Mathematical
Inequalities
& Applications
Volume 29, Number 3 (2026), 257–277 doi:10.7153/mia-2026-29-15

ON THE ZEROS OF A QUATERNIONIC POLYNOMIAL WITH RESTRICTED COEFFICIENTS

TAHIR FAYAZ AND FIRDOSE AHMAD*

(*Communicated by I. Perić*)

Abstract. In recent times, several mathematicians are working on the estimation of zeros of polynomials with quaternionic coefficients using diverse methods. In this paper, we focus on deriving the upper bounds for the location of zeros and establish zero-free regions for specific regular functions of a quaternionic variable with restricted coefficients. To achieve our results, we utilize the Schwarz's lemma and the zero sets of a regular product established in the recently developed theory of regular functions and polynomials of a quaternionic variable. Additionally, our findings lead to generalisations of several well known results in the existing quaternionic literature.

1. Introduction

The concept of complex numbers inspired Irish mathematician Sir William Rowan Hamilton to explore and develop a new set of numbers, known as quaternions, in 1843. He recognized their potential significance in various scientific fields. Over time, quaternion theory has evolved extensively, finding applications not only in contemporary mathematical areas such as algebra, analysis, and geometry but also in diverse fields like computer graphics, control theory, signal processing, physics, and fluid dynamics. In recent years, substantial progress has been made in quaternion research, with numerous scholarly articles published annually across a wide range of disciplines.

A quaternion is a number of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary units i, j, and k satisfy certain properties: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. These numbers form a non-commutative division ring. Any quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ can be divided into its real part $\mathrm{Re}(q) = x_0$ and its imaginary part $\mathrm{Im}(q) = x_1 i + x_2 j + x_3 k$. The conjugate of q is denoted by $\overline{q}$ and defined as $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$. The norm of q is given by $|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. Additionally, the inverse of each nonzero element q of $\mathbb{H}$ is expressed as $q^{-1} = |q|^{-2}\overline{q}$.

The algebraic proof of the Fundamental Theorem of Algebra for quaternionic polynomials with quaternionic coefficients is provided in [20] and [21]. This naturally

Mathematics subject classification (2020): Primary 30E10; Secondary 30G35, 16K20.
Keywords and phrases: Quaternionic polynomial, Eneström-Kakeya theorem, Schwarz's lemma, zero-sets of a regular product.

* Corresponding author.

leads to a fascinating discussion about the placement of zeros in quaternionic polynomials. A noteworthy result, which comprehensively characterizes the zero sets of a regular product of two polynomials in terms of the zero sets of their factors, is derived from [10] (see also [4] and [6]).

THEOREM 1. *Let f and g be quaternionic polynomials. Then, $(f*g)(q_0) = 0$ if and only if $f(q_0) = 0$ or if $f(q_0) \neq 0$, it implies that $g\left(f(q_0)^{-1}q_0 f(q_0)\right) = 0$.*

Gentili and Struppa [6] gave a necessary and sufficient condition for a regular quaternionic power series to have a zero at a point in the following result.

THEOREM 2. *Let $f(q) = \sum_{v=0}^{\infty} q^v a_v$ be regular power series with radius of convergence R, and let $p \in B(0,R)$. Then $f(p) = 0$ if and only if there exists a quaternionic power series $g(q)$ with radius of convergence R such that*

$$f(q) = (q-p)*g(q).$$

Maximum modulus theorem for regular functions of quaternionic variable has been introduced by Gentili and Struppa [5], as stated in the following result.

THEOREM 3. (Maximum Modulus Theorem) *Let $B = B(0,r)$ be a ball in $\mathbb{H}$ centered at 0 with radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ attains a relative maximum at a point $a \in B$, then f must be a constant over the entire ball B.*

Recently, Gardner and Taylor [7] used Theorem 3 and extended Schwarz's lemma from the complex to quaternionic setting in following form.

THEOREM 4. *Let $f(q) = \sum_{v=0}^{\infty} q^v a_v$ be a regular in $|q| \leqslant R$ with quaternionic coefficients a_v, $0 \leqslant v < \infty$ and variable q. Suppose $f(0) = 0$, then*

$$|f(q)| \leqslant \frac{M|q|}{R}, \quad \text{for} \quad |q| \leqslant R,$$

where $M = \max_{|q|=R} |f(q)|$.

One of the most intriguing problems in algebra is determining the zeros of a polynomial. However, as the degree of the polynomial increases, identifying all its zeros becomes increasingly challenging. This highlights the importance of studying the location of polynomial zeros within geometric function theory. A well-known classical result in this area is the Eneström-Kakeya theorem, which provides insights into the location of zeros for polynomials with restricted coefficients. This theorem states that:

THEOREM 5. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leqslant 1.$$

We get the following equivalent form of Theorem 5 by applying it to $z^n T(\frac{1}{z})$.

THEOREM 6. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_0 \geqslant a_1 \geqslant \ldots \geqslant a_{n-1} \geqslant a_n > 0,$$

then $T(z)$ *does not vanish in*

$$|z| \leqslant 1.$$

Aziz and Mohammad extended Theorem 6 to a class of analytic functions in the following form.

THEOREM 7. *Let* $T(z) = \sum_{v=0}^{\infty} a_v z^v$ *be analytic in* $|z| \leqslant t,\ t > 0$. *If*

$$a_v > 0 \quad \text{and} \quad a_{v-1} - t a_v \geqslant 0, \quad v = 0, 1, \ldots,$$

then $T(z)$ *does not vanish in*

$$|z| < t.$$

The Eneström-Kakeya theorem and its variants find numerous applications in complex analysis, including the study of the stability of Brown (K,L) methods. The study of the location of zeros of polynomials plays a fundamental role in complex analysis and has important applications in stability theory. In particular, classical results such as the Eneström-Kakeya theorem are widely used to estimate the location of zeros of polynomials with restricted coefficients. Such results are closely related to the stability analysis of numerical methods, including Brown (K,L) methods. The stability of these methods depends on the location of the zeros of associated characteristic polynomials in the complex plane. Therefore, obtaining precise regions containing these zeros is of significant importance. In this context, the present result provides an explicit disc containing the region defined by a certain inequality. This can be viewed as a refinement of classical bounds and may be useful in studying stability properties of numerical schemes. Moreover, the method suggests possible extensions to more general settings, including quaternionic analysis. While this classical result pertains to the geometry of polynomials, it also has clear implications for numerical methods. Additionally, the theorem has significant applications in functional calculus, operator theory, and quantum physics, making it a pivotal contribution to geometric function theory.

For a deeper exploration of the various extensions and a thorough understanding of the Eneström-Kakeya theorem, the comprehensive works of Marden [11] and Milovanović et al. [14] are excellent resources. Joyal, Labelle, and Rahman [9] further extended Theorem 5, removing the assumption of non-negativity, as demonstrated in the following result.

THEOREM 8. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *(where* z *is a complex variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $T(z)$ *lie in*

$$|z| \leqslant \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zarger [2] relaxed the hypothesis of Theorem 5 and proved the following result.

