Mathematical
Inequalities
& Applications
Volume 29, Number 3 (2026), 295–312 doi:10.7153/mia-2026-29-17

ORDER BOUNDEDNESS, ESSENTIAL NORM AND COMPACTNESS OF GENERALIZED STEVIĆ–SHARMA OPERATORS ON DERIVATIVE HARDY SPACES

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(*Communicated by I. Perić*)

Abstract. We investigate the order boundedness of generalized Stević-Sharma operators on derivative Hardy spaces. We also establish characterizations for the boundedness, essential norm and compactness of these operators mapping from derivative Hardy spaces to Bloch-type spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{N}$ the set of positive integers. Denote by $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the family of all analytic self-maps of $\mathbb{D}$.

For $0 < p < \infty$, the Hardy space H^p is defined as the collection of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^p}^p = \sup_{0 \leqslant r < 1} \int_{\partial\mathbb{D}} |f(r\xi)|^p \mathrm{d}m(\xi) < \infty,$$

see [6] for details. The derivative Hardy space, denoted by $\mathscr{S}^p$, is the set of all $f \in H(\mathbb{D})$ whose derivative f' lies in H^p. For $1 < p < \infty$, this space is contained in the disk algebra and forms a Banach space with respect to the norm

$$\|f\|_{\mathscr{S}^p}^p = |f(0)| + \|f'\|_{H^p}^p.$$

For studies concerning operators on the space $\mathscr{S}^p$, see [3, 9, 14, 23, 26] and the references therein.

Let μ be a radial weight, namely a strictly positive continuous function on $\mathbb{D}$ satisfying $\mu(z) = \mu(|z|)$ for all $z \in \mathbb{D}$. The Bloch-type space associated with μ, denoted by $\mathscr{B}_\mu$, is the set of all functions $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

Mathematics subject classification (2020): 47B38, 42B30, 30H30.
Keywords and phrases: Stević-Sharma operator, derivative Hardy space, order bounded, essential norm.
This research is supported by Natural Science Foundation of Henan (No. 262300420319) and the Key Scientific Research Projects of the Higher Education Institutions of Henan (No. 26B110004).
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Endowed with the norm

$$\|f\|_{\mathscr{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)|,$$

$\mathscr{B}_\mu$ is a Banach space. A canonical example is the α-Bloch space, which is exactly the Bloch-type space $\mathscr{B}_\mu$ with the weight $\mu(z) = (1-|z|^2)^\alpha$ for $\alpha > 0$. In particular, the case $\alpha = 1$ yields the classical Bloch space. A natural and important subclass of Bloch-type space is the little Bloch-type space, denoted by $\mathscr{B}_{\mu,0}$. It consists of all functions $f \in \mathscr{B}_\mu$ satisfying the vanishing condition

$$\lim_{|z|\to 1} \mu(z)|f'(z)| = 0.$$

Clearly, $\mathscr{B}_{\mu,0}$ is a closed subspace of $\mathscr{B}_\mu$. One may consult [8, 11, 17, 18, 27, 30, 32, 40, 43, 46] for further studies of these spaces and various concrete operators defined on them.

For $n \in \mathbb{N}\cup\{0\}$, the nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^n f(z) = f^{(n)}(z), \quad f \in H(\mathbb{D}),$$

where $f^{(0)} = f$, and the case $n = 1$ gives the classical differentiation operator D. A well-known property of D is its unboundedness on many analytic function spaces, which has motivated extensive research on product-type operators involving differential operations. Initial research on product-type operators involving D focused on the operators DC_φ and $C_\varphi D$ (see, for instance, [13, 17, 18, 25, 30, 31, 32]). Here C_φ is the composition operator, which is defined by

$$C_\varphi f(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

For $u \in H(\mathbb{D})$ the multiplication operator M_u is defined by

$$M_u f(z) = u(z)f(z), \quad f \in H(\mathbb{D}).$$

The product $W_{u,\varphi} := M_u C_\varphi$ of these two operators is known as the weighted composition operator, and has been investigated a lot in the theory of analytic function spaces (see, for example, [4]). Sharma in [27] considered the six linear operators $M_uC_\varphi D$, M_uDC_φ, $C_\varphi M_u D$, DM_uC_φ, $C_\varphi DM_u$, $DC_\varphi M_u$ induced by products of C_φ, M_u and D from weighted Bergman-Nevanlinna to Bloch-type spaces. To unify the study of these product-type operators, Stević et al. in [38, 39] introduced the following so-called Stević-Sharma operator:

$$T_{u,v,\varphi} f(z) = u(z)f(\varphi(z)) + v(z)f'(\varphi(z)),$$

where $u, v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$. By making appropriate choices of the involved symbols, one can easily obtain the general product-type operators:

$$M_uC_\varphi = T_{u,0,\varphi}, \quad C_\varphi M_u = T_{u\circ\varphi,0,\varphi}, \quad M_uD = T_{0,u,id}, \quad DM_u = T_{u',u,id}, \quad C_\varphi D = T_{0,1,\varphi},$$
$$DC_\varphi = T_{0,\varphi',\varphi}, \quad M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_u D = T_{0,u\circ\varphi,\varphi},$$
$$DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi}, \quad DC_\varphi M_u = T_{\varphi'(u'\circ\varphi),\varphi'(u\circ\varphi),\varphi}.$$

