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BEREZIN NUMBER AND BEREZIN NORM INEQUALITIES: NEW AND IMPROVED RESULTS FOR $n \times n$ OPERATOR MATRICES

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Abstract. Let $T = [T_{jk}]$ be an $n \times n$ operator matrix with $T_{jk} \in \mathscr{CR}(\mathbb{H}_k, \mathbb{H}_j)$, for all $j,k = 1,2,\cdots,n$. In this article, we obtain new and improved upper bounds for Berezin number and Berezin norm of $n \times n$ operator matrices. In particular, we obtain a bound for 2×2 operator matrices,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\Big(\min\Big\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{12}T_{12}^\dagger + T_{21}^\dagger T_{21}\|_{ber}^{\frac{1}{2}},$$
$$\||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}}\|T_{12}^\dagger T_{12} + T_{21}T_{21}^\dagger\|_{ber}^{\frac{1}{2}}\Big\}\Big),$$

which improves the existing bound. Among other estimates, we obtain strengthened Berezin number estimates for $n \times n$ operator matrices and as an application, a Berezin number bound for closed range operators, namely,

$$\mathbf{ber}(T) \leqslant \left(\frac{1}{4}\||T^*|^4 + |T|^4\|_{ber} + \frac{1}{2}\mathbf{ber}(|T^*|^2|T|^2)\right)^{\frac{1}{4}}\left(\frac{1}{4}\|TT^\dagger + T^\dagger T\|_{ber} + \frac{1}{2}\mathbf{ber}(T^\dagger T^2 T^\dagger)\right)^{\frac{1}{4}},$$

which refines the existing bound. Also, we obtain new upper bound for Berezin norm of 2×2 operator matrices and provide an example to show that it is sharper than the existing bound.

1. Introduction

The Berezin range and Berezin number play a significant role in the study of operators on reproducing kernel Hilbert spaces. Since evaluating these quantities is generally difficult, several authors have proposed various upper bounds to estimate the Berezin number (see [3, 8, 9, 10, 13]). Motivated by these developments, we present new bounds for $n \times n$ operator matrices, which enhance and generalize the known results in the literature. For clarity and completeness, we begin by recalling the essential notations and terminology used throughout the article.

Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators acting on complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot,\cdot \rangle$ and the corresponding norm $\|\cdot\|$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be positive whenever $\langle Tx, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, the Moore-Penrose inverse of T is denoted by $T^\dagger$. If the range of T

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is closed then $T^\dagger \in \mathscr{B}(\mathscr{H})$. The set of all operators with closed range is denoted by $\mathscr{C}R(\mathscr{H})$. For $T \in \mathscr{C}R(\mathscr{H})$, $T^\dagger$ is the unique operator in $\mathscr{B}(\mathscr{H})$ which satisfies the following:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

The operator norm of T, denoted by $\|T\|$, is defined as $\|T\| = \sup\{\|Tx\| : x \in \mathscr{H}, \|x\| = 1\}$. The numerical range and the numerical radius of T are denoted by $W(T)$ and $w(T)$, respectively, and are defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\},$$

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}.$$

Recall that $w(\cdot)$ defines a norm on $\mathscr{B}(\mathscr{H})$ that is equivalent to the usual operator norm. More precisely, for any $T \in \mathscr{B}(\mathscr{H})$,

$$\frac{\|T\|}{2} \leqslant w(T) \leqslant \|T\|.$$

We refer the reader to [1, 7, 11, 19] for an extensive discussion on the numerical radius and associated inequalities.

Let X be a non-empty set. A reproducing kernel Hilbert space (RKHS) $\mathbb{H} = \mathbb{H}(X)$ is a Hilbert space consisting of complex-valued functions defined on a set X such that every point evaluation is a continuous linear functional. In other words, for each $\mu \in X$, the mapping $E_\mu : \mathbb{H} \to \mathbb{C}$ given by $E_\mu(f) = f(\mu)$ is bounded (see [20]). According to the Riesz Representation Theorem, for every $\mu \in X$, there exists a unique vector $k_\mu \in \mathbb{H}$ satisfying

$$E_\mu(f) = \langle f, k_\mu\rangle \quad \text{for all} \quad f \in \mathbb{H}.$$

The collection $\{k_\mu : \mu \in X\}$ is known as the reproducing kernel of $\mathbb{H}$, while the family $\left\{\hat{k}_\mu = \frac{k_\mu}{\|k_\mu\|} : \mu \in X\right\}$ is called the normalized reproducing kernel of $\mathbb{H}$. Given an operator $T \in \mathscr{B}(\mathbb{H})$, the mapping $\widetilde{T} : X \to \mathbb{C}$ defined by $\widetilde{T}(\mu) = \langle T\hat{k}_\mu, \hat{k}_\mu\rangle$, is termed the Berezin symbol of T (see [5, 6]). The Berezin set, the Berezin number and the Berezin norm of T are, respectively, given by

$$\mathbf{Ber}(T) := \left\{\widetilde{T}(\mu) : \mu \in X\right\},$$

$$\mathbf{ber}(T) := \sup\left\{|\widetilde{T}(\mu)| : \mu \in X\right\}$$

and

$$\|T\|_{ber} := \sup\left\{\left|\langle T\hat{k}_\mu, \hat{k}_\nu\rangle\right| : \mu, \nu \in X\right\},$$

as discussed in [2, 17]. It is easy to observe that $\mathbf{Ber}(T) \subseteq W(T)$, $\mathbf{ber}(T) \leqslant w(T)$ and $\mathbf{ber}(T) \leqslant \|T\|_{ber} \leqslant \|T\|$. According to [10, Proposition 2.11], the Berezin norm of a positive operator T equals its Berezin number, that is, $\|T\|_{ber} = \mathbf{ber}(T)$. Note that the mapping $\mathbf{ber}(\cdot):\mathscr{B}(\mathbb{H})\to\mathbb{R}$ is not a norm in general. If the reproducing kernel Hilbert space $\mathbb{H}$ satisfies the "Ber" property, that is, if for all $A,B\in\mathscr{B}(\mathbb{H})$,

$$\widetilde{A}(\mu)=\widetilde{B}(\mu) \quad \text{for all} \quad \mu\in X \implies A=B,$$

then $\mathbf{ber}(\cdot)$ defines a norm on $\mathscr{B}(\mathbb{H})$.