THEOREM 9. *If* $T(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *(where* z *is a complex variable) with real coefficients such that*

$$a_n \geqslant a_{n-2} \geqslant \ldots \geqslant a_3 \geqslant a_1 > 0,$$

and

$$a_{n-1} \geqslant a_{n-3} \geqslant \ldots \geqslant a_2 \geqslant a_0 > 0, \qquad \text{if } n \text{ is odd}$$

or

$$a_n \geqslant a_{n-2} \geqslant \ldots \geqslant a_2 \geqslant a_0 > 0,$$

and

$$a_{n-1} \geqslant a_{n-3} \geqslant \ldots \geqslant a_3 \geqslant a_1 > 0, \qquad \text{if } n \text{ is even}$$

then all the zeros of $T(z)$ *lie in*

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leqslant 1 + \frac{a_{n-1}}{a_n}.$$

Recently, Carney et al. [3] extended the Eneström-Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials. They first established the quaternionic analogue of Theorem 5 as follows.

THEOREM 10. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then all the zeros of $T(q)$ *lie in*

$$|q| \leqslant 1.$$

In the same paper, they also established.

THEOREM 11. *Let* $T(q) = \sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$*,* $v = 0, 1, 2, \ldots, n$ *such that*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \alpha_{n-2} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0 \geqslant 0,$$

then all the zeros of $T(q)$ *lie in*

$$|q| \leqslant 1 + \frac{2}{\alpha_n}\left[\sum_{v=0}^{n} \left(|\beta_v| + |\gamma_v| + |\delta_v|\right)\right].$$

Besides proving some interesting results on quaternionic polynomials Tripathi [22] (see also [1], Corollary 3.2) also established the following generalization of Theorem 10,

THEOREM 12. *If* $T(q) = \sum_{v=0}^{n} q^v a_v$ *is a polynomial of degree* n *(where* q *is a quaternionic variable) with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant a_0,$$

then all the zeros of $T(q)$ *lie in*

$$|q| \leqslant \frac{|a_0| - a_0 + a_n}{|a_n|}.$$

Meanwhile, Milovanović et al. [8] generalized Theorem 7 and proved the following result.

THEOREM 13. *Let* $f : B(0,R) \to \mathbb{H}$ *be a regular power series in the quaternionic variable* q*, i,e.,* $f(q) = \sum_{v=0}^{\infty} q^v a_v$*, for all* $q \in B(0,R)$*. If* a_v*,* $v = 0, 1, 2, \ldots, n$ *are real and positive satisfying for some* $\lambda_0 \geqslant 1$ *and* $0 < t < R$

$$\lambda_0 \alpha_0 \geqslant t\alpha_1 \geqslant t^2 \alpha_2 \geqslant \ldots,$$

then $f(q)$ *does not vanish in*

$$\left| q - \frac{(\lambda_0 - 1)t}{2\lambda_0 - 1} \right| < \frac{\lambda_0 t}{2\lambda_0 - 1}.$$

Very recently, Mir and Abrar [13] obtained the generalisation of Theorem 12 in following form.

THEOREM 14. *Let* $T(q)=\sum_{v=0}^{n} q^{v}a_{v}$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_{v}=\alpha_{v}+\beta_{v}i+\gamma_{v}j+\delta_{v}k,\ v=0,1,2,\ldots,n$ *such that for some real number* $\rho\geqslant 1$

$$\rho\alpha_{n}\geqslant\alpha_{n-1}\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_{1}\geqslant\alpha_{0}\geqslant 0,$$

then all the zeros of $T(q)$ *lie in*

$$\left|q+(\rho-1)\frac{\alpha_{n}}{a_{n}}\right|\leqslant\frac{1}{|a_{n}|}\left[\rho\alpha_{n}+2\sum_{v=0}^{n}\left(|\beta_{v}|+|\gamma_{v}|+|\delta_{v}|\right)\right].$$

In the same paper, they also proved.

THEOREM 15. *Let* $T(q)=\sum_{v=0}^{n} q^{v}a_{v}$ *be a polynomial of degree* n*, (where* q *is a quaternionic variable), with real coefficients such that for some real* $\rho\geqslant 1$

$$\rho a_{n}\geqslant a_{n-1}\geqslant a_{n-2}\geqslant\ldots\geqslant a_{1}\geqslant a_{0},$$

then all the zeros of $T(q)$ *lie in*

$$|q+\rho-1|\leqslant\frac{\rho a_{n}+|a_{0}|-a_{0}}{|a_{n}|}.$$

Milovanović and Mir [17] obtained the quaternionic analogue of Theorem 9 by proving the following result.

THEOREM 16. *Let* $T(q)=\sum_{v=0}^{n} q^{v}a_{v}$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_{v}=\alpha_{v}+\beta_{v}i+\gamma_{v}j+\delta_{v}k,\ v=0,1,2,\ldots,n$ *such that*

$$\alpha_{n}\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_{3}\geqslant\alpha_{1}>0,$$

and

$$\alpha_{n-1}\geqslant\alpha_{n-3}\geqslant\ldots\geqslant\alpha_{2}\geqslant\alpha_{0}>0,\qquad \textit{if } n \textit{ is odd}$$

or

$$\alpha_{n}\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_{2}\geqslant\alpha_{0}>0,$$

and

$$\alpha_{n-1}\geqslant\alpha_{n-3}\geqslant\ldots\geqslant\alpha_{3}\geqslant\alpha_{1}>0,\qquad \textit{if } n \textit{ is even}$$

then all the zeros of $T(q)$ *lie in*

$$\left|q+\frac{a_{n-1}}{a_{n}}\right| \leqslant\frac{1}{|a_{n}|}\left[|\alpha_{n}+\alpha_{n-1}+2\sum_{v=0}^{n-2}\left(|\beta_{v}|+|\gamma_{v}|+|\delta_{v}|\right)+|\beta_{n}|+|\beta_{n-1}|+|\gamma_{n}|+|\gamma_{n-1}|+|\delta_{n}|+|\delta_{n-1}|\right].$$

The quaternionic counterparts of the Eneström-Kakeya theorem and its various forms are discussed in recent works such as [1], [15], [16], [18]–[19], and [22]. This paper aims to further extend several Eneström-Kakeya type results in both the complex and quaternionic domains, while also generalizing some well-known results from the existing quaternionic literature. These extensions are achieved by leveraging a recently established maximum modulus theorem (Theorem 3) and insights into the structure of the zero sets of a regular product of two polynomials (Theorem 1) defined over a quaternionic variable. Through these investigations, the paper presents a range of generalizations of Theorems 8–16.

2. Main results

We begin with the generalisation of Theorem 16. In particular, we show that this result gives the quaternionic analogue of Theorem 9.

