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ON THE GENERALIZED n–TH JORDAN–VON NEUMANN CONSTANT AND FIXED POINT PROPERTY

XI WANG, XIANFENG SUN AND CHAO XIA*

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Abstract. In this paper, we introduce a new geometric constant $C_{NJ}^{(n)}(a,X)$ of a Banach space X, which is closely related to the n-th Jordan-Von Neumann constant and analyze some properties of the constant. Subsequently, we present a relationship between the weakly convergent sequence coefficient (WCS) and this new constant. Our main results of the paper improve some known results in the recent literature.

1. Introduction

The concept of normal structure plays a central role in fixed point theory, as its existence profoundly influences the existence and uniqueness of fixed points for nonexpansive mappings in Banach spaces. In order to investigate the geometric nature of Banach spaces, mathematicians introduced various geometric constants. These constants serve as tools to identify whether a Banach space has normal structure, for instance, see [5, 6, 7, 11, 21, 22, 24].

Throughout the paper, let X be a real Banach space with the dual space X^*. As usual, we will denote by $S_X=\{x\in X:\| x \|=1\}$ and $B_X=\{x\in X:\| x \|\leqslant 1\}$ the unit sphere and the unit ball of X, respectively.

Among the numerous geometric constants of Banach spaces, the James constant and the von Neumann-Jordan constant are the two most extensively studied, owing to their profound connections with diverse geometric structures of Banach spaces. The following constant of a Banach space X,

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}:x,y\in X,\ \|x\|+\|y\|>0\right\}$$

is called Jordan-von Neumann constant [2]. It is clear that if is a Hilbert space, $C_{NJ}(X)=1$. Therefore, the Jordan-von Neumann constant is commonly used to quantify the deviation of a general Banach space from the properties of a Hilbert space.

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[*] Corresponding author.

The following constant of a Banach space X,

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:x,y\in B_X\}$$

is called James constants [8], it is utilized to quantify the non-squareness characteristics of a Banach space.

Recently, mathematicians have generalized the aforementioned two constants in the following ways (see [3, 4]): for $0\leqslant a\leqslant 2$,

$$C_{NJ}(a,X)=\sup\left\{\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}:x,y,z\in X,\right.$$
$$\left.\|x\|+\|y\|+\|z\|>0\ \text{ and }\ \|y-z\|\leqslant a\|x\|\right\},$$

$$J(a,X)=\sup\{\min\{\|x+y\|,\|x-z\|\}:x,y,z\in B_X \text{ and } \|y-z\|\leqslant a\|x\|\}.$$

Clearly, $C_{NJ}(0,X)=C_{NJ}(X)$ and $J(0,X)=J(X)$.

Building upon these developments, this paper introduces a further generalization the generalized n-th Jordan-von Neumann constant $C_{NJ}^{(n)}(a,X)$. This constant not only extends $C_{NJ}(a,X)$ to involve multiple variables but also provides a sharper geometric criterion for normal structure and uniform non-squareness in high dimensional settings. We analyze its fundamental properties, its monotonicity and continuity, and how it behaves under ultrapowers. Furthermore, we explore its connection to the weakly convergent sequence coefficient, a quantity measuring how sequences in Banach spaces diverge weakly.

Our results generalize and refine the findings of prior works, including those of [4, 8, 12, 15, 16, 17, 23]. Moreover, our approach employs ultraproduct techniques and functionals constructed on ultrapowers, which have proved powerful in extending fixed point results for nonexpansive maps.

2. Preliminaries

We begin by recalling the classical definition of (weak) normal structure in Banach spaces (see [1]), which plays a central role in the study of fixed point theory.

A Banach space X is said to possess normal structure if for every closed, bounded, and convex subset $K\subset X$ that contains more than one point, there exists a point $x_0\in K$ such that

$$\sup\{\|x_0-y\|:y\in K\}<\operatorname{diam}(K),$$

where the diameter of K, denoted $\operatorname{diam}(K)$, is defined by

$$\operatorname{diam}(K)=\sup\{\|x-y\|:x,y\in K\}.$$

In the case where the Banach space X is reflexive, the notions of normal structure and weak normal structure coincide. It is known that if a Banach space lacks weak normal structure, then there exists a weakly compact, convex subset $C \subset X$ and a sequence $\{x_n\} \subset C$ such that

$$\operatorname{dist}(x_{n+1}, co\{x_k\}_{k=1}^{n}) \to \operatorname{diam}(C) = 1,$$

see [10]. A Banach space is said to have uniform normal structure if there exists a constant $0 < c < 1$ such that for any closed, bounded, convex subset K of X with more than one element, there exists a point $x_0 \in K$ satisfying

$$\sup\{\|x_0 - y\| : y \in K\} < c\operatorname{diam}(K).$$

Kirk [14] established that every reflexive Banach space with normal structure has the fixed point property: for any nonempty, closed, bounded, convex subset C of X and any nonexpansive self-mapping $T : C \to C$, there exists a point $x \in C$ such that $x = Tx$.

In what follows, we briefly recall several fundamental facts concerning ultrapowers of Banach spaces, which form the backbone of the arguments developed in this paper. The ultrapower technique has become an indispensable tool in modern Banach space theory, especially for dealing with asymptotic or structural properties, the reader is directed to [10, 13, 20].

Let F be a filter on $\mathbb{N}$, and let $\{x_n\}$ be a sequence in a Banach space X. We say that $\{x_n\}$ converges to x with respect to the filter F, written $\lim_F x_i = x$, if for every neighborhood of U of x, the set $x, i \in \mathbb{N} : x_i \in U \in F$ belongs to F. A filter $\mathscr{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to inclusion. An ultrafilter is said to be trivial if it contains all subsets of $\{A \subset \mathbb{N} : i_0 \in A\}$ that contain a fixed element $i_0 \in \mathbb{N}$; otherwise, it is called nontrivial. Let $\ell_\infty(X)$ denote the Banach space of all bounded sequences in X, in other words sequences $\sup_n \|x_n\| < \infty$. The norm is defined by

$$\|(x_n)\| := \sup_{n \in \mathbb{N}} \|x_n\| < \infty.$$

Given a nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$, define the subspace

$$N_{\mathscr{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathscr{U}} \|x_n\| = 0\}.$$

Then the ultrapower of X, denoted $\widetilde{X}$, is the quotient space $l_\infty(X)/N_{\mathscr{U}}$, equipped with the quotient norm. The ultrapower $\widetilde{X}$ endowed with this norm is also called **U-spaces**. Elements of $\widetilde{X}$ are denoted by $(x_n)_{\mathscr{U}}$. The norm is given by

$$\|(x_n)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_n\|.$$

If the ultrafilter U is nontrivial, then the original space X embeds isometrically into $\widetilde{X}$. Moreover, if X is super-reflexive, in other words its ultrapower dual satisfies $X^* = \widetilde{X}^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [13]).

