \infty$ *if and only if* $J_0 < \infty$, *where*

$$J_0 := \sup_{r \geqslant 1} \left(\sum_{n=r}^{\infty} \omega_n^\theta \left(\sum_{k=r}^{n} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant r} u_k^{-1}.$$

Moreover, $C \approx J_0$, *where* C *is the best constant in* (1.1).

THEOREM 3.2. *Let* $0 < q < p = 1 \leqslant \theta < \infty$. *Then the inequality* (1.1) *holds, if and only if* $\max\{J_1, J_2\} < \infty$, *where*

$$J_1 := \sup_{k \geqslant 1} \left(\sum_{i=k}^{\infty} \omega_i^\theta \left(\sum_{s=k}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \max_{1 \leqslant i \leqslant k} u_i,$$

$$J_2 := \sup_{k \geqslant 1} \left(\sum_{j=1}^{k} \Delta(\overline{u}_j^{\frac{q}{1-q}}) \left(\sum_{r=j}^{k} \varphi_r^q\right)^{\frac{1}{1-q}}\right)^{\frac{1-q}{q}} \left(\sum_{i=k}^{\infty} \omega_i^\theta\right)^{\frac{1}{\theta}}.$$

Here $\Delta \overline{u}_j = \overline{u}_j - \overline{u}_{j-1}$, $j \geqslant 1$ *and* $\overline{u}_0 = 0$, *where* $\overline{u}_k = \max\limits_{1 \leqslant j \leqslant k} u_j$, $k \geqslant 1$. *Moreover,* $C \approx \max\{J_1, J_2\}$, *where* C *is the best constant in* (1.1).

THEOREM 3.3. *Let* $0 < q < \theta < p = 1$. *Then the inequality* (1.1) *holds, if and only if* $\max\{J_3, J_4\} < \infty$, *where*

$$J_3 := \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \omega_i^\theta \left(\sum_{s=j}^{i} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{1-\theta}} \left(\sum_{i=1}^{j} \Delta(\overline{u}_i^{\frac{q}{1-q}})\right)^{\frac{\theta-q}{q(1-\theta)}} \Delta(\overline{u}_j^{\frac{q}{1-q}})\right)^{\frac{1-\theta}{\theta}},$$

$$J_4 := \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \left(\sum_{s=k}^{j} \varphi_s^q\right)^{\frac{1}{1-q}} \Delta(\overline{u}_k^{\frac{q}{1-q}})\right)^{\frac{\theta(1-q)}{q(1-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^\theta\right)^{\frac{\theta}{1-\theta}} \omega_j^\theta\right)^{\frac{1-\theta}{\theta}}.$$

Here $\Delta \overline{u}_j = \overline{u}_j - \overline{u}_{j-1}$, $j \geqslant 1$ *and* $\overline{u}_0 = 0$, *where* $\overline{u}_k = \max\limits_{1 \leqslant j \leqslant k} u_j$, $k \geqslant 1$. *Moreover,* $C \approx \max\{J_3, J_4\}$, *where* C *is the best constant in* (1.1).

4. Proof of Theorem 3.1

Proof. *Necessity:* Suppose that the inequality (1.1) holds with the best constant $C > 0$. First we prove that $J_0 < \infty$. Now for $1 \leqslant k \leqslant r < \infty$ we assume that $\widetilde{f} = \{\widetilde{f_m}\}_{m=1}^{\infty}$, where $\widetilde{f_m} = 0$ for $m \neq k$ and $\widetilde{f_m} = u_k^{-1}$ for $m = k$, where $u_k \neq 0$. Then

$$\|\widetilde{f}\|_{l_{p,u}} = u_k^{-1} \cdot u_k = 1. \tag{4.1}$$

Substituting $\widetilde{f}$ in the left hand side of inequality (1.1), we can deduce that

$$I(\widetilde{f}) \equiv \left(\sum_{n=1}^{\infty} \omega_n^{\theta} \left(\sum_{s=1}^{n} \left| \varphi_s \sum_{i=1}^{s} \widetilde{f_i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \geqslant \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \left| \varphi_s \sum_{i=1}^{s} \widetilde{f_i} \right|^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}}$$

$$\geqslant \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} u_k^{-1}, \text{ for all } 1 \leqslant k \leqslant r.$$

so that

$$I(\widetilde{f}) \geqslant \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant r} u_k^{-1}, \quad \forall r \geqslant 1.$$

Therefore

$$I(\widetilde{f}) \geqslant \sup_{r \geqslant 1} \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant r} u_k^{-1} = J_0. \tag{4.2}$$

From (1.1), (4.1) and (4.2), we have that

$$J_0 = \sup_{r \geqslant 1} \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant r} u_k^{-1} \leqslant C. \tag{4.3}$$

Sufficiency: Let $J_0 < \infty$. Now, we prove that (1.1) holds for a finite constant C. Let $0 \leqslant f \in l_{p,u}$.

