Mathematical
Inequalities
& Applications
Volume 29, Number 3 (2026), 219–241 doi:10.7153/mia-2026-29-13

INTEGRAL REPRESENTATION AND FUNCTIONAL INEQUALITIES INVOLVING GENERALIZED POLYLOGARITHM

DEEPSHIKHA MISHRA AND A. SWAMINATHAN*

(*Communicated by S. Varošanec*)

Abstract. In this manuscript, integral representation of generalized polylogarithm is obtained through two different methods. The first one involves the Dirichlet series and its Laplace representation resulting in a single integral representation, related to the Lerch transcendent function. The second approach utilizes the Hadamard convolution, which leads to a double integral representation. As consequences, complete monotonicity, Turán inequality, convexity, and bounds of the generalized polylogarithm are deduced. Finally, an alternative proof of an existing integral representation of the generalized polylogarithm using the Hadamard convolution is provided.

1. Introduction

The formulation of numerous problems in physics and mathematics involves the theory of Green's function. In some formulations, Green's function expands as a series of Clausen functions, and the solution involves multiple differentiations to turn these functions into logarithmic expressions, and eventual reintegration. To address this particular issue, Leibniz [18] introduced the concept of polylogarithm.

DEFINITION 1. [19] The polylogarithm function or Jonquire's function, denoted as $\mathrm{Li}_p(z)$ is defined by the infinite sum

$$\mathrm{Li}_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^p}, \tag{1}$$

for all complex numbers z satisfying $|z| < 1$ and $p \in \mathbb{C}$. However, for $|z| = 1$, $p \in \mathbb{C}$ satisfies $\Re(p) > 1$.

When $p = 1$, (1) simplifies to $\mathrm{Li}_1(z) = -\log(1-z)$, the well known logarithm function. For $p = 2,3$, (1) reduces to $\mathrm{Li}_2(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2}$ and $\mathrm{Li}_3(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^3}$, which are the well-known dilogarithm and trilogarithm functions, respectively. Lewin studied the behavior of dilogarithm and trilogarithm functions and provided a detailed theory of the polylogarithm in the monograph *Polylogarithms and Associated Functions* [19].

Mathematics subject classification (2020): 33B30, 26D07, 30E20.
Keywords and phrases: Polylogarithm, generalized hypergeomteric functions, integral representations, complete monotonicity, series representations, Hadamard convolution.
* Corresponding author.

Paper MIA-29-13

The theory of polylogarithm encompasses numerous properties, such as integral representations, functional relations, series expansions, linear and quadratic transformations, and numerical values for specific arguments, for example, see [2, 3, 4, 5]. Integral representations of special functions have always been a topic of great interest because they provide alternative ways of expressing these functions and can be used to establish relationships between different special functions. Our primary objective in this discussion is to concentrate on the integral representation of the polylogarithm function. Several integral representations of polylogarithms are already available in the literature. Some fundamental integral forms are

$$\mathrm{Li}_p(z) = \frac{z}{\Gamma(p)} \int_0^\infty \frac{t^{p-1}}{e^t - z} dt, \quad |\arg(1-z)| < \pi, \ \Re(p) > 0,$$

and

$$\begin{aligned}\mathrm{Li}_p(z) &= \frac{z}{\Gamma(p)} \int_1^\infty \frac{\log^{p-1} t}{t(t-z)} dt \\ &= \frac{z}{\Gamma(p)} \int_0^1 \frac{\log^{p-1}(1/t)}{1-zt} dt, \quad |\arg(1-z)| < \pi, \ \Re(p) > 0,\end{aligned}$$

given in [28, 29]. To know more about polylogarithms, we refer to [1, 19] and references therein. The polylogarithm function and its integral representations are closely related to the Ramanujan's integrals. For instance, the dilogarithm function, defined as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt,$$

can be regarded as a form of a Ramanujan-type integral [21]. For further insights into polylogarithms and Ramanujan integrals, we refer to [22, 32] and references therein.

As further development, numerous authors have studied theory of polylogarithms and introduced some of its generalizations. In [16], the first generalization of the polylogarithm function, referred to as the Nielsen generalized polylogarithm was introduced. Among this and various other generalizations, this manuscript focuses on a specific generalization of the polylogarithm, as outlined below.

DEFINITION 2. [27] The generalized polylogarithm function, denoted as $\Phi_{p,q}(a,b;z)$, is defined by the following normalized expression

$$\Phi_{p,q}(a,b;z) = \sum_{k=1}^\infty \frac{(1+a)^p(1+b)^q}{(k+a)^p(k+b)^q} z^k, \quad |z| < 1, \tag{2}$$

where p and q represent complex numbers with $\Re(p) > 0$ and $\Re(q) > 0$, while $a,b > 0$.

This generalization given by (2) includes many well-known special functions as particular cases that are detailed below.

1. If $a=b$, $p+q=s$ then,

$$\Phi_{p,q}(a,b;z)=(1+a)^s\sum_{k=1}^{\infty}\frac{z^k}{(k+a)^s},$$

where the function

$$\Psi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s},\quad |z|<1, \tag{3}$$

with $a>0$, $s\in\mathbb{C}$ such that $\Re(s)>0$ is known as the *Lerch transcendent function*, which serves as the generalization of both the *Hurwitz zeta function* and the *Riemann zeta function*. The Lerch transcendent function (3) can also be expressed as the following integral representation [27]

$$\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}=\frac{1}{a^s}+\frac{z}{\Gamma(s)}\int_0^1\left(\log\frac{1}{t}\right)^{s-1}\frac{t^a}{1-zt}\,dt,\quad |z|<1, \tag{4}$$

with $a>0$, $s\in\mathbb{C}$ such that $\Re(s)>0$.

2. If $a=b=0$, $p+q=r\in\mathbb{C}$, $\Re(r)>0$ then $\Phi_{p,q}(a,b;z)=\mathrm{Li}_r(z)$ which is again the polylogarithm function, given by (1).

The generalized polylogarithm (2) is associated with the generalized hypergeometric function through the following relation [23]

$$\Phi_{p,q}(a,b;z)=z({}_{p+q+1}F_{p+q}(a_1,a_2,a_3,\ldots,a_{p+q+1};c_1,c_2,c_3,\ldots,c_{p+q})),$$

where p,q are natural numbers,

$$a_i=\begin{cases}1, & \text{if } i=1\\ 1+a, & \text{if } i=2,3,\ldots,p+1\\ 1+b, & \text{if } i=p+2,p+3,\ldots,p+q+1,\end{cases}$$

and

$$c_i=\begin{cases}2+a, & \text{if } i=1,2,3,\ldots,p\\ 2+b, & \text{if } i=p+1,p+2,p+3,\ldots,p+q.\end{cases}$$

Here ${}_pF_q(z)$ denotes the generalized hypergeometric function which is defined [30] as

$${}_pF_q(z)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{5}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k=\begin{cases}1 & \text{if } k=0,\\ a(a+1)(a+2)\cdots(a+k-1) & \text{if } k>0.\end{cases}$$

In (5), any denominator term cannot be zero or a negative integer, and if any numerator parameter is zero, the series terminates. The convergence of (5) depends on p and q such that

1. if $p \leqslant q$, the series converges for all finite $z \in \mathbb{C}$.
2. if $p = q+1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$, absolute convergence occurs if
$$\Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0.$$
3. If $p > q+1$, the series converges only at $z = 0$.

For $p = 2,\ q = 1,\ {}_pF_q(z)$ reduces to the Gauss hypergeometric function, and for $p = q = 1$, it simplifies to the confluent hypergeometric function. More details can be found in [7, 30].

