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PITT–TYPE INEQUALITY FOR $p > q$ WITH GM–CONDITION

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Abstract. In this paper, we examine the results of the Pitt type for the Fourier transform, specifically focusing on scenarios where the parameter p is greater than q. Here, we show that for the Pitt inequality the 'if and only if' condition established by Saucedo and Tikhonov (*The Fourier transform is an extremizer of a class of bounded operators, arXiv.2501.17505*) can be extended under restriction to the class of GM functions.

1. Introduction

Among the various significant types of weighted norm inequalities, the Pitt inequality stands out as particularly important. In particular, we are interested in the inequality

$$\left(\int_{\mathbb{R}} u(\xi)|\hat{f}(\xi)|^q \, d\xi\right)^{1/q} \leqslant C\left(\int_{\mathbb{R}} v(x)|f(x)|^p \, dx\right)^{1/p}, \tag{1.1}$$

where the function $\hat{f}(\xi)$ is defined as the integral $\int_{\mathbb{R}} f(x)e^{ix\xi}\,dx$, with $\xi \in \mathbb{R}$. This expression represents the Fourier transform of a function f that belongs to the space $L^1(\mathbb{R})$. Furthermore, u and v^{-1} are non-increasing weight functions in $\mathbb{R}$. The ranges for p and q are specified as $1 \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$. In this context, the symbol C is used to denote constants that may change from one line to another. Furthermore, $p' = p/(p-1)$ is determined as the conjugate exponent.

When $1 < p \leqslant q < \infty$, Pitt's inequality is well studied. In the 1980s, three groups of authors examined it using different methods, all using weight rearrangements (see [7, 9, 10, 14, 15]).

For $1 < q < p < \infty$, specific weight conditions guarantee that the Fourier transform maps a weighted space L^p to a weighted space L^q, as demonstrated in [2]. These results apply for $2 < q < p$ and $1 < q < p < 2$, but not when $1 < q < 2 < p$. The scenario $1 < p < \infty$, $0 < q < p$ is also studied in [1, 8] and later in [2]. In [17], the authors give counterexamples that illustrate that such simple conditions fail in the case $1 < q < 2 < p$. Then several additional conditions are proposed to re-establish sufficiency.

In all the mentioned papers, only sufficient conditions are established for the case $1 < q < p < \infty$. In [18], the authors present the Pitt inequality with the 'if and only if' condition that inequality (1.1) holds.

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In this paper, we work with a subclass of the BV-space, introduced by Liflyand and Tikhonov [12]. For the functions of this subclass there exist constants $c>0$ and $a>1$ such that for all $x>0$, we have

$$\int_x^{\infty} |df| \leqslant c \int_{x/a}^{\infty} |f(y)| y^{-1} dy, \tag{1.2}$$

with $|f(x)| \to 0$ as $x \to \infty$. We call them general monotone (GM) functions. See also the papers [13, 5]. For Pitt's inequalities under GM condition, see also [6, 4, 3]. For Hardy-Littlewood inequality for trigonometric series, see [16].

Here, we show that for the inequality (1.1) the condition if and only if established in [18] can be extended, in general, under the restriction to the class of GM functions. Moreover, here we introduce the GM classes depending on the parameters τ and κ. Note that for $\tau \leqslant \kappa$ this class coincides with one in the paper [19].

Throughout this paper, we use $\lesssim$ to ignore the constant in $A \leqslant cB$, and $A \simeq B$ denotes $c_1 B \leqslant A \leqslant c_2 B$ for some positive constants c_1 and c_2. In addition, we employ the following notation

$$|x|^{s,t} := \begin{cases} |x|^s, & |x| < 1; \\ |x|^t, & |x| \geqslant 1. \end{cases}$$

2. General monotone functions and main result

We begin this section by providing definitions for the class of BV functions.

DEFINITION 1. We define a space of $BV(a,b)$ functions for $a,b \in \mathbb{R}$ characterized by their total variation

$$\int_a^b |df| := \sup_{P \in \mathscr{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)| < \infty,$$

where $\mathscr{P}$ denotes a partition of the interval $[a,b]$:

$$\mathscr{P} = \{P = \{x_i\} \mid a < x_0 < x_1 < \ldots < x_{n_P} < b\}.$$

A function f belongs to $BV_0(\mathbb{R} \setminus 0)$ if, for all $a>0$, it holds that f is in $BV((-\infty, -a) \cup (a, +\infty))$, and

$$|f(x)| \to 0 \text{ as } |x| \to \infty. \tag{2.1}$$

In this paper, we denote $\mathbb{R}_* := \mathbb{R} \setminus 0$.