Let $f_1 \neq 0$. If $f_i = 0$, $1 < i < k$, $f_k \neq 0$, then $\sum_{i=k}^{\infty} f_i = \sum_{i=1}^{\infty} f_{k-1+i} = \sum_{i=1}^{\infty} \widetilde{f_i}$ and therefore $\widetilde{f_1} \neq 0$.

Let

$$\sup\{k \in Z : 2^k \leqslant f_1\} = k_1,$$

then

$$2^{k_1} \leqslant f_1 < 2^{k_1+1}.$$

Hence,

$$k_{\infty} := \sup\{k \geqslant 1 : 2^{k_1+k-1} \leqslant \sum_{i=1}^{\infty} f_i\}.$$

If $\sum_{i=1}^{\infty} f_i < \infty$, then $k_{\infty} < \infty$. If $\sum_{i=1}^{\infty} f_i = \infty$, then $k_{\infty} = \infty$.

We consider the sequence $\{j_k\}$, where j_k are defined by

$$j_k := \min\{j \geqslant 1 : \sum_{i=1}^{j} f_i \geqslant 2^{k_1+k-1}\}, \;\; 1 \leqslant k \leqslant k_\infty.$$

We note that

$$j_1 = \min\{j \geqslant 1 : \sum_{i=1}^{j} f_i \geqslant 2^{k_1}\} = 1,$$

and then obviously $j_{k_\infty} = \infty$ if $k_\infty = \infty$ and if $k_\infty < \infty$, then

$$2^{k_1+k_\infty-1} \leqslant \sum_{i=1}^{\infty} f_i < 2^{k_1+k_\infty}.$$

For all $k \geqslant 1$ it yields that

$$\sum_{i=1}^{j_k-1} f_i < 2^{k_1+k-1} \leqslant \sum_{i=1}^{j_k} f_i. \tag{4.4}$$

Therefore the set of natural numbers $\mathbb{N}$ can be written

$$\mathbb{N} = \bigcup_{k=2}^{k_\infty} [j_{k-1}, j_k - 1].$$

Moreover,

$$2^{k_1+m-1} \leqslant \sum_{i=1}^{j_m} f_i = \sum_{i=1}^{j_{m-1}-1} f_i + \sum_{i=j_{m-1}}^{j_m} f_i < 2^{k_1+m-2} + \sum_{i=j_{m-1}}^{j_m} f_i, \;\; m \geqslant 3.$$

$$2^{k_1+m-2} \leqslant \sum_{i=j_{m-1}}^{j_m} f_i, \;\; m \geqslant 3.$$

By substituting m by $m-1$ we find that

$$2^{k_1+m-3} \leqslant \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \;\; m \geqslant 4.$$

Then we obtain that

$$2^{k_1+m-1} \leqslant 4 \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \;\; m \geqslant 4. \tag{4.5}$$

Let us consider special cases: if $m = 2$, then we have that

$$2^{k_1+2-1} = 2^{k_1+1} = 2^{k_1} \cdot 2 \leqslant 2 f_1 \leqslant 2 \sum_{i=j_0}^{j_1} f_i, \;\; f_{j_o} = 0,$$

$$2^{k_1+2-1} \leqslant 4 \sum_{i=j_0}^{j_1} f_i, \tag{4.6}$$

if $m = 3$, we apply (4.4) and get that

$$2^{k_1+3-1} = 2 \cdot 2^{k_1+2-1} \leqslant 2 \sum_{i=1}^{j_2} f_i \leqslant 4 \sum_{i=j_{3-2}}^{j_{3-1}} f_i. \tag{4.7}$$

By combining (4.5)–(4.7) we obtain that

$$2^{k_1+m-1} \leqslant 4 \sum_{i=j_{m-2}}^{j_{m-1}} f_i, \quad m \geqslant 2. \tag{4.8}$$