The Berezin symbol serves as an essential analytical tool in the study of operators on RKHS and has been extensively applied to Toeplitz and Hankel operators on Hardy and Bergman spaces. Further details can be found in [16, 17, 18].

Let $\{X_j\}_{j=1}^n$ be nonempty sets and $\mathbb{H}_j=\mathbb{H}(X_j)$ be their respective reproducing kernel Hilbert spaces. Then $\mathbb{H}=\oplus_{j=1}^n\mathbb{H}_j$ forms a reproducing kernel Hilbert space on $X_1\times\cdots\times X_n$, and each $T\in\mathscr{B}(\mathbb{H})$ has a block representation $T=[T_{jk}]$ with $T_{jk}\in\mathscr{B}(\mathbb{H}_k,\mathbb{H}_j)$. Suppose that $\mathscr{CR}(\mathbb{H}_k,\mathbb{H}_j)$ denotes the space of all bounded linear operators with closed range from $\mathbb{H}_j$ to $\mathbb{H}_k$. Berezin number inequalities for $n\times n$ operator matrices have been widely explored by numerous researchers (see[8, 14]). In [8], the authors derived the following Berezin radius bound:

$$\mathbf{ber}\left(\begin{bmatrix}0 & T_{12}\\ T_{21} & 0\end{bmatrix}\right)\leqslant\frac{1}{\sqrt{2}}\left(\max\{\||T_{12}|^2+|T_{21}^*|^2\|_{ber}^{\frac12},\||T_{12}^*|^2+|T_{21}|^2\|_{ber}^{\frac12}\}\right). \quad (1)$$

Also, in [9], it was shown that

$$\mathbf{ber}(T)\leqslant\left(\frac14\||T^*|^4+|T|^4\|_{ber}+\frac12\mathbf{ber}(|T^*|^2|T|^2)\right)^{\frac14}. \quad (2)$$

In this article, we present new and improved Berezin number bounds for $n\times n$ operator matrices. Among several estimates obtained here, one of our main results is the following Berezin number bound for a 2×2 operator matrix. For $T_{jk}\in\mathscr{CR}(\mathbb{H}_k,\mathbb{H}_j),\ \forall j,k=1,2,$ we establish:

$$\mathbf{ber}\left(\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\right)\leqslant\frac12\left(\mathbf{ber}(T_{11})+\mathbf{ber}(T_{22})+\sqrt{(\mathbf{ber}(T_{11})-\mathbf{ber}(T_{22}))^2+t_{12}^2}\right),$$

where $t_{12}=\min\{\||T_{12}|^2+|T_{21}^*|^2\|_{ber}^{\frac12}\|T_{12}T_{12}^\dagger+T_{21}^\dagger T_{21}\|_{ber}^{\frac12},\||T_{12}^*|^2+|T_{21}|^2\|_{ber}^{\frac12}\|T_{12}^\dagger T_{12}+T_{21}T_{21}^\dagger\|_{ber}^{\frac12}\}$. As a special case, we obtain

$$\begin{aligned}&\mathbf{ber}\left(\begin{bmatrix}0 & T_{12}\\ T_{21} & 0\end{bmatrix}\right)\\ &\leqslant\frac12\Big(\min\{\||T_{12}|^2+|T_{21}^*|^2\|_{ber}^{\frac12}\|T_{12}T_{12}^\dagger+T_{21}^\dagger T_{21}\|_{ber}^{\frac12},\\ &\qquad \||T_{12}^*|^2+|T_{21}|^2\|_{ber}^{\frac12}\|T_{12}^\dagger T_{12}+T_{21}T_{21}^\dagger\|_{ber}^{\frac12}\}\Big),\end{aligned}$$

which yields a refinement of the bound in (1). In addition, we obtain the estimate

$$\begin{aligned}\mathbf{ber}(T)\leqslant&\left(\frac14\||T^*|^4+|T|^4\|_{ber}+\frac12\mathbf{ber}(|T^*|^2|T|^2)\right)^{\frac14}\\ &\times\left(\frac14\|TT^\dagger+T^\dagger T\|_{ber}+\frac12\mathbf{ber}(T^\dagger T^2T^\dagger)\right)^{\frac14},\end{aligned}$$

which improves the bound in (2). Furthermore, we derive a new and sharper upper bound for the Berezin norm of 2×2 operator matrices.

2. Berezin number inequalities for operator matrices

We commence this section by establishing a novel Berezin-number inequality for $n\times n$ operator matrices, which, in a particular case, refines the existing bound presented in (1). First, we need the following lemmas.

LEMMA 1. [21] *Let $T\in\mathscr{CR}(\mathscr{H})$. Then for any $x,y\in\mathscr{H}$,*

$$|\langle Tx,y\rangle|^2\leqslant\langle|T|^2x,x\rangle\langle TT^\dagger y,y\rangle.$$

LEMMA 2. [15, p. 44] *Let $T=[t_{jk}]$ be an $n\times n$ complex matrix such that $t_{jk}\geqslant 0$ for all $j,k=1,2,\dots,n$, then $w(T)=w\left(\frac{T+T^*}{2}\right)=r\left(\frac{T+T^*}{2}\right)$, where $r(\cdot)$ denotes the spectral radius.*

THEOREM 1. *Let $\mathbb{H}_j$ $(j=1,2,\cdots,n)$ be reproducing kernel Hilbert spaces, and let $T=[T_{jk}]$ be an $n\times n$ operator matrix with $T_{jk}\in\mathscr{CR}(\mathbb{H}_k,\mathbb{H}_j)$ for all $j,k=1,2,\cdots,n$. Then*

$$\boldsymbol{ber}(T)\leqslant w([t_{jk}]),$$

where

$$t_{jk}=\begin{cases}\boldsymbol{ber}(T_{jj}) & \text{when } j=k\\ \min\Big\{\||T_{jk}|^2+|T_{kj}^*|^2\|_{ber}^{\frac12}\|T_{jk}T_{jk}^\dagger+T_{kj}^\dagger T_{kj}\|_{ber}^{\frac12}, & \\ \quad\||T_{jk}^*|^2+|T_{kj}|^2\|_{ber}^{\frac12}\|T_{jk}^\dagger T_{jk}+T_{kj}T_{kj}^\dagger\|_{ber}^{\frac12}\Big\} & \text{when } j<k\\ 0 & \text{otherwise.}\end{cases}$$