LEMMA 1. (see [18]) *If a super-reflexive Banach space X fails to have normal structure, then for $r \in (0,1]$ there are $\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3} \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2}, \widetilde{f_3} \in S_{\widetilde{X}^*}$ such that*

1. *$\|\widetilde{x}_i-\widetilde{x}_j\|=1$ and $\widetilde{f}_i(\widetilde{x}_j)=0$ for all $i\neq j$;*
2. *$\widetilde{f}_i(\widetilde{x}_i)=1$ for $i=1,2,3$;*
3. *$\|\widetilde{x}_3-(\widetilde{x}_2+r\widetilde{x}_1)\|\geqslant\|\widetilde{x}_2+r\widetilde{x}_1\|$.*

3. The constant $C_{NJ}^{(n)}(a,X)$

DEFINITION 1. For any $a\geqslant 0$, define the generalized n-th Jordan-von Neumann constant, $n\geqslant 1$, by

$$C_{NJ}^{(n)}(a,X):=\sup\left\{\frac{\sum_{\theta_j}\|x+\sum_{j=1}^n\theta_j x_j^{(\theta_j)}\|^2}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)},x,x_j^{(\theta_j)}\in X\right.$$
$$\text{are not all zero, where } x_j^{(\theta_j)}=y_j \text{ if } \theta_j=1 \text{ and } x_j^{(\theta_j)}=z_j$$
$$\left.\text{if } \theta_j=-1, \text{ with } \|y_j-z_j\|\leqslant a\|x\| \text{ for } j=1,2,\cdots,n\right\}.$$

REMARK 1.

1. $C_{NJ}^{(1)}(a,X)=C_{NJ}(a,X)$, and $C_{NJ}^{(n)}(0,X)=C_{NJ}^{(n)}(X)$;
2. $C_{NJ}^{(n)}(a,X)$ is a nondecreasing function on a;
3. if there exists $a\geqslant 0$ such that $C_{NJ}^{(n)}(a,X)<n+1$, then we have $C_{NJ}(a,X)<n+1$, hence X is uniformly non-square.

PROPOSITION 1. *For any $a\geqslant 0$, we have $\frac{4+4na+n^2a^2}{4+na^2}\leqslant C_{NJ}^{(n)}(a,X)\leqslant n+1$. Moreover, $C_{NJ}^{(n)}(a,X)=n+1$, for all $a\geqslant 2$.*

Proof. For the inequality on the left hand side, take $x\in S_X$, let $y=\frac{a}{2}x=-z$. Then, $y-z=ax$, hence,

$$\begin{aligned}C_{NJ}^{(n)}(a,X)\geqslant&\frac{\sum_{\theta_j}\|x+\sum_{j=1}^n\theta_j x_j^{(\theta_j)}\|^2}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)}\\ \geqslant&\frac{2^n\cdot\left(1+\frac{na}{2}\right)^2\|x\|^2}{2^n\|x\|^2+2^n n\frac{a^2}{4}\|x\|^2}\\ =&\frac{4+4na+n^2a^2}{4+na^2}.\end{aligned}$$

For the inequality on the right hand side, by the triangle inequality, we have

$$\begin{aligned}\sum_{\theta_j}\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2 \leqslant & 2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ & +2^n\|x\|\sum_{j=1}^{n}(\|y_j\|+\|z_j\|)\\ & +2^{n-1}\sum_{1\leqslant i\neq j\leqslant n}(\|y_i\|\|y_j\|+\|z_i\|\|z_j\|+\|y_i\|\|z_j\|+\|y_j\|\|z_i\|)\\ \leqslant & 2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ & +2^n n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|+\|z_j\|)\\ & +2^{n-1}(n-1)\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ = & (n+1)\left(2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\right),\end{aligned}$$

so $C_{NJ}^{(n)}(a,X)\leqslant n+1$. At last, notice that the function $f(a)=\frac{4+4na+n^2a^2}{4+na^2}$ is strictly increased on the interval $[0,a]$, and take its maximum at $a=2$. Hence, for any $a\geqslant 2$, we have $C_{NJ}^{(n)}(a,X)=n+1$. □

LEMMA 2. *Let H be a Hilbert space, for any two elements $x,y\in H$, the inner product of x,y is denoted by $\langle x,y\rangle$. Then following equality*

$$\begin{aligned}\sum_{\theta_j}\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2 = & 2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ & +2^{n-1}\sum_{j=1}^{n}(\langle x,y_j-z_j\rangle+\langle y_j-z_j,x\rangle)\\ & +2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i,y_j-z_j\rangle\end{aligned}\tag{1}$$

is valid for any positive integer n, where $\theta_j=\pm 1$ and $x_j^{(\theta_j)}=y_j$ if $\theta_j=1$, $x_j^{(\theta_j)}=z_j$ if $\theta_j=-1$ for $j=1,2,\cdots,n$.