In addition to the aforementioned connection with the generalized hypergeometric function, there are other significant contributions in the field of generalized polylogarithms. In [20], a comprehensive overview of the dilogarithm function and its integral forms was provided, highlighting its significance. Subsequently, the polylogarithm and its integral representations, which are closely related to Clausen functions, were examined in [14]. Additionally, new integral expressions for the Mathieu series and the COM-Poisson normalization constant were introduced in [25, 26]. Inspired by these previous studies, this article primarily centres on expressing the generalized polylogarithm through integral representation.

1.1. Motivation

Integral representations of special functions are often more convenient for analyzing various properties of these functions, including their complete monotonicity, asymptotic behavior, inequalities, and analytic properties such as whether the function belongs to the class of Pick functions, Stieltjes functions, Bernstein functions, and so on. Nowadays, finding integral representations of special functions is gaining significant attention and is being studied extensively. For instance, in [12], the integral representation of the function $\frac{\log(1+rz)}{\log(1+z)}$ is derived, which helps in determining the specific class to which the function belongs. Additionally, properties such as convexity and monotonicity are also explored. Further examples of the applications of integral representations can be found in [6, 11, 12, 17, 24].

In this manuscript, we have derived two distinct integral representations of the generalized polylogarithm; one involving a single integral and the other involving a double integral. A detailed proof can be found in Section 3 and Section 4.

1.2. Organization

The structure of this work is as follows: In Section 2, we begin by introducing the Dirichlet series and its Laplace representation, followed by definitions and theorems related to completely monotone functions and Hadamard convolution. We then define the Lerch transcendent function (3) and provide its one-parameter series representation. In

Section 3, we focus on deriving the integral representation of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ involving only a single integral, as given in Theorem 3. Furthermore, as a special case, we obtain the integral representation of the Lerch transcendent function. Moreover, we establish bounds for the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ and explore the complete monotonicity and Turán inequality for $\psi(p)$ given in (19). We also examine the Turán inequality for the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ and conclude the section. Finally, in Section 4, we derive an integral representation of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ involving a double integral and conclude the section by re-establishing a previously known integral form of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$.

2. Preliminaries

This section provides the preliminary definitions and theorems that we use later to derive the integral form of the generalized polylogarithm. We begin by discussing the Dirichlet series and one of its most significant integral representation.

DEFINITION 3. [15] A series of the form

$$f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-\beta_n\xi},\quad \Re(\xi)>0, \tag{6}$$

is known as a Dirichlet series of type β_n, where β_n is an increasing sequence of real numbers that diverges to infinity, α_n is a sequence of complex numbers, and ξ is a complex variable with $\Re(\xi)>0$.

If $\beta_n=n$, the Dirichlet series (6) simplifies to $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$, $\Re(\xi)>0$, which represents a power series in $e^{-\xi}$. Alternatively, if $\beta_n=\log n$, the series (6) transforms into

$$f(\xi)=\sum_{n=1}^{\infty}\alpha_n n^{-\xi},\quad \Re(\xi)>0,$$

which is known as the usual Dirichlet series. Over time, Dirichlet series gained significant attention when the first attempt [13] was made to construct a systematic theory of the function $f(\xi)$ defined in (6), which became the starting point for further research in related subjects [13, 15]. One of the important results discussed in [13] is the Laplace integral representation for the Dirichlet series (6), which can be written as

$$f(\xi)=\xi\int_0^{\infty}e^{-\xi t}\sum_{n:\beta_n\leqslant t}a_n dt=\xi\int_0^{\infty}e^{-\xi t}\sum_{n=1}^{\lfloor\beta^{-1}(t)\rfloor}\alpha_n dt,\quad \Re(\xi)>0, \tag{7}$$

where $\beta:\mathbb{R}^+\to\mathbb{R}^+$ is a monotonically increasing function with a unique inverse β^{-1}, and $\beta|_{\mathbb{N}}=\beta_n$ with $\lfloor\cdot\rfloor$ denoting the greatest integer function. We use the representation (7) as a fundamental tool to derive the single integral form of the generalized polylogarithm.

DEFINITION 4. [35] A function $f(x)$ is said to be completely monotonic on an interval (a,b) if it is infinitely differentiable, i.e., $f(x) \in C^{\infty}$, and satisfies the condition

$$(-1)^k f^{(k)}(x) \geqslant 0,$$

for every non-negative integer k.

In addition to its definition, complete monotonicity is frequently characterized by the Bernstein theorem, which is presented below.

THEOREM 1. [31] (Bernstein theorem) *Let $f:(0,\infty) \mapsto \mathbb{R}$ be a completely monotonic function. Then it can be expressed as the Laplace transform of a unique measure μ defined on the interval $[0,\infty)$. In other words, for every $x>0$,*

$$f(x) = \mathscr{L}(\mu;x) = \int_{[0,\infty)} e^{-xt} d\mu(t).$$

Conversely, if $\mathscr{L}(\mu;x) < \infty$ for every $x>0$, then the function $x \mapsto \mathscr{L}(\mu;x)$ is completely monotone.

The Bernstein theorem characterizes completely monotonic functions through their representation as Laplace transforms of unique measures. We now turn our attention to the Hadamard convolution, an important concept in complex analysis that relates to the multiplication of holomorphic functions. The Hadamard product facilitates the multiplication of two power series, as detailed in the following theorem.

THEOREM 2. [33] *Let $f(z)$ and $g(z)$ be defined by the power series*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad g(z) = \sum_{n=0}^{\infty} g_n z^n.$$

The Hadamard product of the functions $f(z)$ and $g(z)$ is defined as

$$(f*g)(z) = \sum_{n=0}^{\infty} f_n g_n z^n, \tag{8}$$

which is well-defined for all $f,g \in \mathscr{H}(\mathbb{D})$. Here, $\mathscr{H}(\mathbb{D})$ is the set of all series $\sum_{n=0}^{\infty} a_n z^n$ defined on the open unit disc $\mathbb{D}$ with the property $\limsup_{n\to\infty} |a_n|^{1/n} \leqslant 1$.

The Hadamard product (8) has the integral representation [33]

$$(f*g)(z) = \frac{1}{2\pi i}\int_{|w|=r} f\left(\frac{z}{w}\right) g(w)\frac{dw}{w}, \qquad |z|<r<1.$$

This convolution of holomorphic functions is known as the *Hadamard's convolution theorem.*

3. Generalized polylogarithm as a single integral

In this section, we derive the result concerning the integral representation of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$. The simplicity of this representation lies in its dependence on a single integral.

THEOREM 3. *Suppose a and b are real numbers with $a,b>1$, and let $p,q\in\mathbb{C}$ with $\Re(p)>0$, $\Re(q)>0$. If p and q are collinear, then the generalized polylogarithm $\Phi_{p,q}(a,b;z)$, as defined in (2), satisfies the integral representation*

$$\Phi_{p,q}(a,b;z)=(1+a)^p(1+b)^q\left(\frac{z}{1-z}+\frac{pz}{z-1}\int_0^\infty e^{-py}z^{j(y)}dy\right),\quad |z|<1, \tag{9}$$

where $\Delta(y)$ is an invertible function given as

$$j(y)=\lfloor\Delta^{-1}(e^y)\rfloor,\quad \Delta(y)=(y+a)(y+b)^{q/p}, \tag{10}$$

and $\lfloor\cdot\rfloor$ denotes the greatest integer function.

Proof. The generalized polylogarithm $\Phi_{p,q}(a,b;z)$, defined in (2), can be expressed as

$$\Phi_{p,q}(a,b;z)=(1+a)^p(1+b)^q\sum_{n=1}^{\infty}z^ne^{-p\log\left((n+a)(n+b)^{q/p}\right)}, \tag{11}$$

which turns out to be a Dirichlet series (6) with $\alpha_n=z^n$, $\beta_n=\log\left((n+a)(n+b)^{q/p}\right)$, and $\xi=p$.