THEOREM 17. *Let* $T(q)=\sum_{v=0}^{n} q^{v}a_{v}$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k,\ v=0,1,2,\dots,n$ *such that for some non-negative real numbers* $\lambda,\ \tau,\ \kappa,\ \sigma$

$$\alpha_n+\lambda\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_3\geqslant\alpha_1-\tau,$$

and

$$\alpha_{n-1}+\kappa\geqslant\alpha_{n-3}\geqslant\ldots\geqslant\alpha_2\geqslant a_0-\rho,\qquad \text{if } n \text{ is odd}$$

or

$$\alpha_n+\lambda\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_2\geqslant\alpha_0-\tau,$$

and

$$\alpha_{n-1}+\kappa\geqslant\alpha_{n-3}\geqslant\ldots\geqslant\alpha_3\geqslant\alpha_1-\rho,\qquad \text{if } n \text{ is even}$$

then all the zeros of $T(q)$ *lie in*

$$\begin{aligned}&\left|q+\frac{a_{n-1}}{a_n}\right|\\&\leqslant\frac{1}{|a_n|}\Bigg[|\alpha_0|-\alpha_0+|\alpha_1|-\alpha_1+2(\lambda+\tau+\kappa+\rho)+\alpha_n+\alpha_{n-1}\\&\quad+2\sum_{v=0}^{n-2}\big(|\beta_v|+|\gamma_v|+|\delta_v|\big)+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|\Bigg].\end{aligned}$$

Proof. Consider the product

$$\begin{aligned}T(q)*(1-q^2)&=a_0+qa_1+q^2(a_2-a_0)+q^3(a_3-a_1)\ldots+q^{n-1}(a_{n-1}-a_{n-3})\\&\quad+q^n(a_n-a_{n-2})-q^{n+1}a_{n-1}-q^{n+2}a_n\\&=\phi(q)-q^{n+1}a_{n-1}-q^{n+2}a_n,\end{aligned}$$

where

$$\phi(q) = a_0 + qa_1 + q^2(a_2 - a_0) + q^3(a_3 - a_1)\ldots + q^{n-1}(a_{n-1} - a_{n-3}) + q^n(a_n - a_{n-2}).$$

For $|q| = 1$, we get

$$\begin{aligned}
&|\phi(q)| \\
&= |a_0 + qa_1 + q^2(a_2 - a_0) + q^3(a_3 - a_1)\ldots + q^{n-1}(a_{n-1} - a_{n-3}) + q^n(a_n - a_{n-2})| \\
&\leqslant |a_0| + |a_1| + \sum_{v=2}^{n} |a_v - a_{v-1}| \\
&\leqslant |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |\alpha_1| + |\beta_1| + |\gamma_1| + |\delta_1| \\
&\quad + \sum_{v=2}^{n} |\alpha_v - \alpha_{v-1}| + \sum_{v=2}^{n} \left(|\beta_v| + |\beta_{v-1}|\right) + \sum_{v=2}^{n} \left(|\gamma_v| + |\gamma_{v-1}|\right) + \sum_{v=2}^{n} \left(|\delta_v| + |\delta_{v-1}|\right) \\
&= |\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + |\alpha_1| + |\beta_1| + |\gamma_1| + |\delta_1| + |\alpha_2 - \alpha_0 + \rho - \rho| \\
&\quad + |\alpha_3 - \alpha_1 + \tau - \tau| + |\alpha_4 - \alpha_2| + \ldots + |\alpha_{n-2} - \alpha_{n-4}| + |\alpha_{n-1} - \alpha_{n-3} + \kappa - \kappa| \\
&\quad + |\alpha_n - \alpha_{n-2} + \lambda - \lambda| + |\beta_2| + |\beta_0| + |\beta_3| + |\beta_1| + \ldots + |\beta_{n-1}| + |\beta_{n-3}| + |\beta_n| \\
&\quad + |\beta_{n-2}| + |\gamma_2| + |\gamma_0| + |\gamma_3| + |\gamma_1| + \ldots + |\gamma_{n-1}| + |\gamma_{n-3}| + |\gamma_n| + |\gamma_{n-2}| \\
&\quad + |\delta_2| + |\delta_0| + |\delta_3| + |\delta_1| + \ldots + |\delta_{n-1}| + |\delta_{n-3}| + |\delta_n| + |\delta_{n-2}| \\
&\leqslant |\alpha_0| + |\alpha_1| + |\alpha_2 - (\alpha_0 - \rho)| + \rho + |\alpha_3 - (\alpha_1 - \tau)| + \tau + |\alpha_4 - \alpha_2| + \ldots \\
&\quad + |\alpha_{n-2} - \alpha_{n-4}| + |(\kappa + \alpha_{n-1}) - \alpha_{n-3}| + \kappa + |(\lambda + \alpha_n) - \alpha_{n-2}| + \lambda \\
&\quad + 2\sum_{v=0}^{n-2} \left(|\beta_v| + |\gamma_v| + |\delta_v|\right)| + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| \\
&= |\alpha_0| + |\alpha_1| + \alpha_2 - \alpha_0 + \rho + \rho + \alpha_3 - \alpha_1 + \tau + \tau + \alpha_4 - \alpha_2 + \ldots + \alpha_{n-2} - \alpha_{n-4} \\
&\quad + \kappa + \alpha_{n-1} - \alpha_{n-3} + \kappa + \lambda + \alpha_n - \alpha_{n-2} + \lambda + 2\sum_{v=0}^{n-1} \left(|\beta_v| + |\gamma_v| + |\delta_v|\right)| + |\beta_n| \\
&\quad + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}| \\
&= |\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda + \tau + \kappa + \rho) + \alpha_n + \alpha_{n-1} + 2\sum_{v=0}^{n-2} \left(|\beta_v| + |\gamma_v| + |\delta_v|\right) \\
&\quad + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|.
\end{aligned}$$

Notice that

$$\begin{aligned}
\max_{|q|=1} \left| q^n * \phi\left(\frac{1}{q}\right) \right| &= \max_{|q|=1} \left| q^n \phi\left(\frac{1}{q}\right) \right| = \max_{|q|=1} \left| \phi\left(\frac{1}{q}\right) \right| \\
&= \max_{|q|=1} |\phi(q)|,
\end{aligned}$$

it follows that $q^n * \phi(1/q)$ has the same bound on $|q| = 1$ as ϕ i.e.,

$$\begin{aligned}\left|q^n * \phi\left(\frac{1}{q}\right)\right| = \left|q^n \phi\left(\frac{1}{q}\right)\right| &\leqslant |\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda+\tau+\kappa+\rho) + \alpha_n + \alpha_{n-1}\\ &\quad + 2\sum_{v=0}^{n-2}\left(|\beta_v| + |\gamma_v| + |\delta_v|\right) + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|,\\ &\qquad \text{for } |q| = 1.\end{aligned}$$

Since $q^n * \phi(1/q)$ is a polynomial and so is regular in $|q| \leqslant 1$, it follows by the Maximum Modulus Theorem 3, that

$$\begin{aligned}&\left|q^n \phi\left(\frac{1}{q}\right)\right|\\ &\leqslant |\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda+\tau+\kappa+\rho) + \alpha_n + \alpha_{n-1} + 2\sum_{v=0}^{n-2}\left(|\beta_v| + |\gamma_v| + |\delta_v|\right)\\ &\quad + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|, \quad \text{for} \quad |q| \leqslant 1.\end{aligned}$$

Replacing q by $1/q$, we have for $|q| \geqslant 1$,

$$\begin{aligned}|\phi(q)| \leqslant |q|^n \Bigg[& |\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda+\tau+\kappa+\rho) + \alpha_n + \alpha_{n-1}\\ & + 2\sum_{v=0}^{n-2}\left(|\beta_v| + |\gamma_v| + |\delta_v|\right) + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|\Bigg].\end{aligned}$$