We are currently prepared to present the novel class of general monotone functions in detail.

DEFINITION 2. Let $\tau, \kappa \in \mathbb{R}$. Suppose that $f \in BV_0(\mathbb{R}_*)$. Then we write $f \in GM(\mathbb{R}_*; \tau, \kappa)$ (general monotone with parameters τ and κ) if there exists a constant $m>1$ such that

$$\int_{|y| \geqslant |x|} |df(y)| \lesssim \int_{|y| \geqslant |x/m|} \frac{|f(y)|}{|y|^{\tau,\kappa}} dy \tag{2.2}$$

is true for all $|x| > 0$.

In what follows, we need the following property.

PROPERTY 1. [12] *Let* $f \in BV_0(\mathbb{R}_*)$. *Then*

$$|f(x)| \lesssim \int\limits_{|t| \geqslant |x|} |df(t)|,$$

for any $|x| > 0$.

REMARK 1. For $\tau \leqslant \kappa$, a class of functions $GM(\mathbb{R}_*; \tau, \kappa)$ coincides with one in [19].

Now we present the Pitt inequality in the $L^p - L^q$ setting from [18], with the 'if and only if' condition that the inequality (1.1) holds.

THEOREM 1. [18, Theorem 5.1 (ii)] *Let* $1 < q < p \leqslant 2$, *or* $2 \leqslant q < p < \infty$. *Suppose that* u *and* v^{-1} *are positive non-increasing functions. Then the inequality* (1.1) *holds if and only if*

$$\int_0^\infty u(x) \left(\int_0^x u(t)dt \right)^{r/p} \left(\int_0^{1/x} v^{\frac{1}{1-p}}(t)dt \right)^{r/p'} dx < \infty. \tag{2.3}$$

In the case $q < 2 < p < \infty$, we have the following.

THEOREM 2. [18, Theorem 5.1 (iii)] *Let* $q < 2 < p < \infty$. *Suppose that* u *and* v^{-1} *are positive non-increasing functions. Then the inequality* (1.1) *holds if and only if in addition to* (2.3) *we have*

$$\int_1^\infty u^{\frac{2}{2-q}}(t)dt < \infty, \tag{2.4}$$

and

$$\int_0^\infty u(t)U^{\frac{q}{2-q}}(t)\xi^{\frac{q}{q-2}}(t)t^{-q/2} \left(\int_t^\infty u(y)U^{\frac{q}{2-q}}(y)\xi^{\frac{q}{q-2}}(y)y^{-q/2}dy \right)^{\frac{q}{p-q}}$$
$$\times \left(\int_{t^{-1}}^\infty v^{\frac{2}{2-p}}(y) \left(\frac{V(y)}{yv(y)} \right)^{\frac{p}{2-p}} dy \right)^{\frac{q(p-2)}{2(p-q)}} dt < \infty, \tag{2.5}$$

where

$$U(x) = \int_0^x u(y)dy,\ V(x) = \int_0^x v(y)dy,$$
$$\xi(x) = U(x) + x^{q/2} \left(\int_x^\infty u^{2/(2-q)}(y)dy \right)^{(2-q)/2}.$$

The following remark addresses specific examples of Theorems 1 and 2.

REMARK 2. Let us take weight functions such as $u(t) := |t|^{\alpha_0,\alpha_1}$ and $v(t) := |t|^{\beta_0,\beta_1}$ with $\alpha_0,\alpha_1 \leqslant 0$ and $\beta_0,\beta_1 \geqslant 0$. Then the condition (2.3) is equivalent to

$$\begin{cases} -1 < \alpha_0 \leqslant 0,\ \alpha_1 \leqslant 0, \\ \alpha_1 < \frac{q}{p}(p-1)-1, \\ 0 \leqslant \beta_0 < \min(p-1-\frac{p}{q}(\alpha_1+1), p-1), \\ \beta_1 > p-1-\frac{p}{q}(\alpha_0+1), \\ \beta_1 \geqslant 0. \end{cases}$$