Proof. We prove this lemma by induction of positive integer n. First, for $n=1$, we have

$$\begin{aligned}\|x+y_1\|^2+\|x-z_1\|^2 &= \langle x+y_1,x+y_1\rangle+\langle x-z_1,x-z_1\rangle\\ &= 2\|x\|^2+\|y_1\|^2+\|z_1\|^2+\langle x,y_1\rangle+\langle y_1,x\rangle-\langle x,z_1\rangle-\langle z_1,x\rangle\\ &= 2\|x\|^2+\|y_1\|^2+\|z_1\|^2+\langle x,y_1-z_1\rangle+\langle y_1-z_1,x\rangle.\end{aligned}$$

Suppose this equality is hold for positive integer n, i.e.

$$\begin{aligned}\sum_{\theta_j}\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2 =&2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ &+2^{n-1}\sum_{j=1}^{n}(\langle x,y_j-z_j\rangle+\langle y_j-z_j,x\rangle)\\ &+2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i,y_j-z_j\rangle,\end{aligned}$$

then, for integer $n+1$ we have,

$$\begin{aligned}&\sum_{\theta_j=\pm 1}\|x+\sum_{j=1}^{n+1}\theta_j x_j^{(\theta_j)}\|^2\\ =&\sum_{\theta_j=\pm 1}\|x+y_{n+1}\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2+\sum_{\theta_j}\|x-z_{n+1}+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2\\ =&2^n\|x+y_{n+1}\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ &+2^{n-1}\sum_{j=1}^{n}(\langle x+y_{n+1},y_j-z_j\rangle+\langle y_j-z_j,x+y_{n+1}\rangle)\\ &+2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i,y_j-z_j\rangle+2^n\|x-z_{n+1}\|^2+2^{n-1}\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ &+2^{n-1}\sum_{j=1}^{n}(\langle x-z_{n+1},y_j-z_j\rangle+\langle y_j-z_j,x-z_{n+1}\rangle)+2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i,y_j-z_j\rangle\\ =&2^{n+1}\|x\|^2+2^n(\|y_{n+1}\|^2+\|z_{n+1}\|^2)+2^n\sum_{j=1}^{n}(\|y_j\|^2+\|z_j\|^2)\\ &+2^n(\langle x,y_{n+1}-z_{n+1}\rangle+\langle y_{n+1}-z_{n+1},x\rangle)+2^n\sum_{j=1}^{n}(\langle x,y_j-z_j\rangle+\langle y_j-z_j,x\rangle)\\ &+2^{n-1}\sum_{j=1}^{n}(\langle y_{n+1}-z_{n+1},y_j-z_j\rangle+\langle y_j-z_j,y_{n+1}-z_{n+1}\rangle)+2^{n-1}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i,y_j-z_j\rangle\\ =&2^{n+1}\|x\|^2+2^n\sum_{j=1}^{n+1}(\|y_j\|^2+\|z_j\|^2)+2^n\sum_{j=1}^{n+1}(\langle x,y_j-z_j\rangle+\langle y_j-z_j,x\rangle)\\ &+2^{n-1}\sum_{1\leqslant i\neq j\leqslant n+1}\langle y_i-z_i,y_j-z_j\rangle.\end{aligned}$$

We complete the proof. □

Now we have the following proposition.

PROPOSITION 2. *Let H be Hilbert space. Then for any number $a\in[0,2]$, we have $C_{NJ}^{(n)}(a,H)=1+\frac{4na+n(n-1)a^2}{4+na^2}$.*

Proof. Suppose that $a \in [0,2], x, y_j, z_j \in H$ with $x \neq 0$ and $\|y_j - z_j\| = \alpha\|x\|$ for $j = 1,2,\cdots,n$, where $\alpha \in [0,a]$. Then, by Lemma 2, we have

$$
\begin{aligned}
&\frac{\sum_{\theta_j} \|x+\sum_{j=1}^n \theta_j x_j^{(\theta_j)}\|^2}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n(\|y_j\|^2+\|z_j\|^2)}\\
=&\frac{1}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)}\cdot\Bigg[2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)\\
&+2^{n-1}\sum_{j=1}^n\left(\langle x, y_j-z_j\rangle+\langle y_j-z_j, x\rangle\right)+2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\langle y_i-z_i, y_j-z_j\rangle\Bigg]\\
\leqslant&1+\frac{2^n\|x\|(\sum_{j=1}^n\|y_j-z_j\|)+2^{n-2}\sum_{1\leqslant i\neq j\leqslant n}\left(\|y_i-z_i\|\|y_j-z_j\|\right)}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)}\\
=&1+\frac{2^n n\alpha\|x\|^2+2^{n-2}n(n-1)\alpha^2\|x\|^2}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^n\left(\|y_j\|^2+\|z_j\|^2\right)}\\
\leqslant&1+\frac{2^n n\alpha\|x\|^2+2^{n-2}n(n-1)\alpha^2\|x\|^2}{2^n\|x\|^2+2^{n-2}\sum_{j=1}^n\left(\|y_j-z_j\|^2+\|y_j+z_j\|^2\right)}\\
\leqslant&1+\frac{2^n n\alpha\|x\|^2+2^{n-2}n(n-1)\alpha^2\|x\|^2}{2^n\|x\|^2+2^{n-2}\sum_{j=1}^n\|y_j-z_j\|^2}\\
=&1+\frac{4n\alpha+n(n-1)\alpha^2}{4+n\alpha^2}\leqslant 1+\frac{4na+n(n-1)a^2}{4+na^2}.
\end{aligned}
$$

Hence, by Proposition 1, we have $C_{NJ}^{(n)}(a,H) = 1+\frac{4na+n(n-1)a^2}{4+na^2}$. $\square$

PROPOSITION 3. *$C_{NJ}^{(n)}(a,X)$ is a continuous function on $[0,\infty)$.*

Proof. Since $C_{NJ}^{(n)}(\cdot,X)$ is non-decreasing, we can assume that there exist $a>0$ such that

$$
\sup_{b<a} C_{NJ}^{(n)}(b,X) = \alpha < \beta < \gamma = \inf_{b>a} C_{NJ}^{(n)}(b,X).
$$

Select $\gamma_m \downarrow a$ and $x_m, y_{j,m}, z_{j,m} \in B_X, j = 1,2,\cdots,n$ with at least one of them belongs to S_X, satisfying $\|y_{j,m}-z_{j,m}\| = \gamma_m\|x_m\|$ and

$$
g(x_m, y_{1,m},\cdots,y_{n,m}, z_{1,m},\cdots,z_{n,m}) \geqslant \beta
$$

for all $m \in \mathbb{N}$. Choose $\eta_m \downarrow 1$, such that $\frac{\gamma_m}{\eta_m} < a$ for all $m \in \mathbb{N}$. Then, we have

$$
g(\eta_m x_m, y_{1,m},\cdots,y_{n,m}, z_{1,m},\cdots,z_{j,m}) = g\left(x_m, \frac{y_{1,m}}{\eta_m},\cdots,\frac{y_{n,m}}{\eta_m},\frac{z_{1,m}}{\eta_m},\cdots,\frac{z_{j,m}}{\eta_m}\right)
$$