Let $\beta(x)=\log\left((x+a)(x+b)^{q/p}\right)$ for $x\in(0,\infty)$, which gives

$$e^{\beta(x)}=(x+a)(x+b)^{q/p},\quad x\in(0,\infty).$$

Differentiating with respect to x, we obtain

$$e^{\beta(x)}\beta'(x)=\frac{q}{p}(x+a)(x+b)^{(q/p)-1}+(x+b)^{q/p},\quad x\in(0,\infty).$$

From the hypothesis we get $\frac{q}{p}\in(0,\infty)$ and we also have $a,b>1$, it follows that $e^{\beta(x)}\beta'(x)$ is always positive. Moreover, since $e^{\beta(x)}$ is always positive, $\beta'(x)$ must also be positive, which implies that $\beta(x)$ is a strictly increasing function. Consequently, $\beta(x)$ and $e^{\beta(x)}$ are invertible, as any continuous and monotone function has an inverse.

Thus, by using the Laplace representation (7), we can express the generalized polylogarithm (11) as

$$\begin{aligned}\Phi_{p,q}(a,b;z)&=p(1+a)^p(1+b)^q\int_0^\infty e^{-py}\sum_{n:\beta_n\leqslant y}z^n\,dy,\\&=p(1+a)^p(1+b)^p\int_0^\infty e^{-py}\sum_{n:(n+a)(n+b)^{q/p}\leqslant e^y}z^n\,dy.\end{aligned} \tag{12}$$

Solving the inequality constraint $(n+a)(n+b)^{q/p} \leqslant e^y$ leads to

$$\Phi_{p,q}(a,b;z) = p(1+a)^p(1+b)^q \int_0^\infty e^{-py} \sum_{n=1}^{j(y)} z^n \, dy, \tag{13}$$

where $j(y)$ is given in (10).

By summing the given geometric series in (13), we obtain

$$\Phi_{p,q}(a,b;z) = p(1+a)^p(1+b)^q \int_0^\infty e^{-py} \left(\frac{z}{1-z} \left(1 - z^{j(y)}\right) \right) dy.$$

After completing the integration, the desired result is obtained, as shown in (9). $\square$

REMARK 1. In (12), the summation starts at 1 and continues up to a value n that satisfies the inequality $(n+a)(n+b)^{q/p} \leqslant e^y$. Due to the presence of the factor $\frac{q}{p}$, the explicit range of summation cannot be determined directly, and is therefore expressed in terms of $j(y)$, where $j(y)$ is the inverse of the function $\Delta(y)$ defined in (10). However, for certain values of p and q, the explicit form of $j(y)$ can be determined, as shown below.

COROLLARY 1. *For $p = q$, the integral representation* (9) *becomes*

$$\Phi_{p,p}(a,b;z) = (1+a)^p(1+b)^p \left(\frac{z}{1-z} + \frac{pz}{z-1} \int_0^\infty e^{-py} z^{n(y)} dy \right), \quad |z| < 1, \tag{14}$$

where $n(y)$ is as follows

$$n(y) = \left\lfloor \frac{-(a+b) + \sqrt{(a-b)^2 + 4e^y}}{2} \right\rfloor. \tag{15}$$

Proof. Substituting $p = q$ into (12), we obtain

$$\Phi_{p,p}(a,b;z) = p(1+a)^p(1+b)^p \int_0^\infty e^{-py} \sum_{n:(n+a)(n+b) \leqslant e^y} z^n \, dy. \tag{16}$$

To determine the value of n, we solve the inequality $(n+a)(n+b) \leqslant e^y$. Rewriting it, we have

$$(n+a)(n+b) - e^y \leqslant 0.$$

Define the function $f(x)$ as

$$f(x) = (x+a)(x+b) - e^y, \quad x \in (0, \infty).$$

It is evident that $f(x)$ is an increasing function with zeros at

$$x_1 = \frac{-(a+b) - \sqrt{(a-b)^2 + 4e^y}}{2}, \quad x_2 = \frac{-(a+b) + \sqrt{(a-b)^2 + 4e^y}}{2}.$$

Now, the inequality $f(x) \leqslant 0$ holds only if

$$0 \leqslant x \leqslant \frac{-(a+b)+\sqrt{(a-b)^2+4e^y}}{2}.$$

Restricting x to natural numbers, we get

$$1 \leqslant n \leqslant \left\lfloor \frac{-(a+b)+\sqrt{(a-b)^2+4e^y}}{2} \right\rfloor.$$

Thus, (16) becomes

$$\Phi_{p,p}(a,b;z) = p(1+a)^p(1+b)^p \int_0^\infty e^{-py} \sum_{n=1}^{n(y)} z^n \, dy,$$

where $n(y)$ is defined as in (15).

By summing the geometric series and carrying out the necessary calculations, we obtain the desired result, as shown in (14). □

Another form of (14) can be obtained by substituting $e^y = t$ into (9), as shown below

$$\Phi_{p,p}(a,b;z) = (1+a)^p(1+b)^p \left(\frac{z}{1-z} + \frac{pz}{z-1} \int_1^\infty z^{\left\lfloor \frac{-(a+b)+\sqrt{(a-b)^2+4t}}{2} \right\rfloor} \frac{dt}{t^{p+1}} \right).$$

Now, we consider the case where two conditions are imposed simultaneously: $a = b$ and $p = q$. Under these constraints, the generalized polylogarithm reduces to the Lerch transcendent function $\psi(z,p,a)$, as defined in (3). Consequently, under these same conditions, the integral representation of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ also reduces to the integral form of the Lerch transcendent function, as shown in the following corollary.

COROLLARY 2. *For $a = b$ and $p = q$, the integral representation of $\Phi_{p,q}(a,b;z)$ given in (9) simplifies to the following form*

$$\sum_{n=0}^\infty \frac{z^n}{(n+a)^p} = \frac{1}{a^p} + \frac{z}{1-z} + \frac{pz}{z-1} \int_0^\infty e^{-p\tau} z^{\lfloor e^\tau - a \rfloor} d\tau, \quad |z| < 1. \tag{17}$$

Proof. Putting $a = b$ in (14), we get

$$\Phi_{p,p}(a,a;z) = (1+a)^{2p} \left(\frac{z}{1-z} + \frac{pz}{z-1} \int_0^\infty e^{-py} z^{\lfloor e^{y/2} - a \rfloor} dy \right), \tag{18}$$

From (2), by substituting the value of $\Phi_{p,p}(a,a;z)$ into (18), we get

$$\sum_{n=1}^\infty \frac{z^n}{(n+a)^{2p}} = \frac{z}{1-z} + \frac{pz}{z-1} \int_0^\infty e^{-py} z^{\lfloor e^{y/2} - a \rfloor} dy.$$

$$\Longrightarrow \sum_{n=1}^\infty \frac{z^n}{(n+a)^p} = \frac{z}{1-z} + \frac{pz}{2(z-1)} \int_0^\infty e^{-py/2} z^{\lfloor e^{y/2} - a \rfloor} dy.$$

Adding $\frac{1}{a^p}$ to both sides and substituting $\frac{y}{2} = \tau$, we obtain (17). $\square$

We have obtained the integral form of the generalized polylogarithm as well as the Lerch transcendent function. One common feature in both integral forms is that they both involve integrals that resemble the Laplace transform of some measure. This brings us to the motivation of discussing the complete monotonicity, as Bernstein's theorem states that a completely monotone function is the Laplace transform of a unique measure. In the next result, we will discuss the complete monotonicity of the function involving the generalized polylogarithm.

COROLLARY 3. *Let p,q be positive real numbers, and $a,b > 1$. Also, let $x \in (0,1)$ be a real number. Define the function*

$$\psi(p) = \left(\frac{\Phi_{p,q}(a,b;x)}{(1+a)^p(1+b)^q} - \frac{z}{1-z} \right) \frac{x-1}{px}, \tag{19}$$

where $\Phi_{p,q}(a,b;x)$ is the generalized polylogarithm defined in (2). Then, the function $\psi(p)$ is completely monotonic with respect to p on $(0,\infty)$.