Moreover, equivalent conditions to (2.3), (2.4), and (2.5) are

$$\begin{cases} q/2-1 \leqslant \alpha_0 \leqslant 0, \\ \alpha_1 < q/2-1, \\ 0 \leqslant \beta_0 < \min(p-1-\frac{p}{q}(\alpha_1+1); p-1), \\ \beta_1 > p-1-\frac{p}{q}(\alpha_0+1)+\alpha_0+1-q/2, \\ \beta_1 \geqslant 0, \end{cases}$$

and

$$\begin{cases} -1 < \alpha_0 < q/2-1, \\ \alpha_1 < q/2-1, \\ 0 \leqslant \beta_0 < p-1-\max(\frac{p}{q}(\alpha_1+1), 0), \\ \beta_1 > p-1-\frac{p}{q}(\alpha_0+1)+\frac{q}{2-q}(\frac{q}{2}-\alpha_0-1), \end{cases}$$

In the following statements, we formulate the special cases of Theorems 1 and 2 for weight functions such as $u(x) = |x|^{\alpha_0,\alpha_1}$ and $v(x) = |x|^{\beta_0,\beta_1}$.

COROLLARY 1. *Let $1 < q < p \leqslant 2$, or $2 \leqslant q < p < \infty$. If $-1 < \alpha_0 \leqslant 0$ and $\alpha_1 < q-1-\frac{q}{p} := \alpha_1^*$, $\alpha_1 \leqslant 0$, then the inequality*

$$\left(\int_{\mathbb{R}} |\xi|^{\alpha_0,\alpha_1} |\hat{f}(\xi)|^q d\xi\right)^{1/q} \lesssim \left(\int_{\mathbb{R}} |x|^{\beta_0,\beta_1} |f(x)|^p dx\right)^{1/p}, \tag{2.6}$$

holds if and only if

$$\begin{cases} 0 \leqslant \beta_0 < \min(p-1-\frac{p}{q}(\alpha_1+1), p-1) := \beta_0^*, \\ \beta_1 > p-1-\frac{p}{q}(\alpha_0+1) := \beta_1^*, \\ \beta_1 \geqslant 0. \end{cases}$$

COROLLARY 2. *Let $q < 2 < p < \infty$. If $\alpha_1 < q/2-1 \leqslant \alpha_0 \leqslant 0$, then the inequality (2.6) holds if and only if*

$$\begin{cases} 0 \leqslant \beta_0 < \min(p-1-\frac{p}{q}(\alpha_1+1), p-1) := \beta_0^*, \\ \beta_1 > p-1-\frac{p}{q}(\alpha_0+1)+\alpha_0+1-q/2 := \beta_1^{**}, \\ \beta_1 \geqslant 0. \end{cases}$$

COROLLARY 3. *Let $q<2<p<\infty$. If $-1<\alpha_0<q/2-1$ and $\alpha_1<q/2-1:=\alpha_1^{**}$, then the inequality (2.6) holds if and only if*

$$\begin{cases} 0\leqslant \beta_0<\min(p-1-\frac{p}{q}(\alpha_1+1),p-1):=\beta_0^*, \\ \beta_1>p-1-\frac{p}{q}(\alpha_0+1)+\frac{q}{2-q}(\frac{q}{2}-\alpha_0-1):=\beta_1^{***}. \end{cases}$$

REMARK 3. We note that

- α_1^* could be either nonnegative or negative;
- $\alpha_1^{**}<0$;
- $\beta_1^{**}\geqslant\beta_1^*$, and $\beta_1^{**}=\beta_1^*$ iff $\alpha_0=q/2-1$;
- $\beta_1^{***}>\max(\beta_1^*,0)$.

The main result of this paper is a Pitt-type inequality, expressed as follows.

THEOREM 3. *Suppose that $f\in GM(\mathbb{R}_*;1,1)$. Let $1<q<p<\infty$. If $-1<\alpha_0\leqslant 0$ and $\alpha_1\leqslant 0$, then the inequality (2.6) holds if and only if*

$$\begin{cases} 0\leqslant \beta_0\leqslant\min(p-1-\frac{p}{q}(\alpha_1+1);p-1)=\beta_0^*, \\ \beta_1\geqslant\max(p-1-\frac{p}{q}(\alpha_0+1);0)=\max(\beta_1^*;0). \end{cases}$$

Now, we compare the results of Corollaries 1-3 and Theorem 3.