for all $m \in \mathbb{N}$, Choose a subsequence $\{m'\}$ of the sequence $\{m\}$ such that all the sequences $\|x_{m'} + \sum_{j=1}^{n} \theta_j x_{j,m'}^{(\theta_j)}\|, \|x_{m'}\|, \|y_{j,m'}\|, \|z_{j,m'}\|$ are convergence. Notice that, for any $\omega \in X$ and $\eta_m \to 1$, we have

$$\|x_m + \omega\| - (\eta_m - 1)\|x_m\| \leqslant \|\eta_m x_m + \omega\| \leqslant \|x_n + \omega\| + (\eta_m - 1)\|x_m\|,$$

hence, $\lim_{m'} \|x_{m'} + \sum_{j=1}^{n} \theta_j x_{j,m'}^{(\theta_j)}\| = \lim_{m'} \|\eta_{m'} x_{m'} + \sum_{j=1}^{n} \theta_j x_{j,m'}^{(\theta_j)}\|$. So,

$$\begin{aligned}\beta - \gamma \leqslant & g(x_{m'}, y_{1,m'}, \cdots, y_{n,m'}, z_{1,m'}, \cdots, z_{n,m'}) \\ & - g(\eta_{m'} x_{m'}, y_{1,m'}, \cdots, y_{n,m'}, z_{1,m'}, \cdots, z_{n,m'}) \to 0,\end{aligned}$$

which is a contradiction to the assumption that $\beta > \gamma$. Hence, $C_{NJ}^{(n)}(a,X)$ is a continuous function for $a > 0$.

When $a = 0$, for a given $\varepsilon > 0$, we choose that $x_m, y_{j,m}, z_{j,m} \in B_X, j = 1,2,\cdots,n$ with at least one of them belongs to S_X, satisfying $\|y_{j,m} - z_{j,m}\| = a_m\|x_m\|, a_m \downarrow 0$, $y_{1,m} = y_{2,m} = \cdots = y_{n,m} = y_m$, $z_{1,m} = z_{2,m} = \cdots = z_{n,m} = z_m$, and

$$C_{NJ}^{(n)}(0^+,X) - \varepsilon := \inf_{a>0} C_{NJ}^{(n)}(a,X) - \varepsilon < \lim_{m\to\infty} g(x_m, y_{1,m}, \cdots, y_{n,m}, z_{1,m}, \cdots, z_{n,m}).$$

Let $\varepsilon_m = 2^n(2n(n+1)a_m + n^2 a_m^2), \gamma_m = n2^{n-1} a_m \|x_m\|(\|y_m\| - a_m\|x_m\|)$. Hence, $\varepsilon_m, \gamma_m \to 0$. Extract the subsequences, we can assume that $\lim_{n\to\infty}(\|x_m\|^2 + n\|y_m\|^2) = b$ exists. By the chosen of $x_m, y_{j,m}, z_{j,m}, j = 1,2,\cdots,n$, we know that $b \neq 0$. For sufficiently large n, we have

$$\begin{aligned} g(x_m, y_{1,m}, \cdots, y_{n,m}, z_{1,m}, \cdots, z_{j,m}) \leqslant & \frac{\sum_{\theta_j=\pm 1} \|x_m + \sum_{j=1}^{n} \theta_j y_m\| + \varepsilon_m}{2^n\|x_m\| + n2^n\|y_m\| - \gamma_m} \\ \leqslant & g(x_m, y_m, \cdots, y_m, y_m, \cdots, y_m) \\ & + \frac{\varepsilon_m + \gamma_m g(x_m, y_m, \cdots, y_m, y_m, \cdots, y_m)}{2^n\|x_m\| + n2^n\|y_m\| - \gamma_m} \\ \leqslant & C_{NJ}^{(n)}(X) + \frac{\varepsilon_m + \gamma_m C_{NJ}^{(n)}(X)}{2^n\|x_m\| + n2^n\|y_m\| - \gamma_m}. \end{aligned}$$

Hence, for any $\varepsilon > 0$, we have $C_{NJ}^{(n)}(0^+,X) - \varepsilon < C_{NJ}^{(n)}(X) \leqslant C_{NJ}^{(n)}(0^+,X)$, so $C_{NJ}^{(n)}(0^+,X) = C_{NJ}^{(n)}(X)$, i.e., $C_{NJ}^{(n)}(\cdot,X)$ is continuous at $a = 0$. We complete the proof. ☐

PROPOSITION 4. *$C_{NJ}^{(n)}(a,X) = C_{NJ}^{(n)}(a,\widetilde{X})$ for $a \geqslant \frac{n-1}{n}$.*

Proof. $C_{NJ}^{(n)}(a,X) \leqslant C_{NJ}^{(n)}(a,\widetilde{X})$ is obvious. Now, we prove $C_{NJ}^{(n)}(a,X) \geqslant C_{NJ}^{(n)}(a,\widetilde{X})$.

For any $\varepsilon > 0$, suppose $\widetilde{x}, \widetilde{y_j}, \widetilde{z_j} \in \widetilde{X}, j = 1,2,\cdots,n$ are not all zero, satisfying $\|\widetilde{y_j} - \widetilde{z_j}\| = \alpha\|x\|, \alpha \in [0,a]$. If $\widetilde{x} = 0$, then $g(\widetilde{x}, \widetilde{y_1}, \cdots, \widetilde{y_n}, \widetilde{z_1}, \cdots, \widetilde{z_n}) \leqslant n \leqslant \frac{4+4na+n^2a^2}{4+na^2} \leqslant$

$C_{NJ}^{(n)}(a,X)$. If $\widetilde{x} \neq 0$, Choose $\varepsilon > 0$ such that $\varepsilon < \delta\|x\|$. As

$$\begin{aligned} c := & \frac{\sum_{\theta_j} \|\widetilde{x} + \sum_{j=1}^{n} \theta_j \widetilde{x}_j^{(\theta_j)}\|^2}{2^n\|\widetilde{x}\|^2 + 2^{n-1}\sum_{j=1}^{n}\left(\|\widetilde{y}_j\|^2 + \|\widetilde{z}_j\|^2\right)} \\ = & \lim_{\mathscr{U}} \frac{\sum_{\theta_j} \|x_m + \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}\|^2}{2^n\|x_m\|^2 + 2^{n-1}\sum_{j=1}^{n}\left(\|y_{j,m}\|^2 + \|z_{j,m}\|^2\right)} \\ := & \lim_{\mathscr{U}} c_m. \end{aligned}$$