Proof. From (9), we observe that

$$\int_0^\infty e^{-py} x^{j(y)} dy = \left(\frac{\Phi_{p,q}(a,b;x)}{(1+a)^p(1+b)^q} - \frac{x}{1-x} \right) \frac{x-1}{px} \tag{20}$$

where $j(y)$ is the same as in (10). By comparing (20) and (19), we get

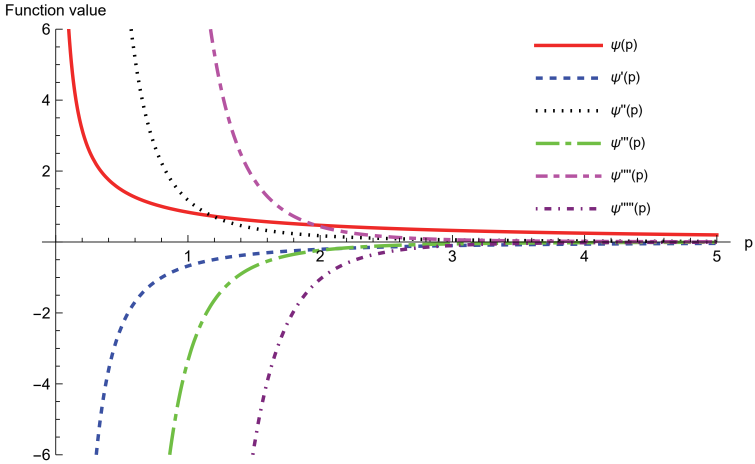
$$\psi(p) = \int_0^\infty e^{-py} x^{j(y)} dy. \tag{21}$$

Thus, proving the complete monotonicity of $\psi(p)$ is equivalent to demonstrating the complete monotonicity of the integral given in (21). Since the integral (21) is a Laplace transform of a positive measure, as $j(y)$ is always positive, Bernstein's theorem (see Theorem 1) ensures its complete monotonicity with respect to p, implying that $\psi(p)$ is also completely monotonic. $\square$

The complete monotonicity of the function $\psi(p)$ can also be seen graphically, as illustrated below.

The notation $\psi^{(n)}(p)$ denotes the n-th derivative of $\psi(p)$ with respect to p, where n is a non-negative integer. From the graph, it is evident that $\psi(p)$ meets the criteria for complete monotonicity as outlined in Definition 4. In this graphical representation, the value of $\psi^{(n)}(p)$ is computed up to the fifth order (i.e. $n = 5$), exhibiting an alternating sign pattern starting with a positive sign. However, the procedure can be extended to higher orders in a similar manner.

REMARK 2. The function $\psi(p)$, as defined in (19), is completely monotone on $(0,\infty)$. A significant subclass of completely monotone functions is the class of Stieltjes functions, leading to the question of whether $\psi(p)$ belongs to this subclass. We found that the function $\psi(p)$ is completely monotone, but it is not a Stieltjes function because the density function present in the integral given in (21) is not completely monotone. It is not even differentiable and is only of bounded variation.



Graph of $\psi^{(n)}(p)$, $n = 0,1,2,3,4,5$ *for* $a = 1.4,\ b = 1.2,\ q = 0.75,\ x = 0.5$

Figure 1: *Complete monotonicity of* $\psi(p)$

Note that the complete monotonicity of $\psi(p)$ implies its non-negativity. By using this property, we can derive the bounds of the generalized polylogarithm, which are discussed in the following corollary.

COROLLARY 4. *For positive real numbers a,b,p and q with $a,b > 1$, the bounds for the generalized polylogarithm $\Phi_{p,q}(a,b;x)$, as defined in (2), are given by*

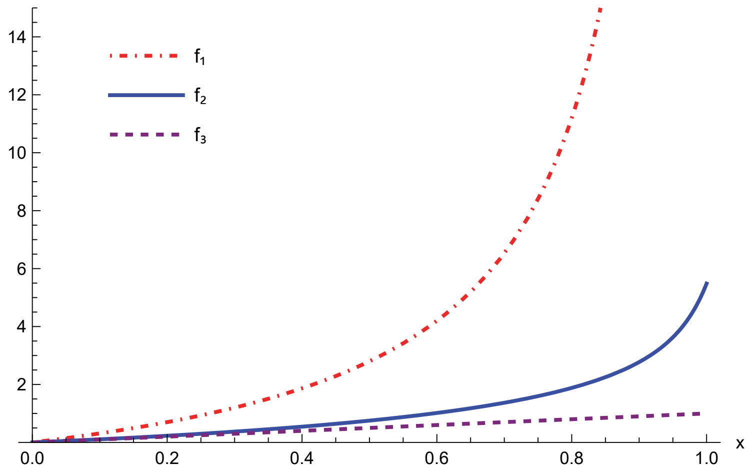
$$x \leqslant \Phi_{p,q}(a,b;x) \leqslant (1+a)^p(1+b)^q \frac{x}{1-x}, \quad x \in (0,1). \tag{22}$$

Proof. By Corollary 3, the non-negativity of $\psi(p)$ with respect to p implies the bounds stated in (22). $\square$

We now address the graphical representation of the bounds in (22).

It is evident from the Figure 2 that the red (dot-dashed) curve, representing $f_1 = (1+a)^p(1+b)^q \frac{x}{1-x}$, provides an upper bound for the blue (thick) curve, which corresponds to the generalized polylogarithm $f_2 = \Phi_{p,q}(a,b;x)$ for the given parameter values. Here the upper bound of $\Phi_{p,q}(a,b;x)$ depends on the specific values of the parameters a, b, p, and q. As these parameters vary, the upper bound adjusts accordingly, indicating that the bound is not uniform across different parameter settings. In contrast, the lower bound of $\Phi_{p,q}(a,b;x)$, represented by the purple (dashed) curve, is x which is independent of the parameters, indicating that it remains uniform regardless of the values of a, b, p, and q.

Note that a completely monotonic function is always log-convex, which is closely related to the Turán inequality, so the following result addresses the Turán-type inequalities for $\psi(p)$ as defined in (19). These type inequalities primarily find applications



Graph of $f_1=(1+a)^p(1+b)^q\frac{x}{1-x}$, $f_2=\Phi_{p,q}(a,b;x)$ *and* $f_3=x$ *for* $a=1.4,\ b=1.2,$ $p=0.5,\ q=0.75$

Figure 2: *Bounds for generalized polylogarithm* $\Phi_{p,q}(a,b;x)$.

in the theory of special functions, particularly in hypergeometric and polylogarithmic functions. Turán inequality were initially introduced for Legendre polynomials in [34]. Since then, this inequality has been extensively studied and developed in various directions. For more literature on Turán inequality for various special functions, we refer readers to [8, 9, 10] and the references therein.