Thus, for $|q| \geqslant 1$, we have

$$\begin{aligned}&|T(q) * (1-q)|\\ &= |\phi(q) - q^{n+1}a_{n-1} - q^{n+2}a_n|\\ &\geqslant |q|^n \Bigg\{ |q||qa_n + a_{n-1}| - \Bigg[|\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda+\tau+\kappa+\rho) + \alpha_n + \alpha_{n-1}\\ &\quad + 2\sum_{v=0}^{n-2}\left(|\beta_v| + |\gamma_v| + |\delta_v|\right) + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|\Bigg]\Bigg\}.\end{aligned}$$

Hence, if

$$\begin{aligned}&\left|q + \frac{a_{n-1}}{a_n}\right|\\ &> \frac{1}{|a_n|}\Bigg[|\alpha_0| - \alpha_0 + |\alpha_1| - \alpha_1 + 2(\lambda+\tau+\kappa+\rho) + \alpha_n + \alpha_{n-1}\\ &\quad + 2\sum_{v=0}^{n-2}\left(|\beta_v| + |\gamma_v| + |\delta_v|\right) + |\beta_n| + |\beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}| + |\delta_n| + |\delta_{n-1}|\Bigg],\end{aligned}$$

then $|T(q)*(1-q)|>0$, that is $T(q)*(1-q)\neq 0$. Since by Theorem 1, the only zeros of $T(q)*(1-q)$ are $q=1$ and the zeros of $T(q)$, therefore, $T(q)\neq 0$ for

$$\begin{aligned}&\left|q+\frac{a_{n-1}}{|a_n|}\right|\\&>\frac{1}{|a_n|}\Big[|\alpha_0|-\alpha_0+|\alpha_1|-\alpha_1+2(\lambda+\tau+\kappa+\rho)+\alpha_n+\alpha_{n-1}\\&\quad+2\sum_{v=0}^{n-2}\left(|\beta_v|+|\gamma_v|+|\delta_v|\right)+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|\Big].\end{aligned}$$

Thus all zeros of $T(q)$ lie in

$$\begin{aligned}&\left|q+\frac{a_{n-1}}{a_n}\right|\\&\leqslant\frac{1}{|a_n|}\Big[|\alpha_0|-\alpha_0+|\alpha_1|-\alpha_1+2(\lambda+\tau+\kappa+\rho)+\alpha_n+\alpha_{n-1}\\&\quad+2\sum_{v=0}^{n-2}\left(|\beta_v|+|\gamma_v|+|\delta_v|\right)+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|\Big].\end{aligned}$$

The proof for even n follows in the same way, which completes the proof of Theorem 17. □

REMARK 1. By taking $\beta_v=\gamma_v=\delta_v=0$, $v=0,1,2,\dots,n$ in Theorem 17, we get the following result.

COROLLARY 1. *Let* $T(q)=\sum_{v=0}^{n}q^v a_v$ *be a polynomial of degree* n*, with real coefficients* a_v*,* $v=0,1,2,\dots,n$ *such that for some non-negative real numbers* λ*,* τ*,* κ*,* σ

$$a_n+\lambda\geqslant a_{n-2}\geqslant\ldots\geqslant a_3\geqslant a_1-\tau,$$

and

$$a_{n-1}+\kappa\geqslant a_{n-3}\geqslant\ldots\geqslant a_2\geqslant a_0-\rho,\qquad \textit{if } n \textit{ is odd}$$

or

$$a_n+\lambda\geqslant a_{n-2}\geqslant\ldots\geqslant a_2\geqslant a_0-\tau,$$

and

$$a_{n-1}+\kappa\geqslant a_{n-3}\geqslant\ldots\geqslant a_3\geqslant a_1-\rho,\qquad \textit{if } n \textit{ is even}$$

then all the zeros of $T(q)$ *lie in*

$$\left|q+\frac{a_{n-1}}{a_n}\right|\leqslant\frac{1}{|a_n|}\Big[|a_0|-a_0+|a_1|-a_1+2(\lambda+\tau+\kappa+\rho)+a_n+a_{n-1}\Big].$$

REMARK 2. Assuming $a_0>0$, $a_1>0$, Corollary 1 ensures the quaternionic analogue of Theorem 9.

REMARK 3. For $\lambda=\tau=\kappa=\sigma=0$ in theorem 17, we get the following result.

COROLLARY 2. *Let* $T(q)=\sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k,\ v=0,1,2,\ldots,n$ *such that*

$$\alpha_n \geqslant \alpha_{n-2} \geqslant \ldots \geqslant \alpha_3 \geqslant \alpha_1,$$

and

$$\alpha_{n-1} \geqslant \alpha_{n-3} \geqslant \ldots \geqslant \alpha_2 \geqslant \alpha_0, \qquad \text{if } n \text{ is odd}$$

or

$$\alpha_n \geqslant \alpha_{n-2} \geqslant \ldots \geqslant \alpha_2 \geqslant \alpha_0,$$

and

$$\alpha_{n-1} \geqslant \alpha_{n-3} \geqslant \ldots \geqslant \alpha_3 \geqslant \alpha_1, \qquad \text{if } n \text{ is even}$$

then all the zeros of $T(q)$ *lie in*

$$\begin{aligned}\left|q+\frac{a_{n-1}}{a_n}\right| \leqslant \frac{1}{|a_n|}\Bigg[&|\alpha_0|-\alpha_0+|\alpha_1|-\alpha_1+\alpha_n+\alpha_{n-1}\\ &+2\sum_{v=0}^{n-2}\left(|\beta_v|+|\gamma_v|+|\delta_v|\right)+|\beta_n|+|\beta_{n-1}|+|\gamma_n|+|\gamma_{n-1}|+|\delta_n|+|\delta_{n-1}|\Bigg].\end{aligned}$$

REMARK 4. For $\alpha_0>0,\ \alpha_1>0$, in Corollary 2, we recover Theorem 16.

REMARK 5. The significance of Theorem 17 lies in its versatility. It applies to a broader class of polynomials and relaxes the conditions required by Theorem 16. For instance, if the monotonicity of the real coefficients is reversed between the first and last components, Theorem 16 cannot provide zero inclusion regions in such cases. To address this limitation and establish non-centric zero inclusion regions under these circumstances, we have developed a framework that accommodates a wider class of polynomials while also generalizing Theorem 16 and other related results. This can be demonstrated through the following example.

EXAMPLE. Consider the polynomial $T(q)=q^5 5+q^4 3+q^3 6+q^2 4+q8+6$. Clearly none of the Theorems 10–16 is applicable to this polynomial. However, the conditions of Theorem 16 seems to be applicable for $\lambda=1,\ \tau=2,\ \kappa=1,\ \sigma=2$ and the zeros of $T(q)$ are observed to lie in $\left|q+0.6\right| \leqslant 4$.