Therefore, in view of (4.4), we have that

$$\begin{aligned} I^{\theta}(f) := \sum_{n=1}^{\infty} \omega_n^{\theta} \left(\sum_{s=1}^{n} \left| \varphi_s \sum_{i=1}^{s} f_i \right|^q \right)^{\frac{\theta}{q}} &= \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=1}^{n} \left| \varphi_s \sum_{i=1}^{s} f_i \right|^q \right)^{\frac{\theta}{q}} \\ &\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \left(\sum_{i=1}^{s} f_i \right)^q \right)^{\frac{\theta}{q}} \\ &\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \left(\sum_{i=1}^{j_m-1} f_i \right)^q \right)^{\frac{\theta}{q}} \\ &\leqslant \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \left(2^{k_1+m-1} \right)^q \right)^{\frac{\theta}{q}}. \end{aligned}$$

Hence, by applying (4.8) we get that

$$I^{\theta}(f) \leqslant 4^{\theta} \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^q \right)^{\frac{\theta}{q}}. \tag{4.9}$$

We must now consider the cases $\theta \leqslant q$ and $\theta > q$ separately.

a) The case $p \in (0,1]$ *and* $p \leqslant \theta \leqslant q$.

We consider the inequality (4.9) and note that $\frac{\theta}{q} \leqslant 1$. By applying the elementary inequality (2.5) and using (4.9), we find that

$$\begin{aligned} I^{\theta}(f) &\leqslant 4^{\theta} \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \sum_{m=2}^{k} \left(\sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \\ &= 4^{\theta} \sum_{k=2}^{k_\infty} \sum_{m=2}^{k} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{\min(n, j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}}. \end{aligned}$$

Thus, by changing the orders of sums, we get that

$$I^{\theta}(f) \leqslant 4^{\theta} \sum_{m=2}^{k_\infty} \sum_{k=m}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}}$$

$$\leqslant 4^{\theta} \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \sum_{n=j_{m-1}}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}}$$

$$\ll \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \cdot u_i \cdot u_i^{-1} \right)^{p\cdot\frac{\theta}{p}} \sum_{n=j_{m-1}}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

By applying (2.5) with $0 < p \leqslant 1$, we obtain that

$$I^{\theta}(f) \leqslant \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\max_{j_{m-2} \leqslant i \leqslant j_{m-1}} u_i^{-1} \right)^{\theta} \sum_{n=j_{m-1}}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}}$$

$$\leqslant \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left(\max_{1 \leqslant k \leqslant j_{m-1}} u_k^{-1} \right)^{\theta} \sum_{n=j_{m-1}}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}}.$$

By using (2.6), we get that

$$I^{\theta}(f) \leqslant \left(\sum_{m=2}^{k_\infty} \sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{m \geqslant 2} \left(\sum_{n=j_{m-1}}^{\infty} \omega_n^{\theta} \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant j_{m-1}} u_k^{-1} \right]^{\theta}$$

$$\leqslant \left(\sum_{i=1}^{\infty} |f_i \cdot u_i|^p \right)^{\frac{\theta}{p}} \left[\sup_{r \geqslant 1} \left(\sum_{n=r}^{\infty} \omega_n^{\theta} \left(\sum_{s=r}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{\theta}} \max_{1 \leqslant k \leqslant r} u_k^{-1} \right]^{\theta} = \left(J_0 \|f\|_{p,u} \right)^{\theta},$$

so that

$$I(f) \ll J_0 \|f\|_{p,u}, \quad \text{if } \ p \leqslant \theta \leqslant q, \tag{4.10}$$

where $p \in (0,1]$.

b) The case $p \in (0,1]$ *and* $p \leqslant q < \theta$.