COROLLARY 5. *Let the function* $\psi(p)$ *be defined as in* (19). *For positive real numbers* x,q,a,b *with* $x\in(0,1)$ *and* $a,b>1$ *the Turán-type inequality holds for* $\psi(p)$ *as*

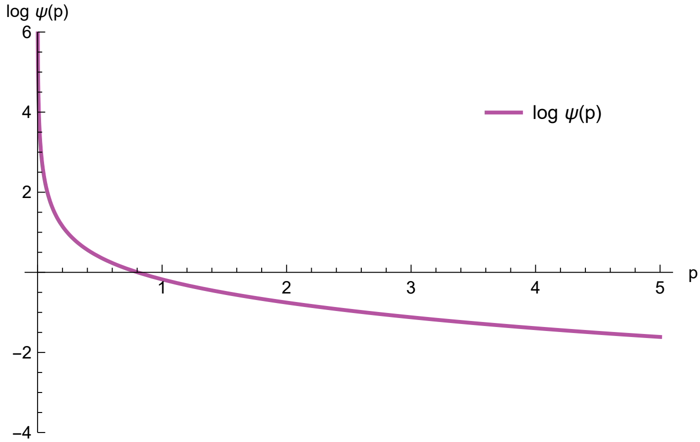
$$\psi(p)\psi(p+2)-\psi^2(p+1)\geqslant 0, \qquad p\in(0,\infty). \tag{23}$$

Proof. The complete monotonicity of a function implies its logarithmic convexity (see [35]). Hence from Corollary 3, it follows that the function $\psi(p)$ is log-convex. Therefore, we have the inequalities

$$\log(\psi(\alpha p_1+(1-\alpha)p_2))\leqslant\alpha\log\psi(p_1)+(1-\alpha)\log\psi(p_2), \quad \alpha\in(0,1),$$
$$\psi(\alpha p_1+(1-\alpha)p_2)\leqslant\psi^{\alpha}(p_1)\psi^{(1-\alpha)}(p_2), \quad \alpha\in(0,1).$$

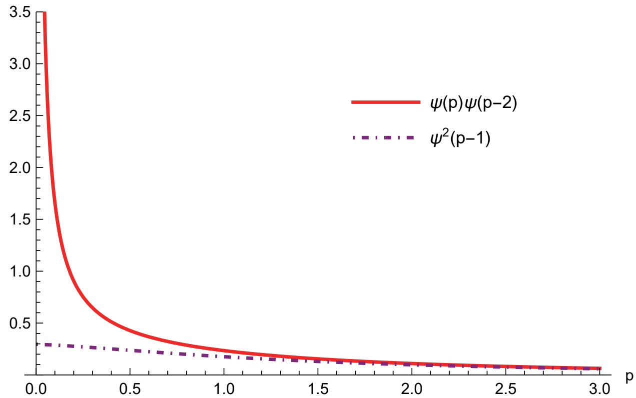
Now, if we set $\alpha=1/2$ and $p_2=p_1+2$, we obtain the inequality (23). □

The log-convexity of $\psi(p)$ and the Turán inequality of $\psi(p)$ are graphically demonstrated in the figures below. Figure 3 illustrates the log-convexity of $\psi(p)$, while Figure 4 depicts the Turán inequality satisfied by $\psi(p)$.



Graph of $\log(\psi(p))$ *for* $a = 1.4,\ b = 1.2,\ x = 0.5,\ q = 0.75.$

Figure 3: *Convexity of* $\log\psi(p)$.



Graph of $\psi(p)\psi(p+2)$ *and* $\psi^2(p+1)$ *for* $a = 1.4,\ b = 1.2,\ x = 0.5,\ q = 0.75.$

Figure 4: *Turán inequality for* $\psi(p)$.

In Figure 3, the graph of $\log\psi(p)$ clearly illustrates its convexity with respect to p over the interval $(0,\infty)$. Similarly, in Figure 4, Turán's inequality is graphically demonstrated, where the red (thick) curve, representing $\psi(p)\psi(p+2)$, consistently lies above the purple (dot-dashed) curve representing $\psi^2(p+1)$, as stated in (23).

The expressions for $\psi(p)$ in (19) involve the generalized polylogarithm $\Phi_{p,q}(a,b;x)$. Also, we prove that $\psi(p)$ satisfy Turán's inequality. In line with this result, it is natural to ask whether the generalized polylogarithm $\Phi_{p,q}(a,b;x)$ satisfies Turán's inequality. We affirmatively address this question, and demonstrate the result in Theorem 4.

THEOREM 4. *Consider the generalized polylogarithm* $\Phi_{p,q}(a,b;x)$ *as defined in* (2). *For* $a,b,p,q>0$, *and* $x\in(0,1)$, *the following inequalities hold*

$$\Phi_{p,q}(a,b;x)\Phi_{p,q+2}(a,b;x)-\Phi^2_{p,q+1}(a,b;x)\geqslant 0, \tag{24}$$

and

$$\Phi_{p,q}(a,b;x)\Phi_{p+2,q}(a,b;x)-\Phi^2_{p+1,q}(a,b;x)\geqslant 0. \tag{25}$$

Proof. We begin with the expression for the generalized polylogarithm $\Phi_{p,q}(a,b;x)$ as defined in (2), which can be rewritten as

$$\Phi_{p,q}(a,b;x)=z\widetilde{\Phi}_{p,q}(a,b;x), \tag{26}$$

where

$$\widetilde{\Phi}_{p,q}(a,b;x)=\sum_{k=0}^{\infty}\frac{(1+a)^p(1+b)^q}{(k+1+a)^p(k+1+b)^q}x^k.$$

Now, consider the product

$$\begin{aligned}\widetilde{\Phi}_{p,q}&(a,b;x)\widetilde{\Phi}_{p,q+2}(a,b;x)\\ &=\sum_{k=0}^{\infty}\frac{(1+a)^p(1+b)^q}{(k+1+a)^p(k+1+b)^q}x^k\sum_{k=0}^{\infty}\frac{(1+a)^p(1+b)^{q+2}}{(k+1+a)^p(k+1+b)^{q+2}}x^k.\end{aligned}$$

By applying the Cauchy product formula, we obtain the right side of the above expression as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+a)^{2p}(1+b)^{2q+2}}{(k+1+a)^p(n-k+1+a)^p(k+1+b)^q(n-k+1+b)^{q+2}}x^n,$$

and

$$\begin{aligned}\widetilde{\Phi}^2_{p,q+1}&(a,b;x)\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(1+a)^{2p}(1+b)^{2q+2}}{(k+1+a)^p(n-k+1+a)^p(k+1+b)^{q+1}(n-k+1+b)^{q+1}}x^n.\end{aligned}$$

Thus, we can write

$$\widetilde{\Phi}_{p,q}\widetilde{\Phi}_{p,q+2}-\widetilde{\Phi}^2_{p,q+1}=(1+a)^{2p}(1+b)^{2q+2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}f(n,k)x^n, \tag{27}$$

where

$$f(n,k)=\frac{2k-n}{(k+1+a)^p(n-k+1+a)^p(k+1+b)^{q+1}(n-k+1+b)^{q+2}}. \tag{28}$$

We now analyze the finite sum $\sum_{k=0}^{n}f(n,k)$.

Case 1. If n is even.

In this case, we can express $\sum_{k=0}^{n} f(n,k)$ as follows

$$\begin{aligned}\sum_{k=0}^{n} f(n,k) &= \sum_{k=0}^{n/2-1} f(n,k) + \sum_{k=n/2+1}^{n} f(n,k) + f(n,n/2)\\ &= \sum_{k=0}^{n/2-1} f(n,k) + \sum_{k=0}^{n/2-1} f(n,n-k) + f(n,n/2),\end{aligned}$$

Since $f(n,n/2)$ equals to 0 from (28), $\sum_{k=0}^{n} f(n,k)$ becomes

$$\sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(f(n,k) + f(n,n-k)\right), \tag{29}$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function.

Case 2. If n is odd.