Hence, the set $\left\{ m \in \mathbb{N} : |c_m - c| < \delta, \|y_{j,m} - z_{j,m}\| \leqslant \alpha\|x_m\| + \varepsilon < (\alpha+\delta)\|x_m\| \right\} \in \mathscr{U}$. In particular, there exists m such that

$$c < g(x_m, y_{1,m}, \cdots, y_{n,m}, z_{1,m}, \cdots, z_{j,m}) + \delta \leqslant C_{NJ}^{(n)}(a+\delta, X) + \delta.$$

By the arbitrariness of δ and continuity of $C_{NJ}^{(n)}(\cdot, X)$, we have $C_{NJ}^{(n)}(a,\widetilde{X}) \leqslant C_{NJ}^{(n)}(a,X)$. □

LEMMA 3. *Let any $a \in [0,2)$, if $C_{NJ}^{(n)}(a,X) = n+1$, then there exist $\{x_m\}$, $\{y_{j,m}\}$, $\{z_{j,m}\} \subset B_X$, $j = 1,2,\cdots n$ satisfying the following conditions*

1. $\|x_m\|, \|y_{j,m}\|, \|z_{j,m}\| \to 1$, *for* $j = 1,2,\cdots,n$;
2. $\|x_m + \sum_{j=1}^{n} x_{j,m}^{(\theta_j)}\| \to n+1$, *where* $x_{j,m}^{(\theta_j)} = y_{j,m}$ *if* $\theta_j = 1$ *and* $x_{j,m}^{(\theta_j)} = z_{j,m}$ *if* $\theta_j = -1$ *for* $j = 1,2,\cdots,n$;
3. $\|y_{j,m} - z_{j,m}\| \leqslant a\|x_m\|$, *for* $j = 1,2,\cdots,n$ *and* $\forall m \in \mathbb{N}$.

Moreover, $\{x_m\}, \{y_{j,m}\}, \{z_{j,m}\}$ *can take value in* S_X, *for* $j = 1,2,\cdots n$.

Proof. Let $a \in [0,2)$, if $C_{NJ}^{(n)}(a,X) = n+1$, then there exists $x_m, y_{j,m}, z_{j,m} \in B_X$ and at least one belongs to S_X, satisfying $\|y_{j,m} - z_{j,m}\| \leqslant a\|x_m\|, \forall n \in \mathbb{N}$, and $g(x_m, y_{1,m}, \cdots, y_{n,m}, z_{1,m}, \cdots, z_{n,m}) \uparrow n+1$, here

$$g(x_m, y_{1,m}, \cdots, y_{n,m}, z_{1,m}, \cdots, z_{n,m}) = \frac{\sum_{\theta_j} \|x_m + \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}\|^2}{2^n\|x\|^2 + 2^{n-1}\sum_{j=1}^{n}\left(\|y_{j,m}\|^2 + \|z_{j,m}\|^2\right)},$$

where $x_{j,m}^{(\theta_j)} = y_{j,m}$ if $\theta_j = 1$ and $x_{j,m}^{(\theta_j)} = z_{j,m}$ if $\theta_j = -1$ for $j = 1,2,\cdots,n$, so the inequality in (3) is hold. Notice that

$$\begin{aligned}
&g(x,y_1,\cdots,y_n,z_1,\cdots,z_n)\\
=&\frac{\sum_{\theta_j}\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\|^2}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}\Big(\|y_j\|^2+\|z_j\|^2\Big)}\\
\leqslant&1+\frac{2^n\|x\|\sum_{j=1}^{n}(\|y_j\|+\|z_j\|)+2^{n-1}\sum_{1\leqslant i\neq j\leqslant n}(\|y_i\|\|y_j\|+\|z_i\|\|z_j\|+\|z_i\|\|y_j\|+\|y_i\|\|z_j\|)}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}\Big(\|y_j\|^2+\|z_j\|^2\Big)}\\
\leqslant&1+\frac{2^n\|x\|\sum_{j=1}^{n}(\|y_j\|+\|z_j\|)+2^{n-1}(n-1)\sum_{1\leqslant i\neq j\leqslant n}(\|y_j\|^2+\|z_j\|^2)}{2^n\|x\|^2+2^{n-1}\sum_{j=1}^{n}\Big(\|y_j\|^2+\|z_j\|^2\Big)}\\
\leqslant&n+\frac{\|x\|\sum_{j=1}^{n}(\|y_j\|+\|z_j\|)-2(n-1)\|x\|^2}{\|x\|^2+\sum_{j=1}^{n}\Big(\|y_j\|^2+\|z_j\|^2\Big)},
\end{aligned}$$

hence,

$$\frac{\|x\|\sum_{j=1}^{n}(\|y_{j,m}\|+\|z_{j,m}\|)-2(n-1)\|x_m\|^2}{2\|x_m\|^2+\sum_{j=1}^{n}\Big(\|y_{j,m}\|^2+\|z_{j,m}\|^2\Big)}\to 1,$$

which means that

$$\frac{\sum_{j=1}^{n}\Big[(\|x_m\|-\|y_{j,m}\|)^2+(\|x_m\|-\|z_{j,m}\|)^2\Big]}{2\|x_m\|^2+\sum_{j=1}^{n}\Big(\|y_{j,m}\|^2+\|z_{j,m}\|^2\Big)}\to 0.$$

Since for any $m\in\mathbb{N}$, at least one of $x_m, y_{j,m}, z_{j,m}$ belongs to S_X, there exist a subsequence of $\{m\}$, saying $\{m'\}$, such that $\|x_{m'}\|, \|y_{j,m'}\|, \|z_{j,m'}\|\to 1$. Hence, $\|x_{m_i}+\sum_{j=1}^{n}x_{j,m_i}^{(\theta_j)}\|\to n+1$, where $x_{j,m_i}^{(\theta_j)}=y_{j,m_i}$ if $\theta_j=1$ and $x_{j,m_i}^{(\theta_j)}=z_{j,m_i}$ if $\theta_j=-1$ for $j=1,2,\cdots,n$; i.e, (1) and (2) are hold.