We start with the inequality (4.9):

$$I^{\theta}(f) \leqslant 4^{\theta} \sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^{\theta} \left(\sum_{m=2}^{k} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \right)^{\frac{\theta}{q}}.$$

First we raise both sides in (4.9) to power $\frac{q}{\theta} \leqslant 1$

$$I^{q}(f) \leqslant 4^{q} \left[\sum_{k=2}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \left(\sum_{m=2}^{k} \omega_n^{q} \sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Next we apply (2.3) in the inner sum with $\sigma = \frac{\theta}{q}$ and obtain that

$$I^q(f) \leqslant 4^q \left[\sum_{k=2}^{k_\infty} \left\{ \sum_{m=2}^{k} \left(\sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \right)^{\frac{q}{\theta}} \right\}^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}},$$

so that, by also using (2.4), we find that

$$I^q(f) \leqslant 4^q \sum_{m=2}^{k_\infty} \left[\sum_{k=m}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{\theta} \right]^{\frac{q}{\theta}}$$

$$= 4^q \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \left[\sum_{k=m}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{\min(n,j_m-1)} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \left[\sum_{n=j_{m-1}}^{j_m-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{k=m+1}^{k_\infty} \sum_{n=j_{k-1}}^{j_k-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{j_m-1} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

Thus, we get that

$$I^q(f) \leqslant \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \left[\sum_{n=j_{m-1}}^{j_m-1} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} + \sum_{n=j_m}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant 4^q \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \right)^{q} \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\ll \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} f_i \cdot u_i \cdot u_i^{-1} \right)^{p\cdot\frac{q}{p}} \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

By applying (2.5) with $0 < p \leqslant 1$, we obtain that

$$I^q(f) \leqslant \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left(\max_{j_{m-2} \leqslant i \leqslant j_{m-1}} u_i^{-1} \right)^{q} \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}$$

$$\leqslant \sum_{m=2}^{k_\infty} \left(\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p \right)^{\frac{q}{p}} \left(\max_{1 \leqslant k \leqslant j_{m-1}} u_k^{-1} \right)^{q} \left[\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q \right)^{\frac{\theta}{q}} \right]^{\frac{q}{\theta}}.$$

By using (2.6), we find that

$$I^q(f) \leqslant \left(\sum_{m=2}^{k_\infty}\sum_{i=j_{m-2}}^{j_{m-1}} |f_i \cdot u_i|^p\right)^{\frac{q}{p}} \left[\sup_{m\geqslant 2}\left(\sum_{n=j_{m-1}}^{\infty} \omega_n^\theta \left(\sum_{s=j_{m-1}}^{n} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \max_{1\leqslant k\leqslant j_{m-1}} u_k^{-1}\right]^q$$

$$\leqslant \left(\sum_{i=1}^{\infty} |f_i \cdot u_i|^p\right)^{\frac{q}{p}} \left[\sup_{r\geqslant 1}\left(\sum_{n=r}^{\infty} \omega_n^\theta \left(\sum_{s=r}^{n} \varphi_s^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \max_{1\leqslant k\leqslant r} u_k^{-1}\right]^q = \left(J_0\|f\|_{p,u}\right)^q,$$

so that

$$I(f) \ll J_0\|f\|_{p,u}, \text{ if } p \leqslant q < \theta, \tag{4.11}$$

where $p \in (0,1]$.

Therefore from the inequalities (4.3), (4.10) and (4.11), we can conclude that $C \approx J_0$, where C is the best constant in (1.1). The proof is complete. □

5. Proof of Theorem 3.2

Proof. Let $0 < q < p = 1 \leqslant \theta$. Let $g = \{g_i\}_{i=1}^\infty$, $g \geqslant 0 \Leftrightarrow g_i \geqslant 0$, $\forall i \geqslant 1$. From (1.1) it follows that

$$C = \sup_{g\geqslant 0} \frac{\left(\sum\limits_{n=1}^{\infty}\left(\omega_n^q \cdot \sum\limits_{k=1}^{n}\left(\varphi_k \sum\limits_{j=1}^{k} g_j\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}}}{\sum\limits_{j=1}^{\infty} |u_j g_j|} < \infty, \tag{5.1}$$

where C is the best constant in (1.1).