REMARK 4. We denote by $I^{GM}(\alpha_0),I^{GM}(\alpha_1),I^{GM}(\beta_0),I^{GM}(\beta_1)$ range intervals of the parameters $\alpha_0,\alpha_1,\beta_0,\beta_1$ in Theorem 3, respectively. Then, we have

$$I^{GM}(\alpha_0)=(-1,0],\ \ I^{GM}(\alpha_1)=(-\infty,0],$$

$$I^{GM}(\beta_0)=[0,\beta_0^*],\ \ I^{GM}(\beta_1)=[0,+\infty)\cap[\beta_1^*,+\infty).$$

By $I^{(i)}(\alpha_0)$, $I^{(i)}(\alpha_1)$, $I^{(i)}(\beta_0),I^{(i)}(\beta_1)$, $i=0,1,2$, we denote range intervals of the parameters $\alpha_0,\alpha_1,\beta_0,\beta_1$ in Corollaries 1, 2, and 3, respectively. Then, we have

- $\begin{cases} I^{(0)}(\alpha_0)=(-1,0]=I^{GM}(\alpha_0), \\ I^{(1)}(\alpha_0)=[q/2-1,0]\subset I^{GM}(\alpha_0), \\ I^{(2)}(\alpha_0)=(-1,q/2-1)\subset I^{GM}(\alpha_0), \end{cases}$
- $\begin{cases} I^{(0)}(\alpha_1)=(-\infty,0]\cap(-\infty,\alpha_1^*)\subseteq I^{GM}(\alpha_1), \\ I^{(1)}(\alpha_1)=I^{(2)}(\alpha_1)=(-\infty,\alpha_1^{**})\subset I^{GM}(\alpha_1), \end{cases}$
- $I^{(0)}(\beta_0)=I^{(1)}(\beta_0)=I^{(2)}(\beta_0)=[0,\beta_0^*)\subset I^{GM}(\beta_0)$,
- $\begin{cases} I^{(0)}(\beta_1)=[0,+\infty)\cap(\beta_1^*,+\infty)\subset I^{GM}(\beta_1), \\ I^{(1)}(\beta_1)=[0,+\infty)\cap(\beta_1^{**},+\infty)\subset I^{GM}(\beta_1), \\ I^{(2)}(\beta_1)=(\beta_1^{***},+\infty)\subset I^{GM}(\beta_1), \end{cases}$

concluding that under the GM condition the range of parameters is wider than in the classical case.

3. Proof of the main result

We start this section by presenting a point-wise estimate for the Fourier transform of a BV function.

LEMMA 1. [12, Lemma 2.2] *Let* $f \in BV_0(\mathbb{R}_*)$. *Then*

$$|\hat{f}(\xi)| \leqslant \int\limits_{|x|<|\xi|^{-1}} |f(x)|dx + \frac{1}{|\xi|} \int\limits_{|x|\geqslant|\xi|^{-1}} |df(x)|,$$

for all $\xi \in \mathbb{R}$ *with* $\xi \neq 0$.

3.1. Hardy inequalities

We now present results directly derived from [11] concerning the Hardy operator

$$(Hf)(x) = \int_0^x f(y)dy,$$

and its conjugate

$$(H^* f)(x) = \int_x^\infty f(y)dy.$$

THEOREM 4. [11] *Let* $0 < q < p < \infty$, $q \neq 1$, $1 < p < \infty$ *be with* $1/p - 1/q = 1/r$. *Let* u *and* v *be weight functions defined in* $\mathbb{R}_+$. *Then for* $f \geqslant 0$ *the following inequality*

$$\left(\int_0^\infty u(x)\left[\int_0^x f(y)dy\right]^q dx\right)^{1/q} \leqslant C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p} \tag{3.1}$$

holds if and only if

$$\int_0^\infty \left(\int_x^\infty u(t)dt\right)^{r/q} \left(\int_0^x v^{1-p'}(t)dt\right)^{r/q'} v^{1-p'}(x)dx < \infty. \tag{3.2}$$

The dual inequality

$$\left(\int_0^\infty u(x)\left[\int_x^\infty f(y)dy\right]^q dx\right)^{1/q} \leqslant C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p} \tag{3.3}$$

holds if and only if

$$\int_0^\infty \left(\int_0^x u(t)dt\right)^{r/q} \left(\int_x^\infty v^{1-p'}(t)dt\right)^{r/q'} v^{1-p'}(x)dx < \infty. \tag{3.4}$$

In the case $p = q$, we have the following.