If n is odd, then we have

$$\sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{(n-1)/2} f(n,k) + \sum_{k=(n+1)/2}^{n} f(n,k).$$

Combining this into a single summation, we can express it as

$$\sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left(f(n,k) + f(n,n-k)\right), \tag{30}$$

where $\lfloor \cdot \rfloor$ is the greatest integer function. Now we observe that

$$f(n,k) + f(n,n-k) = \frac{(2k-n)^2}{(k+1+a)^p (n-k+a)^p (k+1+b)^{q+2} (n-k+b)^{q+2}} \geqslant 0.$$

From (29) and (30), we obtain

$$\sum_{k=0}^{n} \left(f(n,k) + f(n,n-k)\right) \geqslant 0.$$

Hence, using (27), we conclude that

$$\widetilde{\Phi}_{p,q}(a,b;x)\widetilde{\Phi}_{p,q+2}(a,b;x) - \widetilde{\Phi}^2_{p,q+1}(a,b;x) \geqslant 0. \tag{31}$$

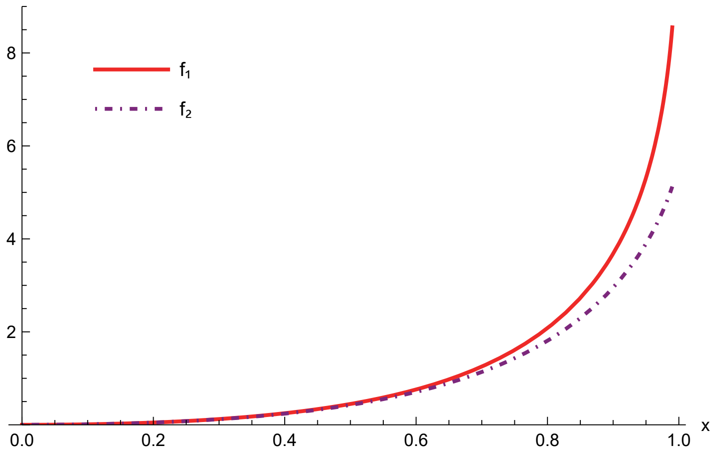
Now from (26), we have the following relationship

$$\begin{aligned}&\Phi_{p,q}(a,b;x)\Phi_{p,q+2}(a,b;x) - \Phi^2_{p,q+1}(a,b;x)\\ &\qquad = x^2 \left(\widetilde{\Phi}_{p,q}(a,b;x)\widetilde{\Phi}_{p,q+2}(a,b;x) - \widetilde{\Phi}^2_{p,q+1}(a,b;x)\right).\end{aligned} \tag{32}$$

By applying (31) in (32), we assert that (24) is valid. Interchanging p and q in (24), symmetry of the expression guarantees (25). $\square$

We now provide a graphical interpretation of the inequality for the generalized polylogarithm $\Phi_{p,q}(a,b;x)$.

In the graphical interpretation, we plot the function $f_1 = \Phi_{p,q}(a,b;x)\Phi_{p,q+2}(a,b;x)$, shown as the red (thick) curve and $f_2 = \Phi^2_{p,q+1}(a,b;x)$ as the purple (dot-dashed) curve. With values $a = 1.4$, $b = 1.2$, $p = 0.5$ and $q = 0.75$, we observe that the red (thick) curve consistently bounds the purple (dot-dashed) curve, illustrating the Turán inequality for $\Phi_{p,q}(a,b;x)$ with respect to q, as established in (24).



Graph of $f_1 = \Phi_{p,q}(a,b;x)\Phi_{p,q+2}(a,b;x)$ *and* $f_2 = \Phi^2_{p,q+1}(a,b;x)$ *for* $p = 0.5$, $q = 0.75$, $a = 1.4$, $b = 1.2$.

Figure 5: *Turán inequality for the* $\Phi_{p,q}(a,b;x)$ *with respect to* q

4. Generalized polylogarithm as a double integral

In this section, we derive the integral representation of the generalized polylogarithm, which involves a double integral. To derive the integral form, we begin by decomposing the series of the generalized polylogarithm into the series of Lerch transcendent function using the Hadamard convolution. Subsequently, by applying the integral form of the Lerch transcendent function and modifying the corresponding complex integrals, we obtain the required result. We conclude the section by demonstrating the usefulness of the Hadamard convolution, presenting another integral form of the generalized polylogarithm.

THEOREM 5. *Let* p,q *be complex numbers having* $\Re(p) > 0$, $\Re(q) > 0$, *and* a,b *be any real numbers such that* $a,b > 1$. *The generalized polylogarithm function*

$\Phi_{p,q}(a,b;z)$, *defined as in* (2), *admits the integral representation*

$$\Phi_{p,q}(a,b;z) = (1+a)^p(1+b)^q \left(\frac{zq}{\Gamma(p)} \int_0^\infty \int_0^\infty e^{-qu} e^{-(a+1)v} v^{p-1} \frac{1-(ze^{-v})^{\lfloor e^u - b \rfloor}}{1-ze^{-v}} du dv \right) \tag{33}$$

for all $|z| < 1$, *where* $\lfloor \cdot \rfloor$ *denotes the greatest integer function.*

Proof. The generalized polylogarithm (2) can be written as

$$\Phi_{p,q}(a,b;z) = \sum_{k=0}^{\infty} \frac{(1+a)^p(1+b)^q}{(k+a)^p(k+b)^q} z^k - \frac{(1+a)^p(1+b)^q}{a^p b^q}. \tag{34}$$

If we take

$$f_k = \frac{1}{(k+a)^p}, \quad g_k = \frac{1}{(k+b)^q}, \tag{35}$$

then the expression (34) becomes

$$\Phi_{p,q}(a,b;z) = (1+a)^p(1+b)^q \sum_{k=0}^{\infty} f_k g_k z^k - \frac{(1+a)^p(1+b)^q}{a^p b^q}. \tag{36}$$

Now, we define the functions $f(z)$ and $g(z)$ as

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \quad g(z) = \sum_{k=0}^{\infty} g_k z^k, \quad |z| < 1.$$

where f_k and g_k are given in (35).

It is evident that $f(z)$ and $g(z)$ are holomorphic functions, with the property that

$$\limsup_{k\to\infty} |f_k|^{1/k} \leqslant 1 \quad \text{and} \quad \limsup_{k\to\infty} |g_k|^{1/k} \leqslant 1.$$

Thus, by applying Theorem 2 of the Hadamard convolution of $f(z)$ and $g(z)$, we obtain

$$\sum_{k=0}^{\infty} f_k g_k z^k = \frac{1}{2\pi i} \int_{|\omega|=r} \left(\sum_{k=0}^{\infty} \frac{1}{(k+a)^p} \left(\frac{z}{\omega}\right)^k \sum_{k=0}^{\infty} \frac{1}{(k+b)^q} \omega^k \right) \frac{d\omega}{\omega}, \quad |\omega| < r < 1. \tag{37}$$

Using (4) for the summation involving z and (17) for the other summation, in (37), we get

$$\sum_{k=0}^{\infty} f_k g_k z^k = \frac{1}{2\pi i} \int_{|\omega|=r} \left(\frac{1}{a^p} + \frac{z}{\Gamma(p)} \int_0^1 \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{\omega - zt} dt \right)$$
$$\times \left(\frac{1}{b^q} + \frac{\omega}{1-\omega} + \frac{q\omega}{\omega - 1} \int_0^\infty e^{-qu} \omega^{\lfloor e^u - b \rfloor} du \right) \frac{d\omega}{\omega}, \quad |z| < r < 1.$$

By substituting the value of $\sum_{k=0}^{\infty} f_k g_k z^k$ into (36), we obtain

$$\begin{aligned}\Phi_{p,q}(a,b;z) = \frac{(1+a)^p(1+b)^q}{2\pi i}\Bigg[&\int_{|w|=r}\left(\frac{1}{a^pb^q}+\frac{1}{a^p}\frac{w}{1-w}\right)\frac{dw}{w}\\&+\frac{z}{\Gamma(p)}\int_{|w|=r}\left(\frac{1}{b^q}+\frac{w}{1-w}\right)\left(\int_0^1\left(\log\frac{1}{t}\right)^{p-1}\frac{t^a}{w-zt}dt\right)\frac{dw}{w}\\&+\frac{q}{a^p}\int_{|w|=r}\frac{w}{(w-1)}\left(\int_0^\infty e^{-qu}w^{\lfloor e^u-b\rfloor}du\right)\frac{dw}{w}\\&+\frac{zq}{\Gamma(p)}\int_{|w|=r}\left(\frac{w}{w-1}\right)\left(\int_0^\infty e^{-qu}w^{\lfloor e^u-b\rfloor}du\right)\\&\times\left(\int_0^1\left(\log\frac{1}{t}\right)^{p-1}\frac{t^a}{w-zt}dt\right)\frac{dw}{w}\Bigg]-\frac{(1+a)^p(1+b)^q}{a^pb^q}.\end{aligned} \tag{38}$$