We raise both sides of (5.1) to power q and get that

$$C^q = \sup_{g\geqslant 0} \frac{\left(\sum\limits_{n=1}^{\infty}\left(\omega_n^q \cdot \sum\limits_{k=1}^{n}\left(\varphi_k \sum\limits_{j=1}^{k} g_j\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{q}{\theta}}}{\left(\sum\limits_{j=1}^{\infty} |u_j g_j|\right)^q}, \tag{5.2}$$

We define $r := \frac{\theta}{q}$, $w_n^r := (\omega_n^q)^{\frac{\theta}{q}}$ and $\Phi_n := \sum\limits_{k=1}^{n}\left(\varphi_k \sum\limits_{j=1}^{k} g_j\right)^q$. Then

$$\left(\sum_{n=1}^{\infty}\left(\omega_n^q \cdot \sum_{k=1}^{n}\left(\varphi_k \sum_{j=1}^{k} g_j\right)^q\right)^{\frac{\theta}{q}}\right)^{\frac{q}{\theta}} = \left(\sum_{n=1}^{\infty} (w_n\Phi_n)^r\right)^{\frac{1}{r}}.$$

Let $h = \{h_i\}_{i=1}^\infty$, $h_i \geqslant 0$, $\forall i \geqslant 1$. Then, by the Hölder inequality,

$$\sum_{n=1}^{\infty} h_n\Phi_n \leqslant \left(\sum_{n=1}^{\infty} |w_n\Phi_n|^r\right)^{\frac{1}{r}} \left(\sum_{k=1}^{\infty} |h_k \cdot w_k^{-1}|^{r'}\right)^{\frac{1}{r'}}, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Since $r > 1$, then

$$\left(\sum_{n=1}^{\infty}(w_n\Phi_n)^r\right)^{\frac{1}{r}} = \sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty} h_n\Phi_n}{\left(\sum\limits_{k=1}^{\infty} h_k^{r'}w_k^{-r'}\right)^{\frac{1}{r'}}} = \sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty} h_n\sum\limits_{k=1}^{n}\left(\varphi_k\sum\limits_{j=1}^{k} g_j\right)^q}{\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}, \tag{5.3}$$

where $r' = \frac{r}{r-1} = \frac{\theta}{\theta-q}$. Now, we insert (5.3) into (5.2) and find that

$$\begin{aligned} C^q &= \sup_{g\geqslant 0}\sup_{h\geqslant 0}\frac{\sum\limits_{n=1}^{\infty} h_n\sum\limits_{k=1}^{n}\left(\varphi_k\sum\limits_{j=1}^{k} g_j\right)^q}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|\right)^q\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}} \\ &= \sup_{g\geqslant 0}\sup_{h\geqslant 0}\frac{\sum\limits_{k=1}^{\infty}\left(\varphi_k\sum\limits_{j=1}^{k} g_j\right)^q\sum\limits_{n=k}^{\infty} h_n}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|\right)^q\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}} \\ &= \sup_{h\geqslant 0}\frac{1}{\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}\sup_{g\geqslant 0}\frac{\sum\limits_{k=1}^{\infty}\left(\varphi_k\sum\limits_{j=1}^{k} g_j\right)^q\sum\limits_{n=k}^{\infty} h_n}{\left(\sum\limits_{j=1}^{\infty}|u_jg_j|\right)^q}. \end{aligned} \tag{5.4}$$

Let $H_k := \sum\limits_{n=k}^{\infty} h_n$. We calculate the second supremum connected to g separately. Now we apply Theorem A to (5.4) related with g and obtain that

$$\sup_{g\geqslant 0}\frac{\left(\sum\limits_{k=1}^{\infty}\left(\sum\limits_{j=1}^{k} g_j\right)^q\varphi_k^qH_k\right)^{\frac{1}{q}}}{\sum\limits_{j=1}^{\infty}|u_jg_j|} \approx \left(\sum_{n=1}^{\infty}\varphi_n^qH_n\left(\sum_{k=n}^{\infty}\varphi_k^qH_k\right)^{\frac{q}{1-q}}\max_{1\leqslant k\leqslant n} u_k^{\frac{q}{1-q}}\right)^{\frac{1-q}{q}}.$$

By using Lemma 2.1, we have that

$$\left(\sum_{n=1}^{\infty}\varphi_n^qH_n\left(\sum_{k=n}^{\infty}\varphi_k^qH_k\right)^{\frac{q}{1-q}}\max_{1\leqslant k\leqslant n} u_k^{\frac{q}{1-q}}\right)^{\frac{1-q}{q}} \approx \left(\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\varphi_k^qH_k\right)^{\frac{1}{1-q}}\Delta(\overline{u}_n^{\frac{q}{1-q}})\right)^{\frac{1-q}{q}}. \tag{5.5}$$

Here $\Delta\overline{u}_n = \overline{u}_n - \overline{u}_{n-1}$ and $\overline{u}_0 \equiv 0$, where $\overline{u}_n = \max\limits_{1\leqslant i\leqslant n} u_i$, $n \geqslant 1$.