In recent years, the study of Stević-Sharma operator has received considerable interest and become increasingly popular. For instance, Guo and Shu in [10] investigated the boundedness and compactness of $T_{u,v,\varphi}$ from Hardy spaces to Stević weighted spaces, which was introduced by Stević in [29] (see also [34]). Zhu et al. in [46] provided some necessary and sufficient conditions for $T_{u,v,\varphi}$ to be bounded or compact from the analytic Besov space into Bloch space. Jiang in [16] characterized the order boundedness of $T_{u,v,\varphi}$ between weighted Dirichlet spaces. Further related results can be found in [1, 2, 5, 8, 9, 11, 36, 37, 40] and the references therein.

Abbasi et al. in [2] generalized the Stević-Sharma operator as follows

$$T^n_{u,v,\varphi}f(z) = u(z)f(\varphi(z)) + v(z)f^{(n)}(\varphi(z)), \quad n \in \mathbb{N},$$

and studied its boundedness, compactness and essential norm from Hardy space into Stević weighted spaces. Note that when $n = 1$, we get the classical Stević-Sharma operator $T_{u,v,\varphi}$. Moreover, If $u \equiv 0$, it is the generalized weighted composition operator $D^n_{v,\varphi}$, which was introduced by Zhu [44] and is also called weighted differentiation composition operator (see previous studies [31, 34], as well as [35] where an n-dimensional analog was introduced and studied). Subsequently, the author in [8] investigated the boundedness, essential norm and compactness of $T^n_{u,v,\varphi}$ from the minimal Möbius invariant space into Bloch-type space. Abbasi and Hassanlou in [1] characterized the boundedness of $T^n_{u,v,\varphi}$ from spaces of fractional Cauchy transforms into weighted Zygmund spaces and gave an estimate for the essential norm. Lo and Loh in [22] characterize the order boundedness of $T^n_{u,v,\varphi}$ between Bergman spaces induced by doubling weights.

Let X be a quasi-Banach space, μ a non-negative Borel measure on $\mathbb{D}$ and

$$L^q_\mu := \left\{ f | f : \mathbb{D} \to \mathbb{C} \text{ is measurable and } \int_{\mathbb{D}} |f(z)|^q \mathrm{d}\mu(z) < \infty \right\}.$$

An operator $T : X \to L^q_\mu$ is said to be order bounded if T maps the closed unit ball B_X of X into an order interval of L^q_μ. More explicitly, there exists a non-negative measurable function $h \in L^q_\mu$ such that $|Tf(z)| \leqslant h(z)$ μ-a.e. on $\mathbb{D}$ for all $f \in B_X$. Order boundedness is a fundamental concept in the theory of ordered vector spaces and operator theory, with close connections to other important properties such as boundedness and compactness. The systematic study of order boundedness for composition operators began with Hardy spaces [15], leading to characterizations on weighted Bergman spaces [12, 21, 22, 41, 42]. Recently, this property has been investigated on several function spaces including weighted Dirichlet spaces and derivative Hardy spaces (see [7, 21, 19, 28]).

For two Banach spaces X and Y, the essential norm of a bounded linear operator $T : X \to Y$ is defined as the distance from T to the compact operators $K : X \to Y$, namely

$$\|T\|_{e,X\to Y} = \inf\left\{ \|T - K\|_{X\to Y} : K \text{ is compact} \right\}.$$

In this paper, we investigate the order boundedness of the generalized Stević-Sharma operators $T^n_{u,v,\varphi}$ between derivative Hardy spaces, and characterize the boundedness, essential norm, compactness of $T^n_{u,v,\varphi}$ acting from derivative Hardy spaces into Bloch-type spaces.

Throughout this work, for any nonnegative quantities X and Y, the notation $X \lesssim Y$ (or $Y \gtrsim X$, which is its equivalent form) is adopted to denote the existence of a positive constant C independent of X and Y such that $X \leqslant CY$. Moreover, the symbol $X \approx Y$ is used to signify that both $X \lesssim Y$ and $Y \lesssim X$ hold at the same time.