Next, for $x\neq 0$, let $x'=\frac{x}{\|x\|}$, by the chosen of $x_m, y_{j,m}, z_{j,m}$ we know that $\|x'_{m'}-x_{m'}\|, \|y'_{j,m'}-y_{j,m'}\|, \|z'_{j,m'}-z_{j,m'}\|\to 2$. As $n+1\geqslant\|x'+y'_1+\cdots y'_n\|\geqslant\|x+y_1+\cdots+y_n\|-\|x'-x\|-\sum_{j=1}^{n}\|y'_j-y_j\|$, we have $\|x_{m'}+\sum_{j=1}^{n}\theta_j x_{j,m'}^{(\theta_j)}\|\to n+1$. At last, $a\|x_{m'}\|\geqslant\|y_{j,m'}-z_{j,m'}\|\geqslant\|y'_{j,m'}-z'_{j,m'}\|-\|y'_{j,m'}-y_{j,m'}\|-\|z'_{j,m'}-z_{j,m'}\|$. Hence, $\limsup_{i\to\infty}\|y'_{j,m'}-z'_{j,m'}\|\leqslant a$, i.e., $\{x_m\}, \{y_{j,m}\}, \{z_{j,m}\}$ can take value in S_X, for $j=1,2,\cdots n$. We complete the proof. □

THEOREM 1. *If X is an U space, then $C_{NJ}^{(n)}(a,X) < n+1, \forall a \in (0, \frac{2}{n})$.*

Proof. If there exists an $\delta > 0$ such that $C_{NJ}^{(n)}(\frac{2-\delta}{n}, X) = n+1$, then there exist sequences $\{x_m\}, \{y_{j,m}\}, \{z_{j,m}\} \subset S_X$ such that $\|x_m + \sum_{j=1} \theta_j x_{j,m}^{(\theta)}\| \to n+1$ and $\|y_{j,m} - z_{j,m}\| < \frac{2-\delta}{n}, \forall m \in \mathbb{N}$.

Choose $f_m \in \Delta_{x_m}$. Since X is an U space, we have $f_m(x_m - \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}) \to 0$. Hence,

$$\begin{aligned}
2^n\|x_m\| = 2^n f_m(x_m) &= \sum_{\theta_j=\pm 1} f_m(x_m - \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}) + 2^{n-1} \sum_{j=1}^{n} f_m(y_{j,m} - z_{j,m}) \\
&\leqslant \sum_{\theta_j=\pm 1} f_m(x_m - \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}) + 2^{n-1} \sum_{j=1}^{n} \|y_{j,m} - z_{j,m}\| \\
&\leqslant \sum_{\theta_j=\pm 1} f_m(x_m - \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta_j)}) + 2^{n-1} n \cdot \frac{2-\delta}{n},
\end{aligned}$$

it means that $2^n \leqslant 2^{n-1}(2-\delta)$, this is contradict to that $\delta > 0$. This complete the proof. ☐

PROPOSITION 5. *For any $\delta > \frac{2}{n}\left(\sqrt{(n+1)C_{NJ}^{(n)}(X)} - 1\right)$, we have $C_{NJ}^{(n)}(2-\delta, X)$ $< n+1$.*

Proof. Suppose that there exists $\delta > \frac{2}{n}\left(\sqrt{(n+1)C_{NJ}^{(n)}(X)} - 1\right)$ such that $C_{NJ}^{(n)}(2-\delta, X) = n+1$, then there exist sequences $\{x_m\}, \{y_{j,m}\}, \{z_{j,m}\} \subset S_X$ such that $\|x_m + \sum_{j=1} \theta_j x_{j,m}^{(\theta)}\| \to n+1$ and $\|y_{j,m} - z_{j,m}\| < 2-\delta, \theta_j = \pm 1, j = 1, 2, \cdots, n, \forall m \in \mathbb{N}$, where $x_{j,m}^{(1)} = y_{j,m}$ and $x_{j,m}^{(-1)} = z_{j,m}$. Hence,

$$\begin{aligned}
\liminf_{m\to\infty} \sum_{\theta_j=\pm 1} \|x_m + \sum_{j=1}^{n} \theta_j y_{j,m}\| \geqslant \lim_{m\to\infty} &\sum_{\theta_j=\pm 1} \|x_m + \sum_{j=1}^{n} \theta_j x_{j,m}^{(\theta)}\| \\
&- 2^{n-1} \limsup_{n\to\infty} \sum_{j=1}^{n} \|y_{j,m} - z_{j,m}\| \\
\geqslant & 2^n(n+1) - 2^{n-1} n(2-\delta) = 2^{n-1}(2+n\delta).
\end{aligned}$$

By the definition of $C_{NJ}^{(n)}(X)$, we have

$$\begin{aligned}
C_{NJ}^{(n)}(X) &\geqslant \liminf_{n\to\infty} \frac{\sum_{\theta_j=\pm 1} \|x_m + \sum_{j=1}^{n} \theta_j y_{j,m}\|^2}{2^n\|x_m\|^2 + 2^n \sum_{j=1}^{n} \|y_{j,m}\|^2} \\
&\geqslant \frac{2^{2n-2}(2+n\delta)^2/2^n}{2^n(n+1)} = \frac{(2+n\delta)^2}{4(n+1)}.
\end{aligned}$$

This is contradict to $\delta > \frac{2}{n}\left(\sqrt{(n+1)C_{NJ}^{(n)}(X)}-1\right)$. Hence, we have $C_{NJ}^{(n)}(2-\delta,X) < n+1$. This complete the proof. $\square$

4. The relationship with the new constant and the weakly convergent sequence coefficient

THEOREM 2. *If X does not have the Schur property, then for $0 \leqslant a \leqslant 1$, we have the following inequalities*

$$[\mathrm{WCS}(X)]^2 \geqslant \frac{1+(1+a)\max\{\frac{((n+1)a+2)}{2(R(1,X)+a)^2}, \frac{na+1}{4}\}}{C_{NJ}^{(n)}(a,X)}, \text{ if } n \text{ is an odd number}$$

and

$$[\mathrm{WCS}(X)]^2 \geqslant \frac{1+\frac{na(1+a)}{\min\{2(R(1,X)+a)^2,4\}}}{C_{NJ}^{(n)}(a,X)}, \text{ if } n \text{ is an even number.}$$

Proof. If $C_{NJ}^{(n)}(a,X) = n+1$, the inequality is evident. If $C_{NJ}^{(n)} < n+1$, then X is uniformly non-square, so it is super-reflexive. Suppose that $x_m \subset S_X$ is a weak zero sequence, with out lose generality, we assume that $d = \lim_{k\neq m}\|x_m - x_k\|$ exist. Choose $\{x_m^*\} \subset S_{X^*}$ satisfying $x_n^*(x_n) = 1$. Notice that X is reflexive, extract the subsequence, we can assume that $x_m^* \xrightarrow{w^*} x^*$.