Our next theorem generalizes the Theorem 12 and other related theorems in various directions. In fact, we prove

THEOREM 18. *Let* $T(q)=\sum_{v=0}^{n} q^v a_v$ *be a polynomial of degree* n*, with quaternionic coefficients* $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k,\ v=0,1,2,\ldots,n$ *such that for some real numbers* $k_j \geqslant 1,\ \alpha_{n-j+1}>0,\ j=1,2,\cdots,r$ *where* $1 \leqslant r \leqslant n$*, satisfying*

$$k_1\alpha_n \geqslant k_2\alpha_{n-1} \geqslant k_3\alpha_{n-2} \geqslant \ldots \geqslant k_r\alpha_{n-r+1} \geqslant \alpha_{n-r} \geqslant \ldots \geqslant \alpha_1 \geqslant \alpha_0,$$

then all the zeros of $T(q)$ lie in

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|$$
$$\leqslant \frac{1}{|a_n|}\left[k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-\alpha_0+|\alpha_0|+M_0\right].$$

where

$$M_0=\sum_{v=0}^{n}|(\beta_{v-1}-\beta_v)|+|(\gamma_{v-1}-\gamma_v)|+|(\delta_{v-1}-\delta_v)| \quad \big(\beta_{-1}=\gamma_{-1}=\delta_{-1}=0\big).$$

Proof. Consider the product

$$\begin{aligned}
&T(q)*(1-q)\\
&=a_0+q(a_1-a_0)+\ldots+q^n(a_n-a_{n-1})-q^{n+1}a_n\\
&=\alpha_0+q(\alpha_1-\alpha_0)+\ldots+q^n(\alpha_n-\alpha_{n-1})-q^{n+1}\alpha_n\\
&\quad+\sum_{v=0}^{n}\big[(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k\big]\\
&=\alpha_0+q(\alpha_1-\alpha_0)+\ldots+q^n\Big(k_1\alpha_n-k_2\alpha_{n-1}-(k_1-1)\alpha_n+(k_2-1)\alpha_{n-1}\Big)\\
&\quad-q^{n+1}a_n+\sum_{v=0}^{n}\big[(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k\big]\\
&=\alpha_0+q(\alpha_1-\alpha_0)+\ldots+q^n(k_1\alpha_n-k_2\alpha_{n-1})-q^n\big((k_1-1)\alpha_n-(k_2-1)\alpha_{n-1})\big)\\
&\quad-q^{n+1}a_n+\sum_{v=0}^{n}\big[(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k\big]\\
&=\phi(q)-q^n\big((k_1-1)\alpha_n-(k_2-1)\alpha_{n-1})\big)-q^{n+1}a_n
\end{aligned}$$

where

$$\begin{aligned}
\phi(q)&=\alpha_0+q(\alpha_1-\alpha_0)+\ldots+q^n(k_1\alpha_n-k_2\alpha_{n-1})\\
&\quad+\sum_{v=0}^{n}(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k\\
&=\alpha_0+q(\alpha_1-\alpha_0)+\ldots+q^{n-r}(\alpha_{n-r}-\alpha_{n-r-1})\\
&\quad+q^{n-r+1}\big(k_r\alpha_{n-r+1}-\alpha_{n-r}-(k_r-1)\alpha_{n-r+1}\big)\\
&\quad+q^{n-r+2}\Big(k_{r-1}\alpha_{n-r+2}-k_r\alpha_{n-r+1}-(k_{r-1}-1)\alpha_{n-r+2}+(k_r-1)\alpha_{n-r+1}\Big)\\
&\quad+\ldots+q^{n-1}\Big(k_2\alpha_{n-1}-k_3\alpha_{n-2}-(k_2-1)\alpha_{n-1}+(k_3-1)\alpha_{n-2}\Big)\\
&\quad+q^n(k_1\alpha_n-k_2\alpha_{n-1})+\sum_{v=0}^{n}(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k.
\end{aligned}$$

For $|q| = 1$, we have

$$
\begin{aligned}
&|\phi(q)| \\
&= |\alpha_0| + |\alpha_1 - \alpha_0| + \ldots + |\alpha_{n-r} - \alpha_{n-r-1}| + |k_r\alpha_{n-r+1} - \alpha_{n-r} - (k_r - 1)\alpha_{n-r+1}| \\
&\quad + |k_{r-1}\alpha_{n-r+2} - k_r\alpha_{n-r+1} - (k_{r-1} - 1)\alpha_{n-r+2} + (k_r - 1)\alpha_{n-r+1}| + \ldots \\
&\quad + |k_2\alpha_{n-1} - k_3\alpha_{n-2} - (k_2 - 1)\alpha_{n-1} + (k_3 - 1)\alpha_{n-2}| + |k_1\alpha_n - k_2\alpha_{n-1}| \\
&\left|\sum_{v=0}^{n} (\beta_v - \beta_{v-1})i + (\gamma_v - \gamma_{v-1})j + (\delta_v - \delta_{v-1})k\right| \\
&\leqslant |\alpha_0| + |\alpha_1 - \alpha_0| + \ldots + |\alpha_{n-r} - \alpha_{n-r-1}| + |k_r\alpha_{n-r+1} - \alpha_{n-r} - (k_r - 1)\alpha_{n-r+1}| \\
&\quad + |k_{r-1}\alpha_{n-r+2} - k_r\alpha_{n-r+1} - (k_{r-1} - 1)\alpha_{n-r+2} + (k_r - 1)\alpha_{n-r+1}| + \ldots \\
&\quad + |k_2\alpha_{n-1} - k_3\alpha_{n-2} - (k_2 - 1)\alpha_{n-1} + (k_3 - 1)\alpha_{n-2}| + |k_1\alpha_n - k_2\alpha_{n-1}| + M_0 \\
&\leqslant |\alpha_0| + |\alpha_1 - \alpha_0| + \ldots + |\alpha_{n-r} - \alpha_{n-r-1}| + |k_r\alpha_{n-r+1} - \alpha_{n-r}| + (k_r - 1)|\alpha_{n-r+1}| \\
&\quad + |k_{r-1}\alpha_{n-r+2} - k_r\alpha_{n-r+1}| + (k_{r-1} - 1)|\alpha_{n-r+2}| + (k_r - 1)|\alpha_{n-r+1}| + \ldots \\
&\quad + |k_2\alpha_{n-1} - k_3\alpha_{n-2}| + (k_2 - 1)|\alpha_{n-1}| + (k_3 - 1)|\alpha_{n-2}| + |k_1\alpha_n - k_2\alpha_{n-1}| + M_0 \\
&= |\alpha_0| + \alpha_1 - \alpha_0 + \ldots + \alpha_{n-r} - \alpha_{n-r-1} + k_r\alpha_{n-r+1} - \alpha_{n-r} - (k_r - 1)|\alpha_{n-r+1}| \\
&\quad + k_{r-1}\alpha_{n-r+2} - k_r\alpha_{n-r+1} - (k_{r-1} - 1)|\alpha_{n-r+2}| + (k_r - 1)|\alpha_{n-r+1}| + \ldots \\
&\quad + k_2\alpha_{n-1} - k_3\alpha_{n-2} + (k_2 - 1)|\alpha_{n-1}| + (k_3 - 1)|\alpha_{n-2}| + k_1\alpha_n - k_2\alpha_{n-1} + M_0 \\
&= k_1\alpha_n + (k_2 - 1)|\alpha_{n-1}| + 2\sum_{j=2}^{r} (k_j - 1)|\alpha_{n-j+1}| - \alpha_0 + |\alpha_0| + M_0.
\end{aligned}
$$