2. Order boundedness

This section is dedicated to the investigation of order boundedness for the operator $T^n_{u,v,\varphi}$ acting on the derivative Hardy spaces $\mathscr{S}^p$. For this purpose, we begin with the following lemma.

LEMMA 1. [14] *Suppose that* $1 < p < \infty$ *and* $k \in \mathbb{N}$, *then*

$$\|f\|_\infty \lesssim \|f\|_{\mathscr{S}^p} \quad \text{and} \quad |f^{(k)}(z)| \lesssim \frac{\|f\|_{\mathscr{S}^p}}{(1-|z|^2)^{\frac{1}{p}+k-1}}$$

for each $f \in \mathscr{S}^p$.

For any $z \in \mathbb{D} \setminus \{0\}$ and $j \in \mathbb{N}$, let

$$f_{j,z}(w) = \frac{(1-|z|^2)^j}{(1-\bar{z}w)^{\frac{1}{p}+j-1}}, \quad w \in \mathbb{D}. \tag{1}$$

It can be verified that $f_{j,z} \in \mathscr{S}^p$ and $\sup_{z\in\mathbb{D}} \|f_{j,z}\|_{\mathscr{S}^p} \lesssim 1$ for all $j \in \mathbb{N}$. Furthermore, $f_{j,z}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|z| \to 1$.

The subsequent lemma is established by the test function construction method introduced in [29, 32, 33] (see also [8, 9]), so we omit the proof.

LEMMA 2. *Let* $1 < p < \infty$, $n \in \mathbb{N}$ *with* $n > 1$. *For any* $z \in \mathbb{D} \setminus \{0\}$ *and each* $i,k \in \{0,1,n,n+1\}$, *there exists a function* $g_{i,z}(w) = \sum_{j=1}^4 c_{i,j} f_{j,z}(w)$ *belonging to* $\mathscr{S}^p$ *such that*

$$g^{(k)}_{i,z}(z) = \frac{\bar{z}^k \delta_{ik}}{(1-|z|^2)^{\frac{1}{p}+k-1}},$$

where $c_{i,j}$ *are constants,* $f_{j,z}$ *are the functions defined by* (1) *and* δ_{ik} *is the Kronecker delta.*

For the sake of notational simplicity, we denote

$$G_1(z) = u(z)\varphi'(z),$$
$$G_n(z) = v'(z),$$
$$G_{n+1}(z) = v(z)\varphi'(z).$$

THEOREM 1. *Let $1 < p,q < \infty$, $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$ with $n > 1$. Then the operator $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{S}^q$ is order bounded if and only if $u \in \mathscr{S}^q$ and for each $i \in \{1,n,n+1\}$,*

$$\int_{\partial\mathbb{D}} \frac{|G_i(\xi)|^q}{(1-|\varphi(\xi)|^2)^{(\frac{1}{p}+i-1)q}} \mathrm{d}m(\xi) < \infty. \tag{2}$$

Proof. Suppose that $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{S}^q$ is order bounded. Then there exists a non-negative function $h \in L^q_m$ such that (see, e.g., [28])

$$|(T^n_{u,v,\varphi}f)'(z)| \leqslant h(z),$$

m-a.e. on $\mathbb{D}$ for all $f \in \mathscr{S}^p$ with $\|f\|_{\mathscr{S}^p} \leqslant 1$. We first take the constant function $p_0(z) = 1 \in \mathscr{S}^p$ as a test function. Substituting it into the above inequality yields

$$h(z) \geqslant |(T^n_{u,v,\varphi}p_0)'(z)| = |u'(z)| \quad m\text{-a.e. on } \mathbb{D}. \tag{3}$$

Since $h \in L^q_m$, this immediately implies $u \in \mathscr{S}^q$. Next, we choose the test function $p_1(z) = \frac{z}{\|z\|_{\mathscr{S}^p}} \in \mathscr{S}^p$, and similarly derive that

$$h(z) \geqslant |(T^n_{u,v,\varphi}p_1)'(z)| \approx |u'(z)\varphi(z) + G_1(z)| \geqslant |G_1(z)| - |u'(z)\varphi(z)|,$$

m-a.e. on $\mathbb{D}$. Combining this with (3) and the fact that $|\varphi(z)| < 1$, we further conclude that

$$|G_1(z)| \lesssim h(z) \quad m\text{-a.e. on } \mathbb{D}. \tag{4}$$

We then take the function $p_n(z) = \frac{z^n}{\|z^n\|_{\mathscr{S}^p}} \in \mathscr{S}^p$, leading to