Proof. For $(\mu_1,\dots,\mu_n)\in X_1\times\cdots\times X_n$, let $\hat{k}_{(\mu_1,\dots,\mu_n)}$ be the corresponding normalized reproducing kernel of $\mathbb{H}_1\oplus\cdots\oplus\mathbb{H}_n$. Then we have

$$\begin{aligned}|\langle T\hat{k}_{(\mu_1,\dots,\mu_n)},\hat{k}_{(\mu_1,\dots,\mu_n)}\rangle| &= \Big|\sum_{j,k=1}^n\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle\Big|\\ &\leqslant\sum_{j,k=1}^n|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|\\ &=\sum_{j=1}^n|\langle T_{jj}k_{\mu_j},k_{\mu_j}\rangle|+\sum_{\substack{j,k=1\\ j\neq k}}^n|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|\\ &=\sum_{j=1}^n|\langle T_{jj}k_{\mu_j},k_{\mu_j}\rangle|+\sum_{\substack{j,k=1\\ j<k}}^n\big(|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|+|\langle T_{kj}k_{\mu_j},k_{\mu_k}\rangle|\big).\end{aligned}$$

Now, for all $j<k$,

$$\begin{aligned}
&\left(|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|+|\langle T_{kj}k_{\mu_j},k_{\mu_k}\rangle|\right)\\
&\leqslant \left(\langle |T_{jk}|^2k_{\mu_k},k_{\mu_k}\rangle^{\frac{1}{2}}\langle T_{jk}T_{jk}^{\dagger}k_{\mu_j},k_{\mu_j}\rangle^{\frac{1}{2}}+\langle |T_{kj}^*|^2k_{\mu_k},k_{\mu_k}\rangle^{\frac{1}{2}}\langle T_{kj}^{\dagger}T_{kj}k_{\mu_j},k_{\mu_j}\rangle^{\frac{1}{2}}\right)\\
&\qquad\qquad\text{(by Lemma 1)}\\
&\leqslant \left(\langle |T_{jk}|^2k_{\mu_k},k_{\mu_k}\rangle+\langle |T_{kj}^*|^2k_{\mu_k},k_{\mu_k}\rangle\right)^{\frac{1}{2}}\left(\langle T_{jk}T_{jk}^{\dagger}k_{\mu_j},k_{\mu_j}\rangle+\langle T_{kj}^{\dagger}T_{kj}k_{\mu_j},k_{\mu_j}\rangle\right)^{\frac{1}{2}}\\
&=\langle (|T_{jk}|^2+|T_{kj}^*|^2)k_{\mu_k},k_{\mu_k}\rangle^{\frac{1}{2}}\langle (T_{jk}T_{jk}^{\dagger}+T_{kj}^{\dagger}T_{kj})k_{\mu_j},k_{\mu_j}\rangle^{\frac{1}{2}}\\
&\leqslant \||T_{jk}|^2+|T_{kj}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}T_{jk}^{\dagger}+T_{kj}^{\dagger}T_{kj}\|_{ber}^{\frac{1}{2}}\|k_{\mu_j}\|\|k_{\mu_k}\|. \qquad (3)
\end{aligned}$$

Similarly, for all $j<k$,

$$\left(|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|+|\langle T_{kj}k_{\mu_j},k_{\mu_k}\rangle|\right)\leqslant \||T_{jk}^*|^2+|T_{kj}|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}^{\dagger}T_{jk}+T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}}\|k_{\mu_j}\|\|k_{\mu_k}\|. \qquad (4)$$

Combining (3) and (4), for all $j<k$, we have

$$\begin{aligned}
&\left(|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle|+|\langle T_{kj}k_{\mu_j},k_{\mu_k}\rangle|\right)\\
&\leqslant \min\{\||T_{jk}|^2+|T_{kj}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}T_{jk}^{\dagger}+T_{kj}^{\dagger}T_{kj}\|_{ber}^{\frac{1}{2}},\\
&\quad \||T_{jk}^*|^2+|T_{kj}|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}^{\dagger}T_{jk}+T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}}\}\|k_{\mu_j}\|\|k_{\mu_k}\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|\langle T\hat{k}_{(\mu_1,\dots,\mu_n)},\hat{k}_{(\mu_1,\dots,\mu_n)}\rangle|\\
&\leqslant \sum_{j=1}^{n}\left|\langle T_{jj}k_{\mu_j},k_{\mu_j}\rangle\right|+\sum_{\substack{j,k=1\\ j<k}}^{n}\left(\left|\langle T_{jk}k_{\mu_k},k_{\mu_j}\rangle\right|+\left|\langle T_{kj}k_{\mu_j},k_{\mu_k}\rangle\right|\right)\\
&\leqslant \sum_{j=1}^{n}\left|\langle T_{jj}k_{\mu_j},k_{\mu_j}\rangle\right|+\sum_{\substack{j,k=1\\ j<k}}^{n}\min\Big\{\||T_{jk}|^2+|T_{kj}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}T_{jk}^{\dagger}+T_{kj}^{\dagger}T_{kj}\|_{ber}^{\frac{1}{2}},\\
&\quad \||T_{jk}^*|^2+|T_{kj}|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}^{\dagger}T_{jk}+T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}}\Big\}\|k_{\mu_j}\|\|k_{\mu_k}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\langle T\hat{k}_{(\mu_1,\dots,\mu_n)},\hat{k}_{(\mu_1,\dots,\mu_n)}\rangle|\\
&\leqslant \sum_{j=1}^{n}\mathbf{ber}(T_{jj})\|k_{\mu_j}\|^2+\sum_{\substack{j,k=1\\ j<k}}^{n}\min\{\||T_{jk}|^2+|T_{kj}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}T_{jk}^{\dagger}+T_{kj}^{\dagger}T_{kj}\|_{ber}^{\frac{1}{2}},\\
&\quad \||T_{jk}^*|^2+|T_{kj}|^2\|_{ber}^{\frac{1}{2}}\|T_{jk}^{\dagger}T_{jk}+T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}}\}\|k_{\mu_j}\|\|k_{\mu_k}\|\\
&=\langle \tilde{T}y,y\rangle,
\end{aligned}$$

where $y = \begin{bmatrix} \|k_{\mu_1}\| \\ \|k_{\mu_2}\| \\ \vdots \\ \|k_{\mu_n}\| \end{bmatrix} \in \mathbb{C}^n$ and $\tilde{T} = [t_{jk}]$ is an $n \times n$ complex matrix with

$$t_{jk} = \begin{cases} \mathbf{ber}(T_{jj}) & \text{when } j = k \\ \min\{\||T_{jk}|^2 + |T_{kj}^*|^2\|_{ber}^{\frac{1}{2}} \|T_{jk}T_{jk}^\dagger + T_{kj}^\dagger T_{kj}\|_{ber}^{\frac{1}{2}}, & \\ \quad \||T_{jk}^*|^2 + |T_{kj}|^2\|_{ber}^{\frac{1}{2}} \|T_{jk}^\dagger T_{jk} + T_{kj}T_{kj}^\dagger\|_{ber}^{\frac{1}{2}}\} & \text{when } j < k \\ 0 & \text{otherwise.} \end{cases}$$