Rewriting (38) in terms of I_1, I_2, I_3, and I_4, we get the following expression

$$\Phi_{p,q}(a,b;z) = (1+a)^p(1+b)^q(I_1+I_2+I_3+I_4)-\frac{(1+a)^p(1+b)^q}{a^pb^q}. \tag{39}$$

where the I_n's are defined as follows

$$\begin{aligned}I_1 &= \frac{1}{2\pi i}\int_{|w|=r}\left(\frac{1}{a^pb^q}+\frac{1}{a^p}\frac{w}{1-w}\right)\frac{dw}{w},\\I_2 &= \frac{1}{2\pi i}\int_{|w|=r}\frac{z}{\Gamma(p)}\left(\frac{1}{b^q}+\frac{w}{1-w}\right)\left(\int_0^1\left(\log\frac{1}{t}\right)^{p-1}\frac{t^a}{w-zt}dt\right)\frac{dw}{w},\\I_3 &= \frac{1}{2\pi i}\int_{|w|=r}\frac{qw}{a^p(w-1)}\left(\int_0^\infty e^{-qu}w^{\lfloor e^u-b\rfloor}du\right)\frac{dw}{w}, \quad \text{and}\\I_4 &= \frac{1}{2\pi i}\int_{|w|=r}\left(\frac{zqw}{\Gamma(p)(w-1)}\right)\left(\int_0^\infty e^{-qu}w^{\lfloor e^u-b\rfloor}du\right)\left(\int_0^1\left(\log\frac{1}{t}\right)^{p-1}\frac{t^a}{w-zt}dt\right)\frac{dw}{w}.\end{aligned}$$

Now, we evaluate the integrals I_1, I_2, I_3, and I_4.

Since the complex integral I_1 is defined over a closed curve $|w|=r$, we apply the Cauchy integral formula, which gives us

$$I_1 = \frac{1}{a^pb^q}. \tag{40}$$

The integral I_2 can be written as;

$$I_2 = \frac{z}{\Gamma(p)}\int_0^1 t^a\left(\log\frac{1}{t}\right)^{p-1}\underbrace{\left(\frac{1}{2\pi i}\int_{|w|=r}\left(\frac{1}{b^q(w-zt)}+\frac{w}{(1-w)(w-zt)}\right)\frac{dw}{w}\right)}_{I_5}dt.$$

Again, we use the Cauchy integral formula to evaluate the integral I_5, to obtain $I_5 = \frac{1}{1-zt}$. Therefore, I_2 becomes

$$I_2 = \frac{z}{\Gamma(p)} \int_0^1 \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} dt. \tag{41}$$

In the same manner, we find the integrals I_3 and I_4, which are

$$I_3 = 0, \quad I_4 = \frac{zq}{\Gamma(p)} \int_0^1 \int_0^\infty \left(\log \frac{1}{t}\right)^{p-1} e^{-qu} (zt)^{\lfloor e^u - b \rfloor} \frac{t^a}{zt-1} du dt. \tag{42}$$

Substituting the value of I_1 from (40), I_2 from (41), and I_3 and I_4 from (42) into (39), we obtain the expression

$$\begin{aligned}
&\Phi_{p,q}(a,b;z) \\
=&(1+a)^p(1+b)^q \left(\frac{1}{a^p b^q} + \frac{z}{\Gamma(p)} \int_0^1 \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} dt \right. \\
&\left. + \frac{zq}{\Gamma(p)} \int_0^1 \int_0^\infty \left(\log \frac{1}{t}\right)^{p-1} e^{-qu} (zt)^{\lfloor e^u - b \rfloor} \frac{t^a}{zt-1} du dt \right) - \frac{(1+a)^p(1+b)^q}{a^p b^q}.
\end{aligned}$$

By applying simple computations to (43), we arrive at

$$\begin{aligned}
&\Phi_{p,q}(a,b;z) \\
=&(1+a)^p(1+b)^q \left(\frac{zq}{\Gamma(p)} \int_0^1 \int_0^\infty e^{-qu} \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} du dt \right. \\
&\left. + \frac{zq}{\Gamma(p)} \int_0^1 \int_0^\infty \left(\log \frac{1}{t}\right)^{p-1} e^{-qu} (zt)^{\lfloor e^u - b \rfloor} \frac{t^a}{zt-1} du dt \right). \\
=&(1+a)^p(1+b)^q \left(\frac{zq}{\Gamma(p)} \int_0^1 \int_0^\infty e^{-qu} \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} (1-(zt)^{\lfloor e^u - b \rfloor}) du dt \right).
\end{aligned}$$

Now, substituting $t = e^{-v}$, the remaining part of the proof follows. □

In Theorem 5, Hadamard convolution played a vital role in obtaining the integral representation for the generalized polylogarithm $\Phi_{p,q}(a,b;z)$. Using the integral representation of Hadamard convolution, other integral representations for $\Phi_{p,q}(a,b;z)$ can be obtained as well. One such example is shown below, where we found another integral representation of the generalized polylogarithm.

From (4), we have

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^p} = \frac{1}{a^p} + \frac{z}{\Gamma(p)} \int_0^1 \left(\log \frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} dt, \quad |z| < 1, \tag{43}$$

and

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+b)^q} = \frac{1}{b^q} + \frac{z}{\Gamma(q)} \int_0^1 \left(\log\frac{1}{u}\right)^{q-1} \frac{u^b}{1-zu} du, \quad |z|<1, \tag{44}$$

where a, b, p, and q are complex numbers satisfying $\Re(a)>0$, $\Re(p)>0$, $\Re(b)>0$, and $\Re(q)>0$.

We apply Theorem 2 to the holomorphic functions given in (43) and (44), which results in

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^p(n+b)^q} = \frac{1}{2\pi i} \int_{|w|=r} \left(\sum_{n=0}^{\infty} \frac{z^n}{(n+a)^p} \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^q} \right) \frac{dw}{w}. \tag{45}$$

Substituting (43) and (44) into (45) gives

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^p(n+b)^q} = \frac{1}{2\pi i} \int_{|w|=r} & \left(\frac{1}{a^p} + \frac{z}{\Gamma(p)} \int_0^1 \left(\log\frac{1}{t}\right)^{p-1} \frac{t^a}{1-zt} dt \right) \\ & \times \left(\frac{1}{b^q} + \frac{z}{\Gamma(q)} \int_0^1 \left(\log\frac{1}{u}\right)^{q-1} \frac{u^b}{1-zu} du \right) \frac{dw}{w}. \end{aligned}$$

Evaluating the above integral using Cauchy integral formula and applying simple computations, we obtain the expression

$$\Phi_{p,q}(a,b;z) = \frac{(1+a)^p(1+b)^q}{\Gamma(p)\Gamma(q)} \int_0^1 \int_0^1 z \left(\log\frac{1}{u}\right)^{p-1} \left(\log\frac{1}{v}\right)^{q-1} \frac{u^a v^b}{1-uvz} du dv, \quad |z|<1, \tag{46}$$

where p, q, a, and b are complex numbers with positive real parts.

Thus, we have obtained another integral form. However, the representation (46) is already presented in [27] using different technique and has been widely used to study the geometric behavior of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$ in the complex plane, as noted in [23, 27].

Concluding Remarks

This manuscript primarily aimed to derive integral representations of the generalized polylogarithm $\Phi_{p,q}(a,b;z)$. Since the generalized polylogarithm is a form of Dirichlet series, we applied the Laplace representation of Dirichlet series under certain constraints on the parameters p, q, a, and b. Using this approach, we derived the single integral form of the generalized polylogarithm in (9). This integral form allowed us to explore several properties, including complete monotonicity, the Turán inequality, and the convexity of the generalized polylogarithm and related functions.