$$\begin{aligned} h(z) &\geqslant |(T^n_{u,v,\varphi}p_n)'(z)| \\ &\approx |u'(z)\varphi(z)^n + nG_1(z)\varphi(z)^{n-1} + G_n(z)n!| \\ &\gtrsim |G_n(z)| - \frac{1}{n!}|u'(z)\varphi(z)^n| - \frac{1}{(n-1)!}|G_1(z)||\varphi(z)|^{n-1}, \end{aligned}$$

m-a.e. on $\mathbb{D}$, from which along with (3), (4) and the fact that $|\varphi(z)| < 1$ it follows that

$$|G_n(z)| \lesssim h(z) \quad m\text{-a.e. on } \mathbb{D}. \tag{5}$$

In the same way, choosing $p_{n+1}(z) = \frac{z^{n+1}}{\|z^{n+1}\|_{\mathscr{S}^p}} \in \mathscr{S}^p$ gives

$$\begin{aligned} h(z) &\geqslant |(T^n_{u,v,\varphi}p_{n+1})'(z)| \\ &\approx |u'(z)\varphi(z)^{n+1} + (n+1)G_1(z)\varphi(z)^n + G_n(z)\varphi(z)(n+1)! + G_{n+1}(z)(n+1)!| \\ &\gtrsim |G_{n+1}(z)| - \frac{1}{(n+1)!}|u'(z)\varphi(z)^{n+1}| - \frac{1}{n!}|G_1(z)||\varphi(z)|^n - |G_n(z)||\varphi(z)|, \end{aligned}$$

m-a.e. on $\mathbb{D}$. From this, together with (3), (4), (5) and the fact that $|\varphi(z)| < 1$, we deduce that

$$|G_{n+1}(z)| \lesssim h(z) \quad \text{a.e. } [m]. \tag{6}$$

Now by Lemma 2, for each $i \in \{1, n, n+1\}$ and $\varphi(z) \neq 0$, there exist constants $c_{i,1}, c_{i,2}, c_{i,3}, c_{i,4}$ such that

$$g_{i,\varphi(z)}(w) = \sum_{j=1}^{4} c_{i,j} f_{j,\varphi(z)}(w) \in \mathscr{S}^p, \tag{7}$$

and

$$g_{i,\varphi(z)}^{(k)}(\varphi(z)) = \frac{\overline{\varphi(z)}^i \delta_{ik}}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}},$$

where $f_{j,\varphi(z)}$ are defined in (1) and $k \in \{0, 1, n, n+1\}$. The order boundedness of $T_{u,v,\varphi}^n$ then yields

$$h(z) \geqslant \left| \left(T_{u,v,\varphi}^n \frac{g_{i,\varphi(z)}}{\|g_{i,\varphi(z)}\|_{\mathscr{S}^p}} \right)'(z) \right| \approx \frac{|G_i(z)||\varphi(z)|^i}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}, \quad \text{a.e. } [m].$$

Thus, for $|\varphi(z)| > \frac{1}{2}$, we have

$$\frac{|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \lesssim h(z), \quad \text{a.e. } [m]. \tag{8}$$

On the other hand, for $|\varphi(z)| \leqslant \frac{1}{2}$, in view of (4), (5), (6), we obtain

$$\frac{|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \lesssim |G_i(z)| \lesssim h(z), \quad \text{a.e. } [m]. \tag{9}$$

Combining (8) with (9) yields (2) holds.

Conversely, assume that $u \in \mathscr{S}^q$ and (2) holds. For any $f \in \mathscr{S}^p$ with $\|f\|_{\mathscr{S}^p} \leqslant 1$, by using Lemma 1 we have

$$\begin{aligned} |(T_{u,v,\varphi}^n f)'(z)| &= \left| u'(z) f(\varphi(z)) + \sum_{i \in \{1,n,n+1\}} G_i(z) f^{(i)}(\varphi(z)) \right| \\ &\lesssim |u'(z)| + \sum_{i \in \{1,n,n+1\}} \frac{|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}. \end{aligned} \tag{10}$$

Since the expression on the right-hand side of (10) belongs to L_m^q, it follows that the operator $T_{u,v,\varphi}^n : \mathscr{S}^p \to \mathscr{S}^q$ is order bounded. $\square$

For the case $n=1$, we have an analogous result to Lemma 2: for any $z \in \mathbb{D} \setminus \{0\}$ and all $i,k \in \{0,1,2\}$, there exist constants $d_{i,j}$ ($j \in \{1,2,3\}$) such that the function

$$h_{i,z}(w) = \sum_{j=1}^{3} d_{i,j} f_{j,z}(w) \tag{11}$$

belongs to $\mathscr{S}^p$, and its k-th derivative at z satisfies

$$h_{i,z}^{(k)}(z) = \frac{\bar{z}^k \delta_{ik}}{(1-|z|^2)^{\frac{1}{p}+k-1}}.$$

By virtue of this result and a minor modification to the proof of Theorem 1, we obtain the following theorem.