Since $\|y\| = 1$, $|\langle T\hat{k}_{(\mu_1,\ldots,\mu_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle| \leqslant w(\tilde{T})$. Taking the supremum over all $(\mu_1, \ldots, \mu_n) \in X^n$, we have $\mathbf{ber}(T) \leqslant w(\tilde{T})$. $\square$

By setting $n = 2$ in Theorem 1, we obtain the following.

COROLLARY 1. *Let $T_{jk} \in \mathscr{CR}(\mathbb{H}_k, \mathbb{H}_j)$ for all $j,k = 1,2$. Then*

$$\boldsymbol{ber}\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\boldsymbol{ber}(T_{11}) + \boldsymbol{ber}(T_{22}) + \sqrt{(\boldsymbol{ber}(T_{11}) - \boldsymbol{ber}(T_{22}))^2 + t_{12}^2}\right),$$

where $t_{12} = \min\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}} \|T_{12}T_{12}^\dagger + T_{21}^\dagger T_{21}\|_{ber}^{\frac{1}{2}}, \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}} \|T_{12}^\dagger T_{12} + T_{21}T_{21}^\dagger\|_{ber}^{\frac{1}{2}}\}$. *In particular, for $T_{11} = T_{22} = 0$, we have*

$$\boldsymbol{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\Big(\min\Big\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}} \|T_{12}T_{12}^\dagger + T_{21}^\dagger T_{21}\|_{ber}^{\frac{1}{2}}, \\ \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}} \|T_{12}^\dagger T_{12} + T_{21}T_{21}^\dagger\|_{ber}^{\frac{1}{2}}\Big\}\Big). \tag{5}$$

Proof. From Theorem 1, we have

$$\begin{aligned} \mathbf{ber}\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) &\leqslant w\left(\begin{bmatrix} \mathbf{ber}(T_{11}) & t_{12} \\ 0 & \mathbf{ber}(T_{22}) \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} \mathbf{ber}(T_{11}) & \frac{t_{12}}{2} \\ \frac{t_{12}}{2} & \mathbf{ber}(T_{22}) \end{bmatrix}\right) \text{ (by Lemma 2)} \\ &= r\left(\begin{bmatrix} \mathbf{ber}(T_{11}) & \frac{t_{12}}{2} \\ \frac{t_{12}}{2} & \mathbf{ber}(T_{22}) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\mathbf{ber}(T_{11}) + \mathbf{ber}(T_{22}) + \sqrt{(\mathbf{ber}(T_{11}) - \mathbf{ber}(T_{22}))^2 + t_{12}^2}\right), \end{aligned}$$

where $t_{12} = \min\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}} \|T_{12}T_{12}^\dagger + T_{21}^\dagger T_{21}\|_{ber}^{\frac{1}{2}}, \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}} \|T_{12}^\dagger T_{12} + T_{21}T_{21}^\dagger\|_{ber}^{\frac{1}{2}}\}$. $\square$

REMARK 1. Note that $\|TT^\dagger + T^\dagger T\|_{ber} \leqslant 2$. Then, we have

$$\begin{aligned}
\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) &\leqslant \frac{1}{2}\Big(\min\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}}\|T_{12}T_{12}^\dagger + T_{21}^\dagger T_{21}\|_{ber}^{\frac{1}{2}}, \\
&\qquad \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}}\|T_{12}^\dagger T_{12} + T_{21}T_{21}^\dagger\|_{ber}^{\frac{1}{2}}\}\Big) \\
&\leqslant \frac{1}{\sqrt{2}}\left(\min\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}}, \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}}\}\right) \\
&\leqslant \frac{1}{\sqrt{2}}\left(\max\{\||T_{12}|^2 + |T_{21}^*|^2\|_{ber}^{\frac{1}{2}}, \||T_{12}^*|^2 + |T_{21}|^2\|_{ber}^{\frac{1}{2}}\}\right). \qquad (6)
\end{aligned}$$

It is easy to observe that the bound in (5) refines the bound in (1).

Next, we derive another Berezin number inequality for $n \times n$ operator matrices that sharpens the bound from [8], supported by a numerical example.

THEOREM 2. *Let $\mathbb{H}_j$ $(j = 1, 2, \cdots, n)$ be reproducing kernel Hilbert spaces, and let $T = [T_{jk}]$ be an $n \times n$ operator matrix with $T_{jk} \in \mathscr{C}\mathscr{R}(\mathbb{H}_k, \mathbb{H}_j)$ for all $j,k = 1, 2, \cdots, n$. Then*

$$\boldsymbol{ber}(T) \leqslant w([t_{jk}]),$$

where

$$t_{jk} = \begin{cases} \boldsymbol{ber}(T_{jj}) & \text{when } j = k \\ \||T_{jk}|^2 + T_{kj}T_{kj}^\dagger\|_{ber}^{\frac{1}{2}} \, \|T_{jk}T_{jk}^\dagger + |T_{kj}|^2\|_{ber}^{\frac{1}{2}} & \text{when } j < k \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $(\mu_1, \ldots, \mu_n) \in X_1 \times \cdots \times X_n$, let $\hat{k}_{(\mu_1,\ldots,\mu_n)}$ be the corresponding normalized reproducing kernel of $\mathbb{H}_1 \oplus \cdots \oplus \mathbb{H}_n$. Then