THEOREM 5. [11] *Let* $1 < p < \infty$ *with* $1/p + 1/p' = 1$. *Let* u *and* v *be weight functions defined in* $\mathbb{R}$. *Then for* $f \geqslant 0$ *the following inequality*

$$\left(\int_0^\infty u(x)\left[\int_x^\infty f(y)dy\right]^p dx\right)^{1/p} \leqslant C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p} \tag{3.5}$$

holds if and only if

$$\sup_{t>0}\left(\int_0^t u(x)dx\right)^{1/p}\left(\int_t^\infty v^{(1-p')}(x)dx\right)^{1/p'} < \infty. \tag{3.6}$$

In the following, we find equivalent conditions (3.2), (3.4), (3.6) in the special case of the weight functions.

REMARK 5. Consider the specific case where the weights are defined by

$$u(x) = |x|^{a_0,a_1}, \quad \text{and} \quad v(x) = |x|^{b_0,b_1},$$

for $a_0, a_1, b_0, b_1 \in \mathbb{R}$.

With these weights, (3.2) becomes

$$\begin{cases} a_1 + 1 < 0, \\ b_0 < \min(\frac{p}{q}(a_0+1) + p - 1, p - 1), \\ b_1 > \frac{p}{q}(a_1+1) + p - 1. \end{cases}$$

For the condition (3.4), we have

$$\begin{cases} a_0 + 1 > 0, \\ b_0 < \frac{p}{q}(a_0+1) + p - 1, \\ b_1 > \max(\frac{p}{q}(a_1+1) + p - 1, p - 1). \end{cases}$$

The condition (3.6) is equivalent to

$$\begin{cases} a_0 + 1 > 0, \\ b_0 \leqslant p + a_0, \\ b_1 > p - 1, \\ b_1 \geqslant p + a_1. \end{cases}$$

3.2. Auxiliary theorems

We are now ready to present auxiliary theorems that will later be employed to establish the main result of this paper.

THEOREM 6. *Let* $0<q<p<\infty,\ q\neq 1,\ 1<p<\infty$. *Let* $\tau,\kappa\in\mathbb{R}$. *Suppose that* $f\in GM(\mathbb{R}_*;\tau,\kappa)$. *If* $\alpha_0>-1,\ \alpha_1<q-1$, *then the inequality*

$$\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}|\hat{f}_1(\xi)|^q d\xi\right)^{1/q}\lesssim\left(\int_{\mathbb{R}}|x|^{\beta_0,\beta_1}|f(x)|^p dx\right)^{1/p},$$

holds for any pair (β_0,β_1) *satisfying the following*

$$\begin{cases}\beta_0\leqslant(1-\tau)p-1+p-\frac{p}{q}\max(\alpha_1+1,0),\\ \beta_1>(1-\kappa)p-1,\\ \beta_1\geqslant(1-\kappa)p-1+p-\frac{p}{q}(\alpha_0+1).\end{cases}$$

Here

$$\hat{f}_1(\xi)=\int\limits_{|x|<|\xi|^{-1}}f(x)e^{ix\xi}dx,$$

for $|\xi|>0$.

Proof. Let us denote by $V(x):=\int\limits_{|y|\geqslant|x|}|df(y)|$, for all $|x|>0$. By the change of variables $\xi\to\xi^{-1}$ and using the Hardy inequality (3.1), we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}\left(\int\limits_{|x|<|\xi|^{-1}}|f(x)|dx\right)^q d\xi\right)^{1/q}\\ &\overset{\text{Property } 1}{\leqslant}\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}\left(\int\limits_{|x|<|\xi|^{-1}}V(x)dx\right)^q d\xi\right)^{1/q}\\ &=\left(\int_{\mathbb{R}}|\xi|^{-\alpha_1-2,-\alpha_0-2}\left(\int\limits_{|x|<|\xi|}V(x)dx\right)^q d\xi\right)^{1/q}\\ &\overset{\text{Remark } 5}{\leqslant}\left(\int_{\mathbb{R}}|\xi|^{\gamma_0,\gamma_1}(V(\xi))^p d\xi\right)^{1/p},\end{aligned}$$

where

$$\begin{cases}\alpha_0+1>0,\\ \gamma_0<\min(\frac{p}{q}(-\alpha_1-1)+p-1,p-1),\\ \gamma_1>\frac{p}{q}(-\alpha_0-1)+p-1.\end{cases}\tag{3.7}$$

Here we are using an equivalent version of (3.2) from Remark 5.