THEOREM 2. *Let $1<p,q<\infty$, $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$. Then the operator $T_{u,v,\varphi}$: $\mathscr{S}^p \to \mathscr{S}^q$ is order bounded if and only if $u \in \mathscr{S}^q$ and*

$$\int_{\partial\mathbb{D}} \frac{|u(z)\varphi'(z)+v'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q}{p}}} \mathrm{d}m(\xi) < \infty,$$
$$\int_{\partial\mathbb{D}} \frac{|v(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{(\frac{1}{p}+1)q}} \mathrm{d}m(\xi) < \infty.$$

Setting $v(z)=0$ in Theorem 2 recovers a result originally due to [19].

COROLLARY 1. *Let $1<p,q<\infty$, $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$. Then the weighted composition operator $W_{u,\varphi} : \mathscr{S}^p \to \mathscr{S}^q$ is order bounded if and only if $u \in \mathscr{S}^q$ and*

$$\int_{\partial\mathbb{D}} \frac{|u(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{\frac{q}{p}}} \mathrm{d}m(\xi) < \infty.$$

3. Boundedness

This section characterizes the boundedness of $T_{u,v,\varphi}^n : \mathscr{S}^p \to \mathscr{B}_\mu(\mathscr{B}_{\mu,0})$, with the classical case $n=1$ treated similarly.

THEOREM 3. *Let $1<p<\infty$, $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$ with $n>1$ and μ be a radial weight. Then the operator $T_{u,v,\varphi}^n : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded if and only if $u \in \mathscr{B}_\mu$ and for each $i \in \{1,n,n+1\}$,*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} < \infty. \tag{12}$$

Proof. Suppose that $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded. We first take the constant function $f_0(z) = 1 \in \mathscr{S}^p$, by the boundedness of $T^n_{u,v,\varphi}$, we have

$$\sup_{z\in\mathbb{D}} \mu(z)|u'(z)| = \sup_{z\in\mathbb{D}} \mu(z)|(T^n_{u,v,\varphi}f_0)'(z)| \leqslant \|T^n_{u,v,\varphi}f_0\|_{\mathscr{B}_\mu} < \infty, \tag{13}$$

which implies $u \in \mathscr{B}_\mu$. Next, we apply $T^n_{u,v,\varphi}$ to the function $f_1(z) = z \in \mathscr{S}^p$. It follows that

$$\begin{aligned}\infty &> \|T^n_{u,v,\varphi}f_1\|_{\mathscr{B}_\mu} \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^n_{u,v,\varphi}f_1)'(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z) + G_1(z)| \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)G_1(z) - \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)|,\end{aligned}$$

Using (13) and the fact that $|\varphi(z)| < 1$, we conclude that

$$\sup_{z\in\mathbb{D}} \mu(z)|G_1(z)| < \infty. \tag{14}$$

Taking the function $f_n(z) = \frac{z^n}{n!} \in \mathscr{S}^p$, we obtain

$$\begin{aligned}\infty &> \|T^n_{u,v,\varphi}f_n\|_{\mathscr{B}_\mu} \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|(T^n_{u,v,\varphi}f_n)'(z)| \\ &= \sup_{z\in\mathbb{D}} \mu(z)\left|u'(z)\frac{\varphi(z)^n}{n!} + G_1(z)\frac{\varphi(z)^{n-1}}{(n-1)!} + G_n(z)\right| \\ &\geqslant \sup_{z\in\mathbb{D}} \mu(z)|G_n(z)| - \frac{1}{n!}\sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)^n| - \frac{1}{(n-1)!}\sup_{z\in\mathbb{D}} \mu(z)|G_1(z)\varphi(z)^{n-1}|,\end{aligned}$$

from which along with (13), (14) and the fact that $|\varphi(z)| < 1$ yields

$$\sup_{z\in\mathbb{D}} \mu(z)|G_n(z)| < \infty. \tag{15}$$

Likewise, taking $f_{n+1}(z) = \frac{z^{n+1}}{(n+1)!} \in \mathscr{S}^p$, we have

$$\begin{aligned}\infty >&\|T^n_{u,v,\varphi}f_{n+1}\|_{\mathscr{B}_\mu} \\ \geqslant&\sup_{z\in\mathbb{D}} \mu(z)|(T^n_{u,v,\varphi}f_{n+1})'(z)| \\ =&\sup_{z\in\mathbb{D}} \mu(z)\left|u'(z)\frac{\varphi(z)^{n+1}}{(n+1)!} + G_1(z)\frac{\varphi(z)^n}{n!} + G_n(z)\varphi(z) + G_{n+1}(z)\right| \\ \geqslant&\sup_{z\in\mathbb{D}} \mu(z)|G_{n+1}(z)| - \frac{1}{(n+1)!}\sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z)^{n+1}| \\ &- \frac{1}{n!}\sup_{z\in\mathbb{D}} \mu(z)|G_1(z)\varphi(z)^n| - \sup_{z\in\mathbb{D}} \mu(z)|G_n(z)\varphi(z)|.\end{aligned}$$