By inserting (5.5) into (5.4) we find that

$$C^q \approx \sup_{h\geqslant 0}\frac{\left(\sum\limits_{n=1}^{\infty}\left(\sum\limits_{k=n}^{\infty}\varphi_k^qH_k\right)^{\frac{1}{1-q}}\Delta(\overline{u}_n^{\frac{q}{1-q}})\right)^{1-q}}{\left(\sum\limits_{k=1}^{\infty}(\omega_k^{-q}h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}}.$$

Next we note that

$$\sum_{k=n}^{\infty} \varphi_k^q H_k = \sum_{k=n}^{\infty} \varphi_k^q \sum_{i=k}^{\infty} h_i = \sum_{i=n}^{\infty} h_i \sum_{k=n}^{i} \varphi_k^q, \tag{5.6}$$

and define

$$U_n^{\frac{1}{1-q}} := \Delta(\overline{u}_n{}^{\frac{q}{1-q}}) \text{ and } W_k := \omega_k^{-q}. \tag{5.7}$$

Accordingly, we get that

$$C^q \approx \sup_{h\geqslant 0} \frac{\left(\sum\limits_{n=1}^{\infty}\left(\sum\limits_{i=n}^{\infty} h_i \sum\limits_{k=n}^{i} \varphi_k^q\right)^{\frac{1}{1-q}} U_n^{\frac{1}{1-q}}\right)^{1-q}}{\left(\sum\limits_{k=1}^{\infty}(W_k h_k)^{\frac{\theta}{\theta-q}}\right)^{\frac{\theta-q}{\theta}}} = \sup_{h\geqslant 0} \frac{\left(\sum\limits_{n=1}^{\infty}\left(U_n \sum\limits_{i=n}^{\infty} a_{i,n} h_i\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}}{\left(\sum\limits_{k=1}^{\infty}(W_k h_k)^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}},$$

where $a_{i,n} = \sum\limits_{k=n}^{i} \varphi_k^q$, $i \geqslant n \geqslant 1$, $\tilde{q} := \frac{1}{1-q}$ and $\tilde{p} := \frac{\theta}{\theta-q}$. We conclude that

$$\left(\sum_{n=1}^{\infty}\left(U_n \sum_{i=n}^{\infty} a_{i,n} h_i\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}} \leqslant C^q \left(\sum_{k=1}^{\infty}(W_k h_k)^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}, \quad h \geqslant 0. \tag{5.8}$$

Here

$$a_{i,n} = \sum_{k=n}^{i} \varphi_k^q \leqslant \sum_{k=n}^{j} \varphi_k^q + \sum_{k=j}^{i} \varphi_k^q = a_{i,j} + a_{j,n},$$

and

$$a_{i,n} \geqslant \sum_{k=n}^{j} \varphi_k^q, \quad a_{i,n} \geqslant \sum_{k=j}^{i} \varphi_k^q \text{ so that } a_{i,n} \geqslant \frac{1}{2}(a_{i,j} + a_{j,n}).$$

We conclude that

$$\frac{1}{2}(a_{i,j} + a_{j,n}) \leqslant a_{i,n} \leqslant a_{i,j} + a_{j,n}$$

so that

$$a_{i,n} \approx a_{i,j} + a_{j,n}, \quad i \geqslant j \geqslant n,$$

which means that $(a_{i,n})$ satisfies the discrete Oinarov condition and, moreover, $1 < \tilde{p} \leqslant \tilde{q} < \infty$, and for the operator A defined by $(Ah)_n = \sum\limits_{i=n}^{\infty} a_{i,n} h_i$, $n \geqslant 1$, we have that

$$\left(\sum_{n=1}^{\infty}\left(U_n (Ah)_n\right)^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}} \leqslant C^q \left(\sum_{k=1}^{\infty}(W_k h_k)^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}, \quad h \geqslant 0. \tag{5.9}$$