$$\begin{aligned}
&|\langle T\hat{k}_{(\mu_1,\ldots,\mu_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle| \\
&\leqslant \sum_{j=1}^{n} \left|\langle T_{jj}k_{\mu_j}, k_{\mu_j}\rangle\right| + \sum_{\substack{j,k=1 \\ j<k}}^{n} \left(\left|\langle T_{jk}k_{\mu_k}, k_{\mu_j}\rangle\right| + \left|\langle T_{kj}k_{\mu_j}, k_{\mu_k}\rangle\right|\right) \\
&\leqslant \sum_{j=1}^{n} |\langle T_{jj}k_{\mu_j}, k_{\mu_j}\rangle| + \sum_{\substack{j,k=1 \\ j<k}}^{n} \big(\langle |T_{jk}|^2 k_{\mu_k}, k_{\mu_k}\rangle^{\frac{1}{2}} \langle T_{jk}T_{jk}^\dagger k_{\mu_j}, k_{\mu_j}\rangle^{\frac{1}{2}} \\
&\quad + \langle |T_{kj}|^2 k_{\mu_j}, k_{\mu_j}\rangle^{\frac{1}{2}} \langle T_{kj}T_{kj}^\dagger k_{\mu_k}, k_{\mu_k}\rangle^{\frac{1}{2}}\big) \\
&\leqslant \sum_{j=1}^{n} |\langle T_{jj}k_{\mu_j}, k_{\mu_j}\rangle| + \sum_{\substack{j,k=1 \\ j<k}}^{n} \big(\langle |T_{jk}|^2 k_{\mu_k}, k_{\mu_k}\rangle + \langle T_{kj}T_{kj}^\dagger k_{\mu_k}, k_{\mu_k}\rangle\big)^{\frac{1}{2}} \\
&\quad \times \big(\langle T_{jk}T_{jk}^\dagger k_{\mu_j}, k_{\mu_j}\rangle + \langle |T_{kj}|^2 k_{\mu_j}, k_{\mu_j}\rangle\big)^{\frac{1}{2}}
\end{aligned}$$

$$= \sum_{j=1}^{n} |\langle T_{jj}k_{\mu_j}, k_{\mu_j}\rangle| + \sum_{\substack{j,k=1\\ j<k}}^{n} \langle(|T_{jk}|^2 + T_{kj}T_{kj}^{\dagger})k_{\mu_k}, k_{\mu_k}\rangle^{\frac{1}{2}} \langle(T_{jk}T_{jk}^{\dagger} + |T_{kj}|^2)k_{\mu_j}, k_{\mu_j}\rangle^{\frac{1}{2}}$$

$$\leqslant \sum_{j=1}^{n} |\langle T_{jj}k_{\mu_j}, k_{\mu_j}\rangle| + \sum_{\substack{j,k=1\\ j<k}}^{n} \||T_{jk}|^2 + T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}} \|T_{jk}T_{jk}^{\dagger} + |T_{kj}|^2\|_{ber}^{\frac{1}{2}} \|k_{\mu_j}\|\|k_{\mu_k}\|.$$

Therefore,

$$|\langle T\hat{k}_{(\mu_1,\ldots,\mu_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle|$$
$$\leqslant \sum_{j=1}^{n} \mathbf{ber}(T_{jj})\|k_{\mu_j}\|^2 + \sum_{\substack{j,k=1\\ j<k}}^{n} \||T_{jk}|^2 + T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}} \|T_{jk}T_{jk}^{\dagger} + |T_{kj}|^2\|_{ber}^{\frac{1}{2}} \|k_{\mu_j}\|\|k_{\mu_k}\|$$
$$= \langle \tilde{T}y, y\rangle,$$

where $y = \begin{bmatrix} \|k_{\mu_1}\| \\ \|k_{\mu_2}\| \\ \vdots \\ \|k_{\mu_n}\| \end{bmatrix} \in \mathbb{C}^n$ and $\tilde{T} = [t_{jk}]$ is an $n \times n$ complex matrix with

$$t_{jk} = \begin{cases} \mathbf{ber}(T_{jj}) & \text{when } j = k \\ \||T_{jk}|^2 + T_{kj}T_{kj}^{\dagger}\|_{ber}^{\frac{1}{2}} \|T_{jk}T_{jk}^{\dagger} + |T_{kj}|^2\|_{ber}^{\frac{1}{2}} & \text{when } j < k \\ 0 & \text{otherwise.} \end{cases}$$

Since $\|y\| = 1$, $|\langle T\hat{k}_{(\mu_1,\ldots,\mu_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle| \leqslant w(\tilde{T})$. Taking the supremum over all $(\mu_1, \ldots, \mu_n) \in X^n$, we have $\mathbf{ber}(T) \leqslant w(\tilde{T})$. $\square$

Considering $n = 2$, in Theorem 2, we get the following.

COROLLARY 2. *Let $T_{jk} \in \mathscr{CR}(\mathbb{H}_k, \mathbb{H}_j)$ for all $j,k = 1,2$. Then*

$$\mathbf{ber}\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\mathbf{ber}(T_{11}) + \mathbf{ber}(T_{22}) + \sqrt{(\mathbf{ber}(T_{11}) - \mathbf{ber}(T_{22}))^2 + t_{12}^2}\right),$$

where $t_{12} = \||T_{12}|^2 + T_{21}T_{21}^{\dagger}\|_{ber}^{\frac{1}{2}} \||T_{21}|^2 + T_{12}T_{12}^{\dagger}\|_{ber}^{\frac{1}{2}}$. In particular, for $T_{11} = T_{22} = 0$,

$$\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\||T_{12}|^2 + T_{21}T_{21}^{\dagger}\|_{ber}^{\frac{1}{2}} \||T_{21}|^2 + T_{12}T_{12}^{\dagger}\|_{ber}^{\frac{1}{2}}. \tag{7}$$

REMARK 2. In [8], it is stated that

$$\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\||T_{12}| + |T_{21}^*|\|_{ber}^{\frac{1}{2}} \||T_{21}| + |T_{12}^*|\|_{ber}^{\frac{1}{2}}. \tag{8}$$

For example, if we consider $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{C}^2$, $T_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $T_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then the bound in (8) gives $\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant 1.015$ and the bound in (7) gives $\mathbf{ber}\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) \leqslant 1$. Thus, for this example, the bound in (7) is better than the bound in (8).

Next, we obtain a theorem establishing a Berezin-number bound for 2×2 operator matrices, in a special case, it yields an improved version of the bound presented in (2). To proceed further, we require the following lemmas.