As in the proof of Theorem 17, we have for $|q| \geqslant 1$,

$$
|\phi(q)| \leqslant |q|^n \left[k_1 a_n + (k_2 - 1)|\alpha_{n-1}| + 2\sum_{j=2}^{r} (k_j - 1)|\alpha_{n-j+1}| - \alpha_0 + |\alpha_0| + M_0 \right].
$$

Thus, for $|q| \geqslant 1$, we have

$$
\begin{aligned}
&|T(q) * (1 - q)| \\
&= \left| \phi(q) - q^n \big(q a_n + (k_1 - 1)\alpha_n - (k_2 - 1)\alpha_{n-1} \big) \right| \\
&\geqslant |q^n|a_n| \left| q + (k_1 - 1)\frac{\alpha_n}{a_n} - (k_2 - 1)\frac{\alpha_{n-1}}{a_n} \right| - |\phi(q)| \\
&\geqslant |q^n||a_n| \left| q + (k_1 - 1)\frac{\alpha_n}{a_n} - (k_2 - 1)\frac{\alpha_{n-1}}{a_n} \right| \\
&\quad - \left[k_1\alpha_n + (k_2 - 1)|a_{n-1}| + 2\sum_{j=2}^{r} (k_j - 1)|\alpha_{n-j+1}| - \alpha_0 + |\alpha_0| + M_0 \right].
\end{aligned}
$$

Hence, if

$$|a_n|\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|$$
$$>\left[k_1a_n+(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|\alpha_{n-j+1}|-\alpha_0+|\alpha_0|+M_0\right],$$

then $|T(q)*(1-q)|>0$. Equivalently, if

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{a_{n-1}}{a_n}\right|$$
$$>\frac{1}{|a_n|}\left[k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|\alpha_{n-j+1}|-\alpha_0+|\alpha_0|+M_0\right],$$

then $|T(q)*(1-q)|>0$, that is $T(q)*(1-q)\neq 0$. By Theorem 1, the only zeros of $T(q)*(1-q)$ are $q=1$ and the zeros of $T(q)$, therefore, $T(q)\neq 0$ for

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|$$
$$>\frac{1}{|a_n|}\left[k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-\alpha_0+|\alpha_0|+M_0\right]$$

In other words, all zeros of $T(q)$ lie in

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|$$
$$>\frac{1}{|a_n|}\left[k_1\alpha_n+(k_2-1)|\alpha_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-\alpha_0+|\alpha_0|+M_0\right],$$

which completes the proof of Theorem 18. $\square$

The significance of Theorem 18 can be demonstrated through the following example.

EXAMPLE. Consider the polynomial $T(q)=q^3 10+q^2 10+1$. Clearly the Theorem 12 and other mentioned results are not applicable to this polynomial. However, the hypothesis of Theorem 18 seems to be applicable for $\beta_v=\gamma_v=\delta_v=0$, $r=2$, $k_1=\frac{16}{15}$, $k_2=\frac{8}{7}$, then the zeros of $T(q)$ are observed to lie in $\left|q-0.076\right|\leqslant 1.2$

REMARK 6. For $\beta_v=\gamma_v=\delta_v=0$, $v=0,1,2,\dots,n$ in Theorem 18, we get the following result.

COROLLARY 3. *Let $T(q)=\sum_{v=0}^{n}q^v a_v$ be a polynomial of degree n, with real coefficients a_v, $v=0,1,2,\dots,n$ such that for some real numbers $k_j\geqslant 1$, $1\leqslant j\leqslant r$ where $1\leqslant r\leqslant n$, satisfying*

$$k_1a_n\geqslant k_2a_{n-1}\geqslant k_3a_{n-2}\geqslant\dots\geqslant k_ra_{n-r+1}\geqslant\alpha_{n-r}\geqslant\dots\geqslant a_1\geqslant a_0,$$

then all the zeros of $T(q)$ lie in

$$\left|q+k_1-1-(k_2-1)\frac{a_{n-1}}{a_n}\right| \leqslant \frac{1}{|a_n|}\left[k_1a_n+(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\right].$$

REMARK 7. By taking $k_j=1,\ j=1,2,\ldots r$, Corollary 3 subsumes Theorem 12.

For $r=2$ in Theorem 18, we get the following result.

COROLLARY 4. *Let $T(q)=\sum_{v=0}^{n}q^va_v$ be a polynomial of degree n, with quaternionic coefficients $a_v=\alpha_v+\beta_vi+\gamma_vj+\delta_vk,\ v=0,1,2,\ldots,n$ such that for some real numbers $k_1\geqslant 1,\ k_2\geqslant 1$*

$$k_1\alpha_n\geqslant k_2\alpha_{n-1}\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_1\geqslant\alpha_0,$$

then all the zeros of $T(q)$ lie in

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|\leqslant\frac{1}{|a_n|}\left[k_1\alpha_n+(k_2-1)|\alpha_{n-1}|-\alpha_0+|\alpha_0|+M_0\right].$$

REMARK 8. Taking $k_2=1$ in Corollary 4, we get the following generalisation of Theorem 12.

COROLLARY 5. *Let $T(q)=\sum_{v=0}^{n}q^va_v$ be a polynomial of degree n, with quaternionic coefficients $a_v=\alpha_v+\beta_vi+\gamma_vj+\delta_vk,\ v=0,1,2,\ldots,n$ such that for some real number $k_1\geqslant 1$,*

$$k_1\alpha_n\geqslant\alpha_{n-1}\geqslant\alpha_{n-2}\geqslant\ldots\geqslant\alpha_1\geqslant\alpha_0,$$

then all the zeros of $T(q)$ lie in

$$\left|q+(k_1-1)\frac{\alpha_n}{a_n}\right|\leqslant\frac{1}{|a_n|}\left[k_1\alpha_n-\alpha_0+|\alpha_0|+M_0\right].$$

REMARK 9. For $\beta_v=\gamma_v=\delta_v=0,\ v=0,1,2,\ldots,n$, Corollary 5 reduces to Theorem 15. In addition, if we take $\alpha_0>0$ and $k_1=1$, Corollary 5 gives Theorem 10.

It is easy to see that $M_0\leqslant 2\sum_{v=0}^{n}\left(|\beta_v|+|\gamma_v|+|\delta_v|\right)$, using this and taking $\alpha_0>0$ in Corollary 5, we get Theorem 14.

Applying Theorem 18 to polynomial $T(tq)$, we obtain the following result.