Finally, we get

$$
\begin{aligned}
\left(\int_{\mathbb{R}} |\xi|^{\gamma_0,\gamma_1} (V(\xi))^p \, d\xi\right)^{1/p} &= \left(\int_{\mathbb{R}} |\xi|^{\gamma_0,\gamma_1} \left(\int\limits_{|y|\geqslant|\xi|} |df(y)|\right)^p d\xi\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}} |\xi|^{\gamma_0,\gamma_1} \left(\int\limits_{|y|\geqslant|\xi/m|} \frac{|f(y)|}{|y|^{\tau,\kappa}} dy\right)^p d\xi\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}} |\xi|^{\gamma_0,\gamma_1} \left(\int\limits_{|y|\geqslant|\xi|} \frac{|f(y)|}{|y|^{\tau,\kappa}} dy\right)^p d\xi\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}} |x|^{b_0-\tau p, b_1-\kappa p} |f(x)|^p dx\right)^{1/p},
\end{aligned}
$$

using an equivalent version of (3.4) from Remark 5 and asking

$$
\begin{cases}
\gamma_0 + 1 > 0, \\
b_0 \leqslant p + \gamma_0, \\
b_1 > p - 1, \\
b_1 \geqslant p + \gamma_1.
\end{cases} \tag{3.8}
$$

in the last inequality.

Combining (3.7) and (3.8), yields

$$
\begin{cases}
\alpha_0 + 1 > 0, \\
-1 < \gamma_0 < \min(\frac{p}{q}(-\alpha_1 - 1), 0) + p - 1, \\
\gamma_1 > \frac{p}{q}(-\alpha_0 - 1) + p - 1, \\
b_0 \leqslant p + \gamma_0, \\
b_1 > p - 1, \\
b_1 \geqslant p + \gamma_1.
\end{cases}
$$

To get maximal ranges for the pairs of (α_0, α_1) and (b_0, b_1), we choose γ_0 and γ_1 such that

$$
\begin{cases}
\alpha_0 + 1 > 0, \ \alpha_1 < q - 1, \\
b_0 \leqslant p - 1 + p - \frac{p}{q} \max(\alpha_1 + 1, 0), \\
b_1 > p - 1, \\
b_1 \geqslant p - 1 + p - \frac{p}{q}(\alpha_0 + 1). \quad \square
\end{cases} \tag{3.9}
$$

Here, we present the property of BV-functions, which is used in the proof of Theorem 7.

LEMMA 2. *Assume that* $f \in BV_0(\mathbb{R}_*)$. *Then*

$$\int_{|y|\geqslant 2|\xi|^{-1}} |df(y)| \lesssim \int_{|x|\geqslant|\xi|^{-1}} |x|^{-1} \int_{|x|\leqslant|y|<2|x|} |df(y)|dx \lesssim \int_{|y|\geqslant|\xi|^{-1}} |df(y)|,$$

for all $|\xi| > 0$.

Lemma 2 can be easily shown using the Fubini theorem.

THEOREM 7. *Let* $0 < q < p < \infty$, $q \neq 1$, $1 < p < \infty$. *Assume* $\tau, \kappa \in \mathbb{R}$. *Suppose that* $f \in GM(\mathbb{R}_*; \tau, \kappa)$. *If* $\alpha_1 - q + 1 < 0$ *and* $\alpha_0 \in \mathbb{R}$, *then the inequality*

$$\left(\int_{\mathbb{R}} |\xi|^{\alpha_0,\alpha_1} |\hat{f}_2(\xi)|^q d\xi\right)^{1/q} \lesssim \left(\int_{\mathbb{R}} |x|^{\beta_0,\beta_1} |f(x)|^p dx\right)^{1/p} \tag{3.10}$$

holds if and only if

$$\begin{cases} \beta_0 \leqslant (2-\tau)p - \frac{p}{q}(\alpha_1+1) - 1, \\ \beta_1 > (1-\kappa)p - 1, \\ \beta_1 \geqslant (1-\kappa)p - 1 - \frac{p}{q}\min(\alpha_0 - q + 1, 0). \end{cases}$$

Here

$$\hat{f}_2(\xi) = \int_{|x|\geqslant|\xi|^{-1}} f(x) e^{ix\xi}\, dx,$$

for $|\xi| > 0$.