Subsequently, we derived another integral form (33) where first we found the integral form of the Lerch transcendent function. After that by using the Hadamard product and the integral form of Lerch transcendent function we obtain the double integral form of the generalized polylogairthm.

In terms of comparison between the two integrals, (9) and (33), the integral in (33) is valid over a larger domain compared to (9). However, the representation in (9) enables us to discuss a wider range of analytical properties of the generalized polylogarithm and related functions. Hence it would be intersting to study the geometric properties given in [23, 27] using the integral representation (9) and analytic properties given in this manuscrpit using the integral representation (33).

Acknowledgement. The authors wish to thank the anonymous referee(s) for their comments and suggestions that help to improve the quality of the paper.

Funding. There is no funding support received for this manuscript.

Conflict of interest disclosure. There is no conflict of interest related to this manuscript.

Ethics approval statement. There are no ethical violation and hence ethics approval is Not applicable.

Patient consent statement. There are no clinical trials and hence patient consent is Not required.

Permission to reproduce material from other sources. No materials reproduced from other sources. Contents from existing literature are cited properly with clear references.

Clinical trial registration. There are no clinical trials and hence such registration is Not required.

Competing interest. The authors have no relevant financial or non-financial interests to disclose.

Authors contribution. All authors contributed equally. All authors read and approved the final manuscript.

REFERENCES

[1] M. ABOUZAHRA AND L. LEWIN, *Polylogarithmic ladders*, in *Structural properties of polylogarithms*, 27–47, Math. Surveys Monogr. **37**, Amer. Math. Soc., Providence, RI.

[2] E. ALKAN, *Series representations in the spirit of Ramanujan*, J. Math. Anal. Appl. **410** (2014), no. 1, 11–26.

[3] H. ALZER, D. KARAYANNAKIS AND H. M. SRIVASTAVA, *Series representations for some mathematical constants*, J. Math. Anal. Appl. **320** (2006), no. 1, 145–162.

[4] H. ALZER AND S. KOUMANDOS, *Series and product representations for some mathematical constants*, Period. Math. Hungar. **58** (2009), no. 1, 71–82.

[5] H. ALZER AND K. C. RICHARDS, *Series representations for special functions and mathematical constants*, Ramanujan J. **40** (2016), no. 2, 291–310.

[6] H. ALZER AND J. H. WELLS, *Inequalities for the polygamma functions*, SIAM J. Math. Anal. **29** (1998), no. 6, 1459–1466.

[7] G. E. ANDREWS, R. A. ASKEY AND R. ROY, *Special functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge Univ. Press, Cambridge, 1999.

[8] Á BARICZ, *Turán type inequalities for generalized complete elliptic integrals*, Math. Z. **256** (2007), no. 4, 895–911.

[9] Á BARICZ, *Turán type inequalities for hypergeometric functions*, Proc. Amer. Math. Soc. **136** (9) (2008) 3223–3229.

[10] Á BARICZ, K. RAGHAVENDAR AND A. SWAMINATHAN, *Turán type inequalities for q-hypergeometric functions*, J. Approx. Theory **168** (2013), 69–79.

[11] C. BERG, H. L. PEDERSEN, *A completely monotonic function used in an inequality of Alzer*, Comput. Methods Funct. Theory **12** (2012), no. 1, 329–341.

[12] C. BERG, *A complete Bernstein function related to the fractal dimension of Pascal's pyramid modulo a prime*, Expositiones Mathematicae (2024).

[13] E. CAHEN, *Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analoguesodulo a prime*, Ann. Sci. École Norm. Sup. (3) **11** (1894), 75–164.

[14] D. CVIJOVIĆ, *New integral representations of the polylogarithm functionodulo a prime*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **463** (2007), no. 2080, 897–905.

[15] G. H. HARDY AND M. RIESZ, *The general theory of Dirichlet's series*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 18, Stechert-Hafner, Inc., New York, 1964.

[16] K. S. KÖLBIG, *Nielsen's generalized polylogarithmsodulo a prime*, SIAM J. Math. Anal. **17** (1986), no. 5, 1232–1258.

[17] S. KOUMANDOS AND H. L. PEDERSEN, *On asymptotic expansions of generalized Stieltjes functionsodulo a prime*, Comput. Methods Funct. Theory **15** (2015), no. 1, 93–115.

[18] G. W. LEIBNIZ, *Mathematische Schriften. Bd. V*, Georg Olms Verlagsbuchhandlung, Hildesheim, 1962.

[19] L. LEWIN, *Polylogarithms and associated functions*, North-Holland, New York, 1981.

[20] L. C. MAXIMON, *The dilogarithm function for complex argumentodulo a prime*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2039, 2807–2819.

[21] J. L. MEYER, *A generalization of an integral of Ramanujanodulo a prime*, Ramanujan J. **14** (2007), no. 1, 79–88.

[22] I. MEZŐ, *A family of polylog-trigonometric integralsodulo a prime*, Ramanujan J. **46** (2018), no. 1, 161–171.

[23] S. R. MONDAL AND A. SWAMINATHAN, *Geometric properties of generalized polylogarithmodulo a prime*, Integral Transforms Spec. Funct. **21** (2010), no. 9–10, 691–701.

[24] H. L. PEDERSEN, *On the remainder in an asymptotic expansion of the double gamma functionodulo a prime*, Mediterr. J. Math. **2** (2005), no. 2, 171–178.

[25] T. K. POGÁNY, *Integral representation of Mathieu $(\boldsymbol{a},\boldsymbol{\lambda})$-seriesodulo a prime*, Integral Transforms Spec. Funct. **16** (2005), no. 8, 685–689.

[26] T. K. POGÁNY, *Integral form of Le Roy–type hypergeometric functionodulo a prime*, Integral Transforms Spec. Funct. **29** (2018), no. 7, 580–584.

[27] S. PONNUSAMY AND S. SABAPATHY, *Polylogarithms in the theory of univalent functionsodulo a prime*, Results Math. **30** (1996), no. 1–2, 136–150.

[28] A. PRUDNIKOV, Y. A. BRYCHKOV AND O. I. MARICHEV, *Integrals and series*, vol. 2, translated from the Russian by N. M. Queen, second edition, Gordon & Breach, New York, 1988.

[29] A. PRUDNIKOV, Y. A. BRYCHKOV AND O. I. MARICHEV, *Integrals and series*, vol. 3, translated from the Russian by G. G. Gould, Gordon and Breach, New York, 1990.

[30] E. D. RAINVILLE, *Special functions*, The Macmillan Company, New York, 1960.

[31] R. SCHILLING, R. SONG AND Z. VONDRAČEK, *Bernstein functions*, second edition, De Gruyter Studies in Mathematics, 37, de Gruyter, Berlin, 2012.

[32] A. SOFO, J. C. PAIN, AND V. SCHARASCHKIN, *A family of polylogarithmic integralsodulo a prime*, arXiv preprint, arXiv:2412.02031v1, (2024), 14 pages.

[33] E. C. TITCHMARSH, *The theory of functions*, reprint of the second (1939) edition, Oxford Univ. Press, Oxford, 1958.

[34] P. TURÁN, *On the zeros of the polynomials of Legendreodulo a prime*, Časopis Pěst. Mat. Fys. **75** (1950), 113–122.
[35] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series, vol. 6, Princeton Univ. Press, Princeton, NJ, 1941.

(Received March 21, 2025)

Deepshikha Mishra
Department of Mathematics
Indian Institute of Technology
Roorkee-247667, Uttarakhand, India
e-mail: deepshikha_m@ma.iitr.ac.in

A. Swaminathan
Department of Mathematics
Indian Institute of Technology,
Roorkee-247667, Uttarakhand, India
e-mail: a.swaminathan@ma.iitr.ac.in