Accordingly, if C^q is the best constant in (5.9), by Theorem B it yields that

$$C^q \approx \max\{\tilde{J}_1, \tilde{J}_2\},$$

where

$$\tilde{J}_1 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} U_n^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}}\left(\sum_{i=k}^{\infty} a_{i,k}^{\tilde{p}'} W_i^{-\tilde{p}'}\right)^{\frac{1}{\tilde{p}'}},$$

$$\tilde{J}_2 = \sup_{k\geqslant 1}\Big(\sum_{n=1}^{k} a_{k,n}^{\tilde{q}} U_n^{\tilde{q}}\Big)^{\frac{1}{\tilde{q}}}\Big(\sum_{i=k}^{\infty} W_i^{-\tilde{p}'}\Big)^{\frac{1}{\tilde{p}'}},$$

Next we calculate the values of $\tilde{J}_1$ and $\tilde{J}_2$. In fact,

$$\tilde{J}_1 = \sup_{k\geqslant 1}\Big(\sum_{j=1}^{k} U_j^{\tilde{q}}\Big)^{\frac{1}{\tilde{q}}}\Big(\sum_{i=k}^{\infty} a_{i,k}^{\tilde{p}'} W_i^{-\tilde{p}'}\Big)^{\frac{1}{\tilde{p}'}}$$

$$= \sup_{k\geqslant 1}\Big(\sum_{i=k}^{\infty}\omega_i^{\theta}\Big(\sum_{s=k}^{i}\varphi_s^{q}\Big)^{\frac{\theta}{q}}\Big)^{\frac{q}{\theta}}\Big(\sum_{j=1}^{k}\Delta(\overline{u}_j{}^{\frac{q}{1-q}})\Big)^{1-q}$$

$$= \sup_{k\geqslant 1}\Big(\sum_{i=k}^{\infty}\omega_i^{\theta}\Big(\sum_{s=k}^{i}\varphi_s^{q}\Big)^{\frac{\theta}{q}}\Big)^{\frac{q}{\theta}}\Big(\max_{1\leqslant i\leqslant k} u_i\Big)^{q} = J_1^q.$$

Consequently

$$\tilde{J}_1 \approx J_1^q. \tag{5.10}$$

Moreover,

$$\tilde{J}_2 = \sup_{k\geqslant 1}\Big(\sum_{n=1}^{k} a_{k,n}^{\tilde{q}} U_n^{\tilde{q}}\Big)^{\frac{1}{\tilde{q}}}\Big(\sum_{i=k}^{\infty} W_i^{-\tilde{p}'}\Big)^{\frac{1}{\tilde{p}'}}$$

$$= \sup_{k\geqslant 1}\Big(\sum_{j=1}^{k}\Delta(\overline{u}_j{}^{\frac{q}{1-q}})\Big(\sum_{r=j}^{k}\varphi_r^{q}\Big)^{\frac{1}{1-q}}\Big)^{1-q}\Big(\sum_{i=k}^{\infty}\omega_i^{\theta}\Big)^{\frac{q}{\theta}} = J_2^q$$

From $\tilde{J}_2 \approx J_2^q$ and (5.10) it follows that $C^q \approx \max\{\tilde{J}_1, \tilde{J}_2\} \approx \max\{J_1^q, J_2^q\}$. Hence, we can conclude that $C \approx \max\{J_1, J_2\}$ so the proof is complete. □

6. Proof of Theorem 3.3

Proof. The proof of Theorem 3.3 is the same as the proof of Theorem 3.2 up to (5.8). But in Theorem 3.2, we proved that the inequality (5.8) holds for the case $1 < \frac{\theta}{\theta-q} = \tilde{p} \leqslant \tilde{q} = \frac{1}{1-q} < \infty$, from which it follows that $0 < q < 1 \leqslant \theta < \infty$.

Note that, $\frac{\theta}{\theta-q} > \frac{1}{1-q} \Rightarrow \theta - \theta q - \theta + q > 0 \Rightarrow q - q\theta > 0 \Rightarrow \theta < 1,\ 0 < q < 1,\ \theta > q$. Then the case $1 < \frac{1}{1-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$ follows from the case $0 < q < \theta < 1$. Therefore we will consider for the inequality (5.8) for the following case: $1 < \frac{1}{1-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$.