For given any $0 < \varepsilon < 1$, choosing sufficiently large positive integer N such that for any $k > N$,

$$|x^*(X_N)| < \frac{\varepsilon}{2} \text{ and } d-\varepsilon < \|x_k - x_N\| < d+\varepsilon.$$

According to the definition of $R(1,X)$, we have

$$\liminf_{k\to\infty}\left\|\frac{x_k}{d+\varepsilon}+x_N\right\| \leqslant R(1,X).$$

Then, there exists positive integer $M > N$ such that

1. $x_N^*(x_M) < \varepsilon$;
2. $|(x_M^* - x^*)(x_N)| < \frac{\varepsilon}{2}$;
3. $\left\|\frac{x_M}{d+\varepsilon}+x_N\right\| \leqslant R(1,X)+\varepsilon$.

Hence, we obtain that $|x_M^*(x_N)| < |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \varepsilon$.

We denote $R(1,X)$ by R for shotten, let

$$x = \frac{x_M - x_N}{d+\varepsilon},$$
$$y_j = \frac{(1+a)((1+a)x_M + (d+\varepsilon)x_N)}{(d+\varepsilon)(R+a+\varepsilon)^2}$$

and

$$z_j = \frac{(1+a)(x_M + (d+\varepsilon+a)x_N)}{(d+\varepsilon)(R+a+\varepsilon)^2}$$

for $j = 1,2,\cdots,n$.

It is easy to see that $x \in B_X$, $\|y_j - z_j\| \leqslant a\|x\|$, $\|y_j\| \leqslant \frac{1+a}{R+a+\varepsilon}$ and $\|z_j\| \leqslant \frac{1+a}{R+a+\varepsilon}$ for $j = 1,2,\cdots,n$.

For given $\theta_1,\theta_2,\cdots,\theta_n \in \{1,-1\}$, we denote y_i by $x_i^{(\theta_i)}$ if $\theta_i = 1$, and denote z_i by $x_i^{(\theta_i)}$ if $\theta_i = -1$. Let k be the number of θ_j's such that $\theta_j = 1$ and $l = n-k$. Then, we have

$$\begin{aligned}
&(d+\varepsilon)\left\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\right\| \\
=&\left\|\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(k+ka-l)\right)x_M\right. \\
&\left.-\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(ld+l\varepsilon+la-kd-k\varepsilon)\right)x_N\right\|,
\end{aligned} \tag{2}$$

Case 1: n is an odd number. If $k \geqslant l$, then we have

$$\begin{aligned}
&(d+\varepsilon)\left\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\right\| \\
\geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(k+ka-l)\right)x_M^*(x_M) \\
&-\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(ld+l\varepsilon+la-kd-k\varepsilon)\right)x_M^*(x_N) \\
\geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(1+\frac{n+1}{2}a\right)\right)(1-\varepsilon),
\end{aligned} \tag{3}$$

if $k < l$, then we have

$$\begin{aligned}
&(d+\varepsilon)\left\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\right\| \\
\geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(k+ka-l)\right)(-x_N)^*(x_M) \\
&-\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(ld+l\varepsilon+la-kd-k\varepsilon)\right)(-x_M)^*(x_N) \\
\geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(d+\varepsilon+\frac{n+1}{2}a\right)\right)(1-\varepsilon) \\
\geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(1+\frac{n+1}{2}a\right)\right)(1-\varepsilon).
\end{aligned} \tag{4}$$

Case 2: n is an even number. If $k \geqslant l$, then we have

$$\begin{aligned}(d+\varepsilon)&\left\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\right\|\\ \geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(k+ka-l)\right)x_M^*(x_M)\\ &-\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(ld+l\varepsilon+la-kd-k\varepsilon)\right)x_M^*(x_N)\\ \geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(\frac{n}{2}a\right)\right)(1-\varepsilon),\end{aligned}\tag{5}$$

if $k \leqslant l$, then we have

$$\begin{aligned}(d+\varepsilon)&\left\|x+\sum_{j=1}^{n}\theta_j x_j^{(\theta_j)}\right\|\\ \geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(k+ka-l)\right)(-x_N)^*(x_M)\\ &-\left(1+\frac{1+a}{(R+a+\varepsilon)^2}(ld+l\varepsilon+la-kd-k\varepsilon)\right)(-x_M)^*(x_N)\\ \geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(\frac{n}{2}a\right)\right)(1-\varepsilon)\\ \geqslant&\left(1+\frac{1+a}{(R+a+\varepsilon)^2}\left(\frac{n}{2}a\right)\right)(1-\varepsilon).\end{aligned}\tag{6}$$

By the definition of $C_{NJ}^{(n)}(a,X)$, we know that, if n is an odd number, then

$$C_{NJ}^{(n)}(a,X) \geqslant \left(\frac{1-\varepsilon}{d+\varepsilon}\right)^2\left(1+\frac{(1+a)(2+na+a)}{2(R+a+\varepsilon)^2}\right),$$

and if n is an even number, then

$$C_{NJ}^{(n)}(a,X) \geqslant \left(\frac{1-\varepsilon}{d+\varepsilon}\right)^2\left(1+\frac{na(1+a)}{2(R+a+\varepsilon)^2}\right).$$