Using (13), (14), (15) together with the fact that $|\varphi(z)|<1$, we get

$$\sup_{z\in\mathbb{D}}\mu(z)|G_{n+1}(z)|<\infty. \tag{16}$$

For each $i\in\{1,n,n+1\}$ and $\varphi(z)\neq 0$, we use the function $g_{i,\varphi(z)}$ defined in (7). Since $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded, we have $\|T^n_{u,v,\varphi}g_{i,\varphi(z)}\|_{\mathscr{B}_\mu}<\infty$. It follows that

$$\begin{aligned}\infty>&\sup_{w\in\mathbb{D}}\mu(w)|(T^n_{u,v,\varphi}g_{i,\varphi(z)})'(w)|\\ \geqslant&\mu(z)|(T^n_{u,v,\varphi}g_{i,\varphi(z)})'(z)|\\ =&\frac{\mu(z)|G_i(z)||\varphi(z)|^i}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.\end{aligned} \tag{17}$$

For $|\varphi(z)|>\frac{1}{2}$, $|\varphi(z)|^i$ is bounded below by a positive constant, so

$$\sup_{|\varphi(z)|>\frac{1}{2}}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}<\infty. \tag{18}$$

On the other hand, by (14), (15) and (16), we get

$$\sup_{|\varphi(z)|\leqslant\frac{1}{2}}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\lesssim\sup_{|\varphi(z)|\leqslant\frac{1}{2}}\mu(z)|G_i(z)|<\infty. \tag{19}$$

Combining (18) and (19) for all $z\in\mathbb{D}$, we conclude that for each $i\in\{1,n,n+1\}$,

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}<\infty,$$

which is exactly the condition (12).

Conversely, assume that $u\in\mathscr{B}_\mu$ and (12) holds for each $i\in\{1,n,n+1\}$. For any $f\in\mathscr{S}^p$, by Lemma 1 we have

$$\begin{aligned}\mu(z)|(T^n_{u,v,\varphi}f)'(z)|&\leqslant\mu(z)|u'(z)||f(\varphi(z))|+\sum_{i\in\{1,n,n+1\}}\mu(z)|G_i(z)||f^{(i)}(\varphi(z))|\\ &\lesssim\left(\mu(z)|u'(z)|+\sum_{i\in\{1,n,n+1\}}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\right)\|f\|_{\mathscr{S}^p}.\end{aligned} \tag{20}$$

In addition,

$$\begin{aligned}|(T^n_{u,v,\varphi}f)(0)|&=|u(0)f(\varphi(0))+v(0)f^n(\varphi(0))|\\ &\lesssim\left(|u(0)|+\frac{|v(0)|}{(1-|\varphi(0)|^2)^{\frac{1}{p}+n-1}}\right)\|f\|_{\mathscr{S}^p}.\end{aligned} \tag{21}$$

In view of (20) and (21), we conclude that

$$\|T^n_{u,v,\varphi}f\|_{\mathscr{B}_\mu}\lesssim\|f\|_{\mathscr{S}^p},$$

for all $i \in \{1,n,n+1\}$. This means the operator $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded. $\square$

An argument analogous to that in Theorem 3, using the functions defined in (11), yields the following result.

THEOREM 4. *Let $1 < p < \infty$, $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ be a radial weight. Then the operator $T_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded if and only if $u \in \mathscr{B}_\mu$ and*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}} < \infty,$$
$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}} < \infty.$$

We now turn to the boundedness of $T^n_{u,v,\varphi}$ into the little Bloch-type space $\mathscr{B}_{\mu,0}$.