Proof. We start by using Lemma 1

$$\left(\int_{\mathbb{R}} |\xi|^{\alpha_0,\alpha_1} |\hat{f}_2(\xi)|^q d\xi\right)^{1/q} \lesssim \int_{\mathbb{R}} |\xi|^{\alpha_0-q,\alpha_1-q} \left(\int_{|x|\geqslant|\xi|^{-1}} |df(x)|\right)^q d\xi. \tag{3.11}$$

Applying Lemma 2 to the right hand side of the inequality (3.11), one could be obtained

$$\begin{aligned} &\int_{\mathbb{R}} |\xi|^{\alpha_0-q,\alpha_1-q} \left(\int_{|x|\geqslant|\xi|^{-1}} |df(x)|\right)^q d\xi \\ &\leqslant \int_{\mathbb{R}} |\xi|^{\alpha_0-q,\alpha_1-q} \left(\int_{|x|\geqslant|\xi|^{-1}/2} |x|^{-1} V(x) dx\right)^q d\xi. \end{aligned}$$

By changing variables $x^{-1}/2 \to x$ in the inner integral

$$\int\limits_{|x|\geqslant|\xi|^{-1}/2} |x|^{-1}V(x)dx \simeq \int\limits_{|x|<|\xi|} |x|^{-1}V(x^{-1})dx,$$

and applying Hardy inequality (3.1), for

$$\begin{cases} \alpha_1 - q + 1 < 0, \\ \gamma_0 < \min(\frac{p}{q}(\alpha_0 - q + 1) + p - 1, p - 1), \\ \gamma_1 > \frac{p}{q}(\alpha_1 - q + 1) + p - 1, \end{cases} \tag{3.12}$$

one obtains

$$\begin{aligned}
&\left(\int_{\mathbb{R}} |\xi|^{\alpha_0 - q, \alpha_1 - q} \left(\int\limits_{|x|<|\xi|)} |x|^{-1}V(x^{-1})dx \right)^q d\xi \right)^{1/q} \\
&\lesssim \left(\int_{\mathbb{R}} |\xi|^{\gamma_0,\gamma_1} \left(|\xi|^{-1}V(\xi^{-1})\right)^p d\xi \right)^{1/p} \\
&= \left(\int_{\mathbb{R}} |\xi|^{\gamma_0 - p, \gamma_1 - p} \left(V(\xi^{-1})\right)^p d\xi \right)^{1/p} \\
&\simeq \left(\int_{\mathbb{R}} |\xi|^{p-\gamma_1-2, p-\gamma_0-2} (V(\xi))^p d\xi \right)^{1/p}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}} |\xi|^{p-\gamma_1-2, p-\gamma_0-2} (V(\xi))^p d\xi \right)^{1/p} \\
&= \left(\int_{\mathbb{R}} |\xi|^{p-\gamma_1-2, p-\gamma_0-2} \left(\int\limits_{|y|\geqslant|\xi|} |df(y)| \right)^p d\xi \right)^{1/p} \\
&\overset{\text{Definition 2}}{\lesssim} \left(\int_{\mathbb{R}} |\xi|^{p-\gamma_1-2, p-\gamma_0-2} \left(\int\limits_{|y|\geqslant|\xi|/m} \frac{|f(y)|}{|y|^{\tau,\kappa}} dy \right)^p d\xi \right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}} |\xi|^{p-\gamma_1-2, p-\gamma_0-2} \left(\int\limits_{|y|\geqslant|\xi|} \frac{|f(y)|}{|y|^{\tau,\kappa}} dy \right)^p d\xi \right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}} |x|^{b_0 - \tau p, b_1 - \kappa p} |f(x)|^p dx \right)^{1/p},
\end{aligned}$$

using an equivalent version of (3.6) from Remark 5 and asking

$$\begin{cases} p-\gamma_1-1>0, \\ b_0 \leqslant 2p-\gamma_1-2, \\ b_1>p-1, \\ b_1 \geqslant 2p-\gamma_0-2, \end{cases} \tag{3.13}$$

in the last inequality.