COROLLARY 6. *Let $T(q)=\sum_{v=0}^{n} q^v a_v$ be a polynomial of degree n, with quaternionic coefficients $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k,\ v=0,1,2,\dots,n$ such that for some reals $t>0$ and $k_j\geqslant 1,\ 1\leqslant j\leqslant n$, where $1\leqslant r\leqslant n$, satisfying*

$$\begin{aligned} t^n k_1\alpha_n &\geqslant t^{n-1}k_2\alpha_{n-1}\geqslant t^{n-2}k_3\alpha_{n-2}\\ &\geqslant \dots \geqslant t^{n-r+1}k_r\alpha_{n-r+1}\geqslant t^{n-r}\alpha_{n-r}\geqslant\dots\geqslant t\alpha_1\geqslant\alpha_0,\end{aligned}$$

then all the zeros of $T(q)$ lie in

$$\begin{aligned}&\left|q+t(k_1-1)\frac{\alpha_n}{a_n}-(k_2-1)\frac{\alpha_{n-1}}{a_n}\right|\\ &\leqslant \frac{1}{|a_n|}\left[tk_1\alpha_n+(k_2-1)|\alpha_{n-1}|+2\sum_{j=2}^{r}(k_j-1)\frac{|a_{n-j+1}|}{t^{j-2}}+\frac{|\alpha_0|-\alpha_0}{t^{n-1}}+M_0\right].\end{aligned}$$

Taking $r=2,\ \alpha_0>0$, and $\beta_v=\gamma_v=\delta_v=0,\ v=0,1,2,\dots,n$, in Corollary 6, we obtain the following result.

COROLLARY 7. *Let $T(q)=\sum_{v=0}^{n} q^v a_v$ be a polynomial of degree n, with real positive coefficients $a_v,\ v=0,1,2,\dots,n$ such that for some reals $t>0$ and $k_j\geqslant 1,\ j=1,2$, where $1\leqslant r\leqslant n$, satisfying*

$$\begin{aligned} t^n k_1 a_n &\geqslant t^{n-1}k_2 a_{n-1}\geqslant t^{n-2}a_{n-2}\\ &\geqslant \dots \geqslant t^{n-r+1}a_{n-r+1}\geqslant t^{n-r}a_{n-r}\geqslant\dots\geqslant ta_1\geqslant a_0>0,\end{aligned}$$

then all the zeros of $T(q)$ lie in

$$\left|q+t(k_1-1)-(k_2-1)\frac{a_{n-1}}{a_n}\right|\leqslant tk_1+(k_2-1)\frac{a_{n-1}}{a_n}.$$

REMARK 10. For different choices of t, k_1, k_2, several interesting results can be obtained.

Finally, we establish a zero-free region in the form of a disk, not centered at the origin, for a relevant class of power series within the ball $B(0,R),\ R>0$. This result generalizes Theorem 13 and also provides the quaternionic counterpart of Theorem 7.

THEOREM 19. *Let $f:B(0,R)\to\mathbb{H}$ be a regular power series in the quaternionic variable q, i.e., $f(q)=\sum_{v=0}^{\infty} q^v a_v$, for all $q\in B(0,R)$. If $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$, $v=0,1,2,\dots,n$ are quaternionic coefficients such that for some reals $k_1\geqslant 1,\ k_2\geqslant 1$*

$$k_1\alpha_0\geqslant k_2\alpha_1\geqslant\alpha_2\geqslant\dots,$$

then $f(q)$ does not vanish in

$$\left|q-\frac{(k_1-1)\alpha_0-(k_2-1)\alpha_1}{(2k_1-1)(\alpha_0+2(k_2-1)\alpha_1)}\right|<\frac{k_1\alpha_0+(k_2-1)\alpha_1+M}{(2k_1-1)(\alpha_0+2(k_2-1)\alpha_1)},$$

where

$$M=\sum_{v=0}^{\infty}|(\beta_{v-1}-\beta_v)|+|(\gamma_{v-1}-\gamma_v)|+|(\delta_{v-1}-\delta_v)| \quad \big(\beta_{-1}=\gamma_{-1}=\delta_{-1}=0\big).$$

Proof. Consider the product

$$\begin{aligned}
F(q)&=f(q)*(1-q)\\
&=((a_0+qa_1+q^2a_2+\ldots)(1-q)\\
&=\sum_{v=0}^{\infty}q^v(a_{v-1}-a_v) \qquad (a_{-1}=0)\\
&=\sum_{v=0}^{\infty}q^v(\alpha_{v-1}-\alpha_v)+\sum_{v=0}^{\infty}q^v\big[(\beta_{v-1}-\beta_v)i+(\gamma_{v-1}-\gamma_v)j+(\delta_{v-1}-\delta_v)k\big]\\
&=-\alpha_0+q(k_1\alpha_0-k_2\alpha_1)+q^2\big((1-k_1)\alpha_0\\
&\quad+(k_2-1)\alpha_1\big)q+\big((k_2\alpha_1-\alpha_2)-(k_2-1)\alpha_1\big)\\
&\quad+\sum_{v=3}^{\infty}q^v(\alpha_{v-1}-\alpha_v)+\sum_{v=0}^{\infty}q^v\big[(\beta_{v-1}-\beta_v)i+(\gamma_{v-1}-\gamma_v)j+(\delta_{v-1}-\delta_v)k\big]\\
&=-\alpha_0-q\big((k_1-1)\alpha_0-(k_2-1)\alpha_1\big)+\phi(q),
\end{aligned}$$

where

$$\begin{aligned}
\phi(q)=q(k_1\alpha_0-k_2\alpha_1)+q^2\big((k_2\alpha_1-\alpha_2)-(k_2-1)\alpha_1\big)+\sum_{v=3}^{\infty}q^v(\alpha_{v-1}-\alpha_v)\\
+\sum_{v=0}^{\infty}q^v\big[(\beta_{v-1}-\beta_v)i+(\gamma_{v-1}-\gamma_v)j+(\delta_{v-1}-\delta_v)k\big].
\end{aligned}$$

Clearly $\phi(q)$ is regular in $|q|\leqslant 1$ and $|\phi(0)|=0$. Moreover, for $|q|=1$, we have

$$\begin{aligned}
|\phi(q)|&\leqslant|k_1\alpha_0-k_2\alpha_1|+|(k_2\alpha_1-\alpha_2)-(k_2-1)\alpha_1|+\sum_{v=3}^{\infty}|\alpha_{v-1}-\alpha_v|\\
&\quad+\sum_{v=0}^{\infty}\big[|\beta_{v-1}-\beta_v|+|\gamma_{v-1}-\gamma_v|+|\delta_{v-1}-\delta_v|\big]\\
&\leqslant k_1\alpha_0-k_2\alpha_1+|k_2\alpha_1-\alpha_2|+(k_2-1)|\alpha_1|+\sum_{v=3}^{\infty}(\alpha_{v-1}-\alpha_v)+M\\
&=k_1\alpha_0-k_2\alpha_1+k_2\alpha_1-\alpha_2+(k_2-1)\alpha_1+\sum_{v=3}^{\infty}(\alpha_{v-1}-\alpha_v)+M\\
&=k_1\alpha_0+(k_2-1)\alpha_1+M,
\end{aligned}$$

therefore by using Schwarz's lemma Theorem 4

$$|\phi(q)| \leqslant |q|(k_1\alpha_0 + (k_2-1)|\alpha_1| + M), \qquad |q| < 1.$$