THEOREM 5. *Let $1 < p < \infty$, $u,v \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $n \in \mathbb{N}$ with $n > 1$ and μ be a radial weight. Then the operator $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_{\mu,0}$ is bounded if and only if $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded, $u \in \mathscr{B}_{\mu,0}$ and for each $i \in \{1,n,n+1\}$,*

$$\lim_{|z|\to 1} \mu(z)|G_i(z)| = 0. \tag{22}$$

Proof. Suppose that $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_{\mu,0}$ is bounded, which immediately implies that $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$ is bounded and $T^n_{u,v,\varphi} f \in \mathscr{B}_{\mu,0}$ for all $f \in \mathscr{S}^p$. Take $f_0(z) = 1 \in \mathscr{S}^p$, we have

$$\lim_{|z|\to 1} \mu(z)|u'(z)| = \lim_{|z|\to 1} \mu(z)|(T^n_{u,v,\varphi} f_0)'(z)| = 0. \tag{23}$$

This together with $u \in \mathscr{B}_\mu$ (derived from the boundedness of $T^n_{u,v,\varphi} : \mathscr{S}^p \to \mathscr{B}_\mu$) implies $u \in \mathscr{B}_{\mu,0}$. We then apply $T^n_{u,v,\varphi}$ to the function $f_1(z) = z \in \mathscr{S}^p$ and obtain

$$\begin{aligned} 0 = \lim_{|z|\to 1} \mu(z)|(T^n_{u,v,\varphi} f_1)'(z)| &= \lim_{|z|\to 1} \mu(z)|u'(z)\varphi(z) + G_1(z)| \\ &\geqslant \lim_{|z|\to 1} \mu(z)|G_1(z)| - \lim_{|z|\to 1} \mu(z)|u'(z)\varphi(z)|, \end{aligned}$$

which along with (23) and the fact that $|\varphi(z)| < 1$ yields

$$\lim_{|z|\to 1} \mu(z)|G_1(z)| = 0. \tag{24}$$

By using the function $f_n(z) = \frac{z^n}{n!} \in \mathscr{S}^p$, we have

$$\begin{aligned} 0 &= \lim_{|z|\to 1} \mu(z)|(T^n_{u,v,\varphi} f_n)'(z)| \\ &= \lim_{|z|\to 1} \mu(z)\left| u'(z)\frac{\varphi(z)^n}{n!} + G_1(z)\frac{\varphi(z)^{n-1}}{(n-1)!} + G_n(z) \right| \\ &\geqslant \lim_{|z|\to 1} \mu(z)|G_n(z)| - \frac{1}{n!}\lim_{|z|\to 1} \mu(z)|u'(z)\varphi(z)^n| - \frac{1}{(n-1)!}\lim_{|z|\to 1} \mu(z)|G_1(z)\varphi(z)^{n-1}|. \end{aligned}$$

From (23), (24) and the fact that $|\varphi(z)|<1$, we obtain

$$\lim_{|z|\to 1}\mu(z)|G_n(z)|=0. \tag{25}$$

Likewise, we take the function $f_{n+1}(z)=\frac{z^{n+1}}{(n+1)!}\in\mathscr{S}^p$. It holds that

$$\begin{aligned}
0&=\lim_{|z|\to 1}\mu(z)|(T^n_{u,v,\varphi}f_{n+1})'(z)|\\
&=\lim_{|z|\to 1}\mu(z)\left|u'(z)\frac{\varphi(z)^{n+1}}{(n+1)!}+G_1(z)\frac{\varphi(z)^n}{n!}+G_n(z)\varphi(z)+G_{n+1}(z)\right|\\
&\geqslant\lim_{|z|\to 1}\mu(z)|G_{n+1}(z)|-\frac{1}{(n+1)!}\lim_{|z|\to 1}\mu(z)|u'(z)\varphi(z)^{n+1}|\\
&\quad-\frac{1}{n!}\lim_{|z|\to 1}\mu(z)|G_1(z)\varphi(z)^n|-\lim_{|z|\to 1}\mu(z)|G_n(z)\varphi(z)|,
\end{aligned}$$

which along with (23), (24), (25) and the fact that $|\varphi(z)|<1$ implies that

$$\lim_{|z|\to 1}\mu(z)|G_{n+1}(z)|=0. \tag{26}$$

On the contrary, if $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded, $u\in\mathscr{B}_{\mu,0}$ and (22) holds for each $i\in\{1,n,n+1\}$. Taking an arbitrary polynomial p in $\mathscr{S}^p$, we have

$$\mu(z)|(T^n_{u,v,\varphi}p)'(z)|\lesssim\mu(z)|u'(z)|\|p\|_\infty+\sum_{i\in\{1,n,n+1\}}\mu(z)|G_i(z)|\|p^{(i)}\|_\infty.$$