By the arbitrariness of $\{x_m\}$ and ε, we obtain that if n is an odd number, then

$$[\mathrm{WCS}(X)]^2 C_{NJ}^{(n)}(a,X) \geqslant 1+\frac{(1+a)(2+na)}{(R+a)^2},$$

and if n is an even number, then

$$[\mathrm{WCS}(X)]^2 C_{NJ}^{(n)}(a,X) \geqslant 1+\frac{na(1+a)}{2(R+a)^2}.$$

Furthermore, let

$$x = \frac{x_M - x_N}{d+\varepsilon},\ y_j' = \frac{(1+a)((1+a)x_N + (1-a)x_M)}{4(d+\varepsilon)}$$

and

$$z_j' = \frac{(1+a)((1-a)x_N + (1+a)x_M)}{4(d+\varepsilon)}$$

for $j = 1, 2, \cdots, n$. It is easy to see that $x \in B_X$, $\|y_j' - z_j'\| \leqslant a\|x\|$, $\|y_j'\| \leqslant \frac{1+a}{2(d+\varepsilon)} \leqslant \frac{1+a}{2}$ and $\|z_j'\| \leqslant \frac{1+a}{2(d+\varepsilon)} \leqslant \frac{1+a}{2}$ for $j = 1, 2, \cdots, n$. For given $\lambda_1, \lambda_2, \ldots, \lambda_n \in \{1, -1\}$, we denote y_i' by $x_i^{(\lambda_i)}$ if $\theta_i = 1$, and denote z_i' by $x_i^{(\lambda_i)}$ if $\theta_i = -1$. Let k' be the number of λ_j's such that $\lambda_j = 1$ and $l' = n - k'$. Then, we have

$$\begin{aligned}
&(d+\varepsilon)\left\| x + \sum_{j=1}^{n} \lambda_j x_j^{(\lambda_j)} \right\| \\
=&\left\| \left(1 + \frac{1+a}{4}(k + na - l)\right) x_N - \left(1 + \frac{1+a}{4}(na + l - k)\right) x_M \right\|.
\end{aligned} \tag{7}$$

Case 1: n is an odd number. If $k' \geqslant l'$, then we have

$$\begin{aligned}
&(d+\varepsilon)\left\| x + \sum_{j=1}^{n} \lambda_j x_j^{(\lambda_j)} \right\| \\
\geqslant&\left(1 + \frac{1+a}{4}(na + k - l)\right) x_N^*(x_N) - \left(1 + \frac{1+a}{4}(na + l - k)\right) x_N^*(x_M) \\
\geqslant&\left(1 + \frac{(1+a)(na+1)}{4}\right)(1-\varepsilon),
\end{aligned} \tag{8}$$

and if $k' \leqslant l'$, then we have

$$\begin{aligned}
&(d+\varepsilon)\left\| x + \sum_{j=1}^{n} \lambda_j x_j^{(\lambda_j)} \right\| \\
\geqslant&\left(1 + \frac{1+a}{4}(k + na - l)\right)(-x_M)^*(x_N) - \left(1 + \frac{1+a}{4}(l - k + na)\right)(-x_M)^*(x_M) \\
\geqslant&\left(1 + \frac{(1+a)(na+1)}{4}\right)(1-\varepsilon).
\end{aligned} \tag{9}$$

Case 2: n is an even number. If $k' \geqslant l'$, then we have

$$\begin{aligned}
&(d+\varepsilon)\left\| x + \sum_{j=1}^{n} \lambda_j x_j^{(\lambda_j)} \right\| \\
\geqslant&\left(1 + \frac{1+a}{4}(na + k - l)\right) x_N^*(x_N) - \left(1 + \frac{1+a}{4}(na + l - k)\right) x_N^*(x_M) \\
\geqslant&\left(1 + \frac{na(1+a)}{4}\right)(1-\varepsilon),
\end{aligned} \tag{10}$$

if $k' \leqslant l'$, then we have

$$\begin{aligned} &(d+\varepsilon)\left\|x+\sum_{j=1}^{n}\lambda_j x_j^{(\lambda_j)}\right\| \\ \geqslant&\left(1+\frac{1+a}{4}(k+na-l)\right)(-x_M)^*(x_N)-\left(1+\frac{1+a}{4}(l-k+na)\right)(-x_M)^*(x_M) \qquad (11)\\ \geqslant&\left(1+\frac{na(1+a)}{4}\right)(1-\varepsilon). \end{aligned}$$

By the definition of $C_{NJ}^{(n)}(a,X)$, we know that if n is an odd number, then

$$C_{NJ}^{(n)}(a,X) \geqslant \left(\frac{1-\varepsilon}{d+\varepsilon}\right)^2\left(1+\frac{(1+a)(1+na)}{4}\right),$$

and if n is an even number, then

$$C_{NJ}^{(n)}(a,X) \geqslant \left(\frac{1-\varepsilon}{d+\varepsilon}\right)^2\left(1+\frac{na(1+a)}{4}\right).$$

By the arbitrariness of $\{x_m\}$ and ε, we obtain that if n is an odd number, then

$$[\mathrm{WCS}(X)]^2 C_{NJ}^{(n)}(a,X) \geqslant 1+\frac{(1+a)(1+na)}{4}, \qquad (12)$$

add if n is an odd number, then

$$[\mathrm{WCS}(X)]^2 C_{NJ}^{(n)}(a,X) \geqslant 1+\frac{na(1+a)}{4}. \qquad (13)$$

Combining the inequalities (12) and (13), we get the conclusion. □

Since $\mathrm{WCS}(X) > 1$, X has a uniformly normal structure, we have the following corollary.

COROLLARY 1. *If there exists an* $a \in [0,1]$ *such that* $C_{NJ}^{(n)}(a,X) < 1+(1+a)\max\{\frac{((n+1)a+2)}{2(R(1,X)+a)^2}, \frac{na+1}{4}\}$ *for* n *is an odd number, and* $C_{NJ}^{a}(a,X) < 1+\frac{na(1+a)}{\min\{2(R(1,X)+a)^2,4\}}$ *for* n *is an even number, then* X *has normal structure.*

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Xi Wang
School of Mathematics and Statistics
Changchun University of Technology
Changchun, 130012, P. R. China
e-mail: `wangxi@ccut.edu.cn`

Xianfeng Sun
School of Mathematics and Statistics
Changchun University of Technology
Changchun, 130012, P. R. China
e-mail: `sunxianfeng1999@163.com`

Chao Xia
School of Mathematics and Statistics
Changchun University of Technology
Changchun, 130012, P. R. China
e-mail: `xiachao@ccut.edu.cn`