Combining (3.12) and (3.13), yields

$$\begin{cases} \alpha_1-q+1<0, \\ \gamma_0<\min(\frac{p}{q}(\alpha_0-q+1),0)+p-1, \\ p-1>\gamma_1>\frac{p}{q}(\alpha_1-q+1)+p-1, \\ b_0 \leqslant 2p-\gamma_1-2, \\ b_1>p-1, \\ b_1 \geqslant 2p-\gamma_0-2. \end{cases}$$

To get maximal ranges for pairs of (α_0,α_1) and (b_0,b_1), we choose γ_0 and γ_1 such that

$$\begin{cases} \alpha_1-q+1<0,\ \alpha_0\in\mathbb{R}, \\ b_0 \leqslant 2p-\frac{p}{q}(\alpha_1+1)-1, \\ b_1>p-1, \\ b_1 \geqslant p-1-\frac{p}{q}\min(\alpha_0-q+1,0). \quad \square \end{cases} \tag{3.14}$$

3.3. Proof of the main theorem

In the following, we present an extension of our main result. In fact, Theorem 3 is a particular case of the subsequent theorem under the assumptions of Theorems 1 and 2 for classical functions GM.

THEOREM 8. *Let* $0<q<p<\infty$, $q\neq 1$, $1<p<\infty$. *For* $\tau,\kappa\in\mathbb{R}$, *suppose that* $f\in GM(\mathbb{R}_*;\tau,\kappa)$. *If* $\alpha_0>-1$, $\alpha_1<q-1$, *then*

$$\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}|\hat{f}(\xi)|^q d\xi\right)^{1/q} \lesssim \left(\int_{\mathbb{R}}|x|^{\beta_0,\beta_1}|f(x)|^p dx\right)^{1/p},$$

where

$$\begin{cases} \beta_0 \leqslant (1-\tau)p-1+p-\frac{p}{q}\max(\alpha_1+1,0), \\ \beta_1 \geqslant (1-\kappa)p-1-\frac{p}{q}\min(\alpha_0-q+1,0). \end{cases} \tag{3.15}$$

Proof. By Theorems 6 and 7, one has

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}|\hat{f}(\xi)|^q d\xi\right)^{1/q}\\
&\leqslant \left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}|\hat{f}_1(\xi)|^q d\xi\right)^{1/q}+\left(\int_{\mathbb{R}}|\xi|^{\alpha_0,\alpha_1}|\hat{f}_2(\xi)|^q d\xi\right)^{1/q}\\
&\lesssim \left(\int_{\mathbb{R}}|x|^{\beta_0,\beta_1}|f(x)|^p dx\right)^{1/p},
\end{aligned}
$$

where by combining the conditions

$$
\begin{cases}
\alpha_0>-1,\, \alpha_1<q-1,\\
\beta_0\leqslant(1-\tau)p-1+p-\frac{p}{q}\max(\alpha_1+1,0),\\
\beta_1>(1-\kappa)p-1,\\
\beta_1\geqslant(1-\kappa)p-1+p-\frac{p}{q}(\alpha_0+1),
\end{cases}
$$

and

$$
\begin{cases}
\alpha_1-q+1<0,\, \alpha_0\in\mathbb{R},\\
\beta_0\leqslant(2-\tau)p-\frac{p}{q}(\alpha_1+1)-1,\\
\beta_1>(1-\kappa)p-1,\\
\beta_1\geqslant(1-\kappa)p-1-\frac{p}{q}\min(\alpha_0-q+1,0),
\end{cases}
$$

we arrive at

$$
\begin{cases}
\alpha_0>-1,\, \alpha_1<q-1,\\
\beta_0\leqslant(1-\tau)p-1+p-\frac{p}{q}\max(\alpha_1+1,0),\\
\beta_1\geqslant(1-\kappa)p-1-\frac{p}{q}\min(\alpha_0-q+1,0).
\end{cases}
$$

This completes the proof. □

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