LEMMA 3. [4, p. 1001] *Let* $T\in\mathscr{B}(\mathbb{H}_1)$ *and* $S\in\mathscr{B}(\mathbb{H}_2)$. *Then*

$$\boldsymbol{ber}\left(\begin{bmatrix} T & 0\\ 0 & S\end{bmatrix}\right)\leqslant \max\left\{\boldsymbol{ber}(T),\boldsymbol{ber}(S)\right\}.$$

LEMMA 4. [12] *Let* $x,y,z\in\mathscr{H}$ *with* $\|z\|=1$. *Then*

$$|\langle x,z\rangle\langle z,y\rangle|\leqslant\frac{1}{2}\left(\|x\|\|y\|+|\langle x,y\rangle|\right).$$

THEOREM 3. *Let* $T_{jk}\in\mathscr{CR}(\mathbb{H}_k,\mathbb{H}_j)$ *for all* $j,k=1,2$. *Then*

$$\begin{aligned}\boldsymbol{ber}\left(\begin{bmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix}\right)\leqslant{}& \max\{\boldsymbol{ber}(T_{11}),\boldsymbol{ber}(T_{22})\}\\ &+\left(\frac{1}{4}\max\{\||T_{12}^*|^4+|T_{21}|^4\|_{ber},\||T_{12}|^4+|T_{21}^*|^4\|_{ber}\}\right.\\ &\left.+\frac{1}{2}\max\{\boldsymbol{ber}(|T_{12}^*|^2|T_{21}|^2),\boldsymbol{ber}(|T_{21}^*|^2|T_{12}|^2)\}\right)^{\frac{1}{4}}\\ &\times\left(\frac{1}{4}\max\{\|T_{12}T_{12}^\dagger+T_{21}^\dagger T_{21}\|_{ber},\|T_{12}^\dagger T_{12}+T_{21}T_{21}^\dagger\|_{ber}\}\right.\\ &\left.+\frac{1}{2}\max\{\boldsymbol{ber}(T_{21}^\dagger T_{21}T_{12}T_{12}^\dagger),\boldsymbol{ber}(T_{12}^\dagger T_{12}T_{21}T_{21}^\dagger)\}\right)^{\frac{1}{4}}.\end{aligned}$$

Proof. Let $M=\begin{bmatrix} 0 & T_{12}\\ T_{21} & 0\end{bmatrix}$. For $(\mu_1,\mu_2)\in X_1\times X_2$, let $\hat{k}_{(\mu_1,\mu_2)}$ be a normalized reproducing kernel of $\mathbb{H}_1\oplus\mathbb{H}_2$. Then, by using the Lemma 3, we have

$$\left|\left\langle\begin{bmatrix} T_{11} & 0\\ 0 & T_{22}\end{bmatrix}\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\right\rangle\right|\leqslant\mathbf{ber}\left(\begin{bmatrix} T_{11} & 0\\ 0 & T_{22}\end{bmatrix}\right)=\max\{\mathbf{ber}(T_{11}),\mathbf{ber}(T_{22})\}.\quad(9)$$

Again, by using Lemma 1, we have

$$\begin{aligned}&\left|\left\langle M\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\right\rangle\right|\\ &\leqslant\langle|M|^2\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\rangle^{\frac{1}{4}}\langle MM^\dagger\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\rangle^{\frac{1}{4}}\langle|M^*|^2\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\rangle^{\frac{1}{4}}\\ &\quad\times\langle M^\dagger M\hat{k}_{(\mu_1,\mu_2)},\hat{k}_{(\mu_1,\mu_2)}\rangle^{\frac{1}{4}}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\frac{1}{2}\||M|^2\hat{k}_{(\mu_1,\mu_2)}\|\||M^*|^2\hat{k}_{(\mu_1,\mu_2)}\| + \frac{1}{2}|\langle |M|^2\hat{k}_{(\mu_1,\mu_2)}, |M^*|^2\hat{k}_{(\mu_1,\mu_2)}\rangle|\right)^{\frac{1}{4}} \\
&\quad \times \left(\frac{1}{2}\|MM^\dagger\hat{k}_{(\mu_1,\mu_2)}\|\|M^\dagger M\hat{k}_{(\mu_1,\mu_2)}\| + \frac{1}{2}|\langle MM^\dagger\hat{k}_{(\mu_1,\mu_2)}, M^\dagger M\hat{k}_{(\mu_1,\mu_2)}\rangle|\right)^{\frac{1}{4}} \\
&\hspace{20em}\text{(by Lemma 4)} \\
&\leqslant \left(\frac{1}{4}\langle (|M|^4+|M^*|^4)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\rangle + \frac{1}{2}|\langle |M^*|^2|M|^2\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\rangle|\right)^{\frac{1}{4}} \\
&\quad \times \left(\frac{1}{4}\langle (MM^\dagger+M^\dagger M)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\rangle + \frac{1}{2}|\langle M^\dagger M^2 M^\dagger\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\rangle|\right)^{\frac{1}{4}} \\
&= \left(\frac{1}{4}\left\langle \begin{bmatrix} |T_{12}^*|^4+|T_{21}|^4 & 0 \\ 0 & |T_{12}|^4+|T_{21}^*|^4 \end{bmatrix}\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\right\rangle\right. \\
&\quad \left. + \frac{1}{2}\left|\left\langle \begin{bmatrix} |T_{12}^*|^2|T_{21}|^2 & 0 \\ 0 & |T_{21}^*|^2|T_{12}|^2 \end{bmatrix}\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\right\rangle\right|\right)^{\frac{1}{4}} \\
&\quad \times \left(\frac{1}{4}\left\langle \begin{bmatrix} T_{12}T_{12}^\dagger+T_{21}^\dagger T_{21} & 0 \\ 0 & T_{12}^\dagger T_{12}+T_{21}T_{21}^\dagger \end{bmatrix}\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\right\rangle\right. \\
&\quad \left. + \frac{1}{2}\left|\left\langle \begin{bmatrix} T_{21}^\dagger T_{21}T_{12}T_{12}^\dagger & 0 \\ 0 & T_{12}^\dagger T_{12}T_{21}T_{21}^\dagger \end{bmatrix}\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\right\rangle\right|\right)^{\frac{1}{4}} \\
&\leqslant \left(\frac{1}{4}\max\{\||T_{12}^*|^4+|T_{21}|^4\|_{ber}, \||T_{12}|^4+|T_{21}^*|^4\|_{ber}\}\right. \\
&\quad \left. + \frac{1}{2}\max\{\mathbf{ber}(|T_{12}^*|^2|T_{21}|^2), \mathbf{ber}(|T_{21}^*|^2|T_{12}|^2)\}\right)^{\frac{1}{4}} \\
&\quad \times \left(\frac{1}{4}\max\{\|T_{12}T_{12}^\dagger+T_{21}^\dagger T_{21}\|_{ber}, \|T_{12}^\dagger T_{12}+T_{21}T_{21}^\dagger\|_{ber}\}\right. \\
&\quad \left. + \frac{1}{2}\max\{\mathbf{ber}(T_{21}^\dagger T_{21}T_{12}T_{12}^\dagger), \mathbf{ber}(T_{12}^\dagger T_{12}T_{21}T_{21}^\dagger)\}\right)^{\frac{1}{4}}.
\end{aligned} \tag{10}
$$