Hence, for $|q| < 1$

$$|F(q)| \geqslant |\alpha_0 + q\big((k_1-1)\alpha_0 - (k_2-1)\alpha_1\big)| - |q|\big(k_1\alpha_0 + (k_2-1)\alpha_1 + M\big) > 0,$$

if

$$\big|q\big((k_1-1)\alpha_0 - (k_2-1)\alpha_1\big) + \alpha_0\big| > |q|\big(k_1\alpha_0 + (k_2-1)\alpha_1 + M\big),$$

that is, $F(q)$ and therefore by Theorem 2, $f(q)$ does not vanish in

$$|q|\big(k_1\alpha_0 + (k_2-1)\alpha_1 + M\big) < \big|q\big((k_1-1)\alpha_0 - (k_2-1)\alpha_1\big) + \alpha_0\big|.$$

Set

$$A = k_1\alpha_0 + (k_2-1)\alpha_1 + M,$$
$$B = (k_1-1)\alpha_0 - (k_2-1)\alpha_1,$$
$$C = \alpha_0.$$

Then the given inequality becomes

$$A|q| < |Bq + C|.$$

Squaring both sides, we obtain

$$A^2|q|^2 < |Bq + C|^2.$$

Using the identity

$$|w|^2 = w\overline{w},$$

we expand:

$$|Bq + C|^2 = (Bq + C)\overline{(Bq + C)}.$$

Since $B, C \in \mathbb{R}$,

$$|Bq + C|^2 = (Bq + C)(B\overline{q} + C).$$

Multiplying,

$$|Bq + C|^2 = B^2 q\overline{q} + BC(q + \overline{q}) + C^2.$$

Since

$$q\overline{q} = |q|^2,$$

and

$$q + \overline{q} = 2\,\mathrm{Re}(q),$$

we get

$$|Bq + C|^2 = B^2|q|^2 + 2BC\,\mathrm{Re}(q) + C^2.$$

Hence,
$$A^2|q|^2 < B^2|q|^2 + 2BC\,\mathrm{Re}(q) + C^2.$$

Rearranging,
$$(A^2 - B^2)|q|^2 - 2BC\,\mathrm{Re}(q) - C^2 < 0.$$

Now compute
$$A^2 - B^2 = (A-B)(A+B).$$

We have
$$A - B = \alpha_0 + 2(k_2-1)\alpha_1 + M,$$
and
$$A + B = (2k_1 - 1)\alpha_0 + M.$$

Thus,
$$A^2 - B^2 = (\alpha_0 + 2(k_2-1)\alpha_1 + M)((2k_1-1)\alpha_0 + M).$$

Assuming positivity conditions, divide by $A^2 - B^2$:
$$|q|^2 - \frac{2BC}{A^2-B^2}\,\mathrm{Re}(q) < \frac{C^2}{A^2-B^2}.$$

Let
$$\beta = \frac{BC}{A^2-B^2}.$$

Then
$$|q|^2 - 2\beta\,\mathrm{Re}(q) < \frac{C^2}{A^2-B^2}.$$

Adding β^2 to both sides,
$$|q|^2 - 2\beta\,\mathrm{Re}(q) + \beta^2 < \frac{C^2}{A^2-B^2} + \beta^2.$$

Using
$$|q-\beta|^2 = |q|^2 - 2\beta\,\mathrm{Re}(q) + \beta^2,$$
we get
$$|q-\beta|^2 < \frac{C^2}{A^2-B^2} + \left(\frac{BC}{A^2-B^2}\right)^2.$$

Simplifying,
$$|q-\beta|^2 < \frac{A^2C^2}{(A^2-B^2)^2}.$$

Taking square roots,
$$|q-\beta| < \frac{AC}{A^2-B^2}.$$

Substituting the values of A, B, C, we obtain
$$\left| q - \frac{(k_1-1)\alpha_0 - (k_2-1)\alpha_1}{(2k_1-1)(\alpha_0 + 2(k_2-1)\alpha_1)} \right| < \frac{k_1\alpha_0 + (k_2-1)\alpha_1 + M}{(2k_1-1)(\alpha_0 + 2(k_2-1)\alpha_1)}.$$

Hence the inequality represents an open disc.

$$\left| q - \frac{(k_1-1)\alpha_0 - (k_2-1)\alpha_1}{(2k_1-1)(\alpha_0+2(k_2-1)\alpha_1)} \right| < \frac{k_1\alpha_0 + (k_2-1)\alpha_1 + M}{(2k_1-1)(\alpha_0+2(k_2-1)\alpha_1)},$$

which completes the proof of Theorem 19. □

REMARK 11. For $\beta_v = \gamma_v = \delta_v = 0$, $\alpha_v > 0$, $v = 0,1,2,\dots$ in Theorem 19, we get the following result.

COROLLARY 8. *$f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i,e., $f(q) = \sum_{v=0}^{\infty} q^v a_v$, for all $q \in B(0,R)$. If a_v, $v = 0,1,2,\dots,n$ are positive real coefficients such that for $k_1 \geqslant 1$, $k_2 \geqslant 1$*

$$k_1 a_0 \geqslant k_2 a_1 \geqslant a_2 \geqslant \dots,$$

then $f(q)$ does not vanish in

$$\left| q - \frac{(k_1-1)a_0 - (k_2-1)a_1}{(2k_1-1)(a_0+2(k_2-1)a_1)} \right| < \frac{k_1 a_0 + (k_2-1)a_1}{(2k_1-1)(a_0+2(k_2-1)a_1)}.$$

Applying Theorem 19 to $f(tq)$, we obtain the following result.

COROLLARY 9. *Let $f : B(0,R) \to \mathbb{H}$ be a regular power series in the quaternionic variable q, i,e., $f(q) = \sum_{v=0}^{\infty} q^v a_v$, for all $q \in B(0,R)$. If $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$, $v = 0,1,2,\dots,n$ are quaternionic coefficients such that for some positive reals t, $0 < t < R$, $k_1 \geqslant 1$ and $k_2 \geqslant 1$*

$$k_1\alpha_0 \geqslant k_2 t\alpha_1 \geqslant t^2\alpha_2 \geqslant \dots,$$

then $f(q)$ does not vanish in

$$\left| q - \frac{(k_1-1)t\alpha_0 - (k_2-1)t^2\alpha_1}{(2k_1-1)(\alpha_0+2(k_2-1)t\alpha_1)} \right| < \frac{k_1 t\alpha_0 + (k_2-1)t^2\alpha_1 + M}{(2k_1-1)(\alpha_0+2(k_2-1)t\alpha_1)}.$$

REMARK 12. For $\beta_v = \gamma_v = \delta_v = 0$, $\alpha_v > 0$, $v = 0,1,2,\dots$ and $k_2 = 1$, Corollary 9 subsumes Theorem 13.

REMARK 13. The essence of Theorem 19 lies in its ability to offer an equivalent version of the Eneström-Kakeya theorem for regular power series. It gives an estimate of a zero-free region in the form of a non-central disc for a regular power series when its coefficients are subjected to certain conditions.

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(Received October 21, 2025)

Tahir Fayaz
Department of Mathematics
University of Kashmir
Srinagar, 190006, India
e-mail: dtahir272@gmail.com

Firdose Ahmad
Department of Mathematics
University of Kashmir
Srinagar, 190006, India
e-mail: firdosewagay99@gmail.com