Let $1 < \frac{1}{1-q} = \tilde{q} < \tilde{p} = \frac{\theta}{\theta-q} < \infty$. We can estimate the value of best constant C^q in the inequality (5.8), where $(a_{i,j})$ satisfies the discrete Oinarov condition. In fact, by using Theorem C, we have that $C^q \approx \max\{\tilde{J}_3, \tilde{J}_4\}$, where

$$\tilde{J}_3 = \Big(\sum_{j=1}^{\infty}\Big(\sum_{i=j}^{\infty} a_{i,j}^{\tilde{p}'} W_i^{-\tilde{p}'}\Big)^{\frac{\tilde{q}(\tilde{p}-1)}{\tilde{p}-\tilde{q}}}\Big(\sum_{i=1}^{j} U_i^{\tilde{q}}\Big)^{\frac{\tilde{q}}{\tilde{p}-\tilde{q}}} U_j^{\tilde{q}}\Big)^{\frac{\tilde{p}-\tilde{q}}{\tilde{p}\tilde{q}}},$$

$$\tilde{J}_4 = \Big(\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{j} a_{j,k}^{\tilde{q}} U_k^{\tilde{q}}\Big)^{\frac{\tilde{p}}{\tilde{p}-\tilde{q}}}\Big(\sum_{i=j}^{\infty} W_i^{-\tilde{p}'}\Big)^{\frac{\tilde{p}(\tilde{q}-1)}{\tilde{p}-\tilde{q}}} W_j^{-\tilde{p}'}\Big)^{\frac{\tilde{p}-\tilde{q}}{\tilde{p}\tilde{q}}}.$$

Next we rewrite the values of $\tilde{J}_3$ and $\tilde{J}_4$ by using (5.6) and (5.7). In fact,

$$\tilde{J}_3 = \left(\sum_{j=1}^{\infty} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \left(\sum_{s=j}^{i} \varphi_s^q \right)^{\frac{\theta}{q}} \right)^{\frac{1}{1-\theta}} \left(\sum_{i=1}^{j} \Delta(\overline{u}_i^{\frac{q}{1-q}}) \right)^{\frac{\theta-q}{q(1-\theta)}} \Delta(\overline{u}_j^{\frac{q}{1-q}}) \right)^{\frac{q(1-\theta)}{\theta}} = J_3^q,$$

and

$$\tilde{J}_4 = \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{j} \left(\sum_{s=k}^{j} \varphi_s^q \right)^{\frac{1}{1-q}} \Delta(\overline{u}_k^{\frac{q}{1-q}}) \right)^{\frac{\theta(1-q)}{q(1-\theta)}} \left(\sum_{i=j}^{\infty} \omega_i^{\theta} \right)^{\frac{\theta}{1-\theta}} \omega_j^{\theta} \right)^{\frac{q(1-\theta)}{\theta}} = J_4^q.$$

Hence $C^q \approx \max\{\tilde{J}_3, \tilde{J}_4\} \approx \{J_3^q, J_4^q\}$ so that $C \approx \{J_3, J_4\}$, where C is the best constant in (1.1). The proof is complete. ☐

7. Final remarks

REMARK 7.1. Some related results, but with the dual operator H_{φ}^*, defined by $(H_{\varphi}^* f)_n := \left(\sum_{k=n}^{\infty} \left| \varphi_k \sum_{i=n}^{\infty} f_i \right|^q \right)^{\frac{\theta}{q}}$, involved can be found in the paper [24]. However, the proof of this result is simpler and Theorems A-C, Lemma A and our new Lemma 2.1 is not required.

REMARK 7.2. As mentioned in our introduction Hardy-type inequalities are important for various applications both in mathematics and engineering sciences. There are a huge number of papers where such applications of Hardy-type inequalities are pointed out, see e.g. the books [13], [14], [16] and [17] and the list of publications in these books. Here we just mention the following example: As a motivation for the book [16] on pages 1–3 an important nonlinear differential equation was considered and it was proved how a simple form of the Hardy inequality could help to solve it. Moreover, we inform about the fact that there are a huge number of PhD theses in this area, even around 20 related to the authors of this paper. Just to mention two late of them where this aspect is central we pronounce [32] and [35]. In particular, in the 2024 PhD thesis [35] on pages 110–113 some important such applications are described and in the 2021 PhD thesis [32] applications to structural analysis related to security problems in bridges are described and already now used by companies in Norway. Finally, we mention the recent paper [22] in this Journal by R. Oinarov, A. Kalybay and L. E. Persson, where it was even necessary to first prove a NEW Hardy-type inequality to be able to solve the problem at hand, i.e. to describe oscillatory and spectral properties of a class of fourth-order differential equations.

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