Combining (9) and (10), we have

$$
\begin{aligned}
&\left|\left\langle \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)}\right\rangle\right| \\
&\leqslant \max\{\mathbf{ber}(T_{11}), \mathbf{ber}(T_{22})\} \\
&\quad + \left(\frac{1}{4}\max\{\||T_{12}^*|^4+|T_{21}|^4\|_{ber}, \||T_{12}|^4+|T_{21}^*|^4\|_{ber}\}\right. \\
&\quad \left. + \frac{1}{2}\max\{\mathbf{ber}(|T_{12}^*|^2|T_{21}|^2), \mathbf{ber}(|T_{21}^*|^2|T_{12}|^2)\}\right)^{\frac{1}{4}}
\end{aligned}
$$

$$\times\left(\frac{1}{4}\max\{\|T_{12}T_{12}^{\dagger}+T_{21}^{\dagger}T_{21}\|_{ber},\|T_{12}^{\dagger}T_{12}+T_{21}T_{21}^{\dagger}\|_{ber}\}\right.$$
$$\left.+\frac{1}{2}\max\{\mathbf{ber}(T_{21}^{\dagger}T_{21}T_{12}T_{12}^{\dagger}),\mathbf{ber}(T_{12}^{\dagger}T_{12}T_{21}T_{21}^{\dagger})\}\right)^{\frac{1}{4}}.$$

Taking the supremum over all $(\mu_1,\mu_2)\in X_1\times X_2,$ we get the desired result. $\square$

COROLLARY 3. *Let $T_{jk}\in\mathscr{CR}(\mathbb{H}_k,\mathbb{H}_j)$ for all $j,k=1,2$. Then*

$$\boldsymbol{ber}\left(\begin{bmatrix}0 & T_{12}\\ T_{21} & 0\end{bmatrix}\right)\leqslant\left(\frac{1}{4}\max\{\||T_{12}^{*}|^4+|T_{21}|^4\|_{ber},\||T_{12}|^4+|T_{21}^{*}|^4\|_{ber}\}\right.$$
$$\left.+\frac{1}{2}\max\{\boldsymbol{ber}(|T_{12}^{*}|^2|T_{21}|^2),\boldsymbol{ber}(|T_{21}^{*}|^2|T_{12}|^2)\}\right)^{\frac{1}{4}}$$
$$\times\left(\frac{1}{4}\max\{\|T_{12}T_{12}^{\dagger}+T_{21}^{\dagger}T_{21}\|_{ber},\|T_{12}^{\dagger}T_{12}+T_{21}T_{21}^{\dagger}\|_{ber}\}\right.$$
$$\left.+\frac{1}{2}\max\{\boldsymbol{ber}(T_{21}^{\dagger}T_{21}T_{12}T_{12}^{\dagger}),\boldsymbol{ber}(T_{12}^{\dagger}T_{12}T_{21}T_{21}^{\dagger})\}\right)^{\frac{1}{4}}.$$

REMARK 3. For $T_{12}=T_{21}=T\in\mathscr{CR}(\mathbb{H})$, Corollary 3 gives

$$\mathbf{ber}(T)\leqslant\left(\frac{1}{4}\||T^{*}|^4+|T|^4\|_{ber}+\frac{1}{2}\mathbf{ber}(|T^{*}|^2|T|^2)\right)^{\frac{1}{4}}$$
$$\times\left(\frac{1}{4}\|TT^{\dagger}+T^{\dagger}T\|_{ber}+\frac{1}{2}\mathbf{ber}(T^{\dagger}T^2T^{\dagger})\right)^{\frac{1}{4}}. \tag{11}$$

Clearly,

$$\left(\frac{1}{4}\|TT^{\dagger}+T^{\dagger}T\|_{ber}+\frac{1}{2}\mathbf{ber}(T^{\dagger}T^2T^{\dagger})\right)^{\frac{1}{4}}\leqslant 1.$$

Thus, the bound in (11) refines the bound in (2).

We conclude this article with a new upper bound for the Berezin norm of 2×2 operator matrices and provide a numerical example to show that our bound is sharper than the corresponding bound in [8]. First, we need the following lemma.

LEMMA 5. [8] *Let $T\in\mathscr{B}(\mathbb{H}_2,\mathbb{H}_1)$ and $S\in\mathscr{B}(\mathbb{H}_1,\mathbb{H}_2)$. Then*

$$\left\|\begin{bmatrix}0 & T\\ S & 0\end{bmatrix}\right\|_{ber}\leqslant\max\left\{\|T\|_{ber},\|S\|_{ber}\right\}.$$

THEOREM 4. *Let $T_{jk} \in \mathscr{C}\mathscr{R}(\mathbb{H}_k, \mathbb{H}_j)$ for all $j,k = 1,2$. Then*

$$
\begin{aligned}
&\left\| \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right\|_{ber} \\
\leqslant\ & \max\{\boldsymbol{ber}(|T_{11}|^2 + iT_{21}^{\dagger}T_{21}), \boldsymbol{ber}(|T_{22}|^2 + iT_{12}^{\dagger}T_{12})\} \\
& \times \max\{\boldsymbol{ber}(|T_{12}^*|^2 + iT_{11}T_{11}^{\dagger}), \boldsymbol{ber}(|T_{21}^*|^2 + iT_{22}T_{22}^{\dagger})\} \\
& + \frac{1}{2}\max\{\||T_{11}|^2 + |T_{21}|^2\|_{ber}, \||T_{22}|^2 + |T_{12}|^2\|_{ber}\} \\
& + \max\{\|T_{21}^*T_{22}\|_{ber}, \|T_{12}^*T_{11}\|_{ber}\}.
\end{aligned}
$$

Proof. Let $M = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}$ and $N = \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}$. For $(\mu_1,\mu_2),(\nu_1,\nu_2),\in X^2$, let $\hat{k}_{(\mu_1,\mu_2)}$ and $\hat{k}_{(\nu_1,\nu_2)}$ be two normalized reproducing kernels of $\mathbb{H}_1 \oplus \mathbb{H}_2$. Then by using Lemma 1, we have

$$
\begin{aligned}
&\left| \left\langle \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \right\rangle \right|^2 \\
=\ & \left| \langle M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle + \langle N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right|^2 \\
\leqslant\ & \left(\left| \langle M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| + \left| \langle N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| \right)^2 \\
=\ & \left| \langle M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right|^2 + \left| \langle N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right|^2 \\
& +2\left| \langle M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| \left| \langle N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| \\
\leqslant\ & \langle |M|^2\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \langle MM^{\dagger}\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \\
& + \langle |N^*|^2\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \langle N^{\dagger}N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \\
& + \left\| M\hat{k}_{(\mu_1,\mu_2)} \right\| \left\| N\hat{k}_{(\mu_1,\mu_2)} \right\| + \left| \langle M\hat{k}_{(\mu_1,\mu_2)}, N\hat{k}_{(\mu_1,\mu_2)} \rangle \right| \ \text{(by Lemma 4)} \\
\leqslant\ & \left(\langle |M|^2\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle^2 + \langle N^{\dagger}N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle^2 \right)^{\frac{1}{2}} \\
& \times \left(\langle MM^{\dagger}\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle^2 + \langle |N^*|^2\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle^2 \right)^{\frac{1}{2}} \\
& + \frac{1}{2}\left(\langle (|M|^2 + |N|^2)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right) + \left| \langle N^*M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right| \\
=\ & \left| \langle |M|^2\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle + i\langle N^{\dagger}N\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right| \\
& \times \left| \langle |N^*|^2\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle + i\langle MM^{\dagger}\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| \\
& + \frac{1}{2}\left(\langle (|M|^2 + |N|^2)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right) + \left| \langle N^*M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right| \\
=\ & \left| \langle (|M|^2 + iN^{\dagger}N)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right| \left| \langle (|N^*|^2 + iMM^{\dagger})\hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \rangle \right| \\
& + \frac{1}{2}\left(\langle (|M|^2 + |N|^2)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right) + \left| \langle N^*M\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \rangle \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\left\langle \begin{bmatrix} |T_{11}|^2 + iT_{21}^{\dagger}T_{21} & 0 \\ 0 & |T_{22}|^2 + iT_{12}^{\dagger}T_{12} \end{bmatrix} \hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle\right| \\
&\quad \times \left|\left\langle \begin{bmatrix} |T_{12}^*|^2 + iT_{11}T_{11}^{\dagger} & 0 \\ 0 & |T_{21}^*|^2 + iT_{22}T_{22}^{\dagger} \end{bmatrix} \hat{k}_{(\nu_1,\nu_2)}, \hat{k}_{(\nu_1,\nu_2)} \right\rangle\right| \\
&\quad + \frac{1}{2}\left(\left\langle \begin{bmatrix} |T_{11}|^2 + |T_{21}|^2 & 0 \\ 0 & |T_{22}|^2 + |T_{12}|^2 \end{bmatrix} \hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right) \\
&\quad + \left|\left\langle \begin{bmatrix} 0 & T_{21}^*T_{22} \\ T_{12}^*T_{11} & 0 \end{bmatrix} \hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle\right| \\
&\leqslant \mathbf{ber}\left(\begin{bmatrix} |T_{11}|^2 + iT_{21}^{\dagger}T_{21} & 0 \\ 0 & |T_{22}|^2 + iT_{12}^{\dagger}T_{12} \end{bmatrix} \right) \mathbf{ber}\left(\begin{bmatrix} |T_{12}^*|^2 + iT_{11}T_{11}^{\dagger} & 0 \\ 0 & |T_{21}^*|^2 + iT_{22}T_{22}^{\dagger} \end{bmatrix} \right) \\
&\quad + \frac{1}{2}\mathbf{ber}\left(\begin{bmatrix} |T_{11}|^2 + |T_{21}|^2 & 0 \\ 0 & |T_{22}|^2 + |T_{12}|^2 \end{bmatrix} \right) + \mathbf{ber}\left(\begin{bmatrix} 0 & T_{21}^*T_{22} \\ T_{12}^*T_{11} & 0 \end{bmatrix} \right) \\
&\leqslant \max\{\mathbf{ber}(|T_{11}|^2 + iT_{21}^{\dagger}T_{21}), \mathbf{ber}(|T_{22}|^2 + iT_{12}^{\dagger}T_{12})\} \\
&\quad \times \max\{\mathbf{ber}(|T_{12}^*|^2 + iT_{11}T_{11}^{\dagger}), \mathbf{ber}(|T_{21}^*|^2 + iT_{22}T_{22}^{\dagger})\} \\
&\quad + \frac{1}{2}\max\{\||T_{11}|^2 + |T_{21}|^2\|_{ber}, \||T_{22}|^2 + |T_{12}|^2\|_{ber}\} + \max\{\|T_{21}^*T_{22}\|_{ber}, \|T_{12}^*T_{11}\|_{ber}\} \\
&\quad \text{(by Lemma 3 and Lemma 5).}
\end{aligned}
$$

Taking the supremum over all $(\mu_1,\mu_2),\ (\nu_1,\nu_2) \in X^2,$ we get the required bound. □

REMARK 4. In [8], it is stated that

$$
\begin{aligned}
&\left\| \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right\|_{ber} \\
&\leqslant \max\{\mathbf{ber}(|T_{11}| + i|T_{21}|), \mathbf{ber}(|T_{22}| + i|T_{12}|)\} \\
&\quad \times \max\{\mathbf{ber}(|T_{11}^*| + i|T_{12}^*|), \mathbf{ber}(|T_{22}^*| + i|T_{21}^*|)\} \\
&\quad + \frac{1}{2}\max\{\||T_{11}|^2 + |T_{21}|^2\|_{ber}, \||T_{22}|^2 + |T_{12}|^2\|_{ber}\} \\
&\quad + \max\{\|T_{21}^*T_{22}\|_{ber}, \|T_{12}^*T_{11}\|_{ber}\}.
\end{aligned} \tag{12}
$$

For example, if we consider $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{C}^2,\ T_{11} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix},\ T_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ T_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $T_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then the bound in (12) gives $\left\| \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right\|_{ber} \leqslant 3.91$ whereas the bound in Theorem 4 gives $\left\| \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right\|_{ber} \leqslant 3.58.$ Thus, the bound in Theorem 4 is sharper than the bound in (12).

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