Letting $|z|\to 1$ on both sides, we get

$$\lim_{|z|\to 1}\mu(z)|(T^n_{u,v,\varphi}p)'(z)|=0.$$

This together with $T^n_{u,v,\varphi}p\in\mathscr{B}_\mu$ implies $T^n_{u,v,\varphi}p\in\mathscr{B}_{\mu,0}$. Then we use the density of polynomials in the derivative Hardy space $\mathscr{S}^p$: for each $f\in\mathscr{S}^p$, there exists a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\|f-p_k\|_{\mathscr{S}^p}=0$. It follows that

$$\|T^n_{u,v,\varphi}f-T^n_{u,v,\varphi}p_k\|_{\mathscr{B}_\mu}\leqslant\|T^n_{u,v,\varphi}\|\cdot\|f-p_k\|_{\mathscr{S}^p}\to 0$$

as $k\to\infty$. Recall that $\mathscr{B}_{\mu,0}$ is a closed subspace of $\mathscr{B}_\mu$, so we have $T^n_{u,v,\varphi}f\in\mathscr{B}_{\mu,0}$. Combined with the boundedness of $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$, we conclude that $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ is bounded. □

The following theorem can be proved similar to Theorem 5. We omit the proof.

THEOREM 6. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight. Then the operator $T_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ is bounded if and only if $T_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded, $u\in\mathscr{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1}\mu(z)|u(z)\varphi'(z)+v'(z)|=\lim_{|z|\to 1}\mu(z)|v(z)\varphi'(z)|=0.$$

4. Essential norm and compactness

In this section, we estimate the essential norm of the operator $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ and characterize its compactness. To this end, we require the following lemma, which follows by standard arguments (see [20] and [4, Proposition 3.11]).

LEMMA 3. *Let $1<p<\infty$ and μ be a radial weight such that a linear operator $T:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded. Then $T:\mathscr{S}^p\to\mathscr{B}_\mu$ is compact if and only if $\|Tf_k\|_{\mathscr{B}_\mu}\to 0$ as $k\to\infty$ for each bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $\mathscr{S}^p$ which converges to zero uniformly on compact subsets of $\mathbb{D}$.*

In addition, we need the following two results, which will be used in the proof of our main theorems.

LEMMA 4. [45] *Let $1<p<\infty$. Then every norm bounded sequence in $\mathscr{S}^p$ has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in $\mathscr{S}^p$.*

The following lemma is proved similar to the corresponding result in [24] and have been used in many investigations so far.

LEMMA 5. *A closed set Q in $\mathscr{B}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in Q}\mu(z)|f'(z)|=0.$$

THEOREM 7. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $n\in\mathbb{N}$ with $n>1$ and μ be a radial weight such that $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded. Then*

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu}\approx\max_{i\in\{1,n,n+1\}}\left\{\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\right\}.$$

Proof. Let $\{z_j\}_{j\in\mathbb{N}}$ be any sequence in $\mathbb{D}$ such that $|\varphi(z_j)|\to 1$ as $j\to\infty$. Without loss of generality, we may assume $|\varphi(z_j)|\geqslant\frac{1}{2}$ for all $j\in\mathbb{N}$. For each fixed $i\in\{1,n,n+1\}$, consider the sequence of test functions $\{g_{i,\varphi(z_j)}\}_{j\in\mathbb{N}}\subset\mathscr{S}^p$, where $g_{i,\varphi(z_j)}$ is defined as in (7). It is known that $g_{i,\varphi(z_j)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$. Since $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded, for any compact operator $K:\mathscr{S}^p\to\mathscr{B}_\mu$ and for each $i\in\{1,n,n+1\}$, Lemma 3 together with (17) yields

$$\begin{aligned}\|T^n_{u,v,\varphi}-K\|_{\mathscr{S}^p\to\mathscr{B}_\mu}&\gtrsim\limsup_{j\to\infty}\|(T^n_{u,v,\varphi}-K)g_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu}\\&\geqslant\limsup_{j\to\infty}\|T^n_{u,v,\varphi}g_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu}-\limsup_{j\to\infty}\|Kg_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu}\\&=\limsup_{j\to\infty}\|T^n_{u,v,\varphi}g_{i,\varphi(z_j)}\|_{\mathscr{B}_\mu}\\&\gtrsim\limsup_{j\to\infty}\frac{\mu(z_j)|G_i(z_j)|}{(1-|\varphi(z_j)|^2)^{\frac{1}{p}+i-1}}.\end{aligned}$$

Taking the infimum over all compact operators K on the left-hand side gives

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu} \gtrsim \limsup_{j\to\infty} \frac{\mu(z_j)|G_i(z_j)|}{(1-|\varphi(z_j)|^2)^{\frac{1}{p}+i-1}}.$$

By the arbitrariness of the sequence $\{z_j\}_{j\in\mathbb{N}}$, we may take the supremum over all such sequences to obtain

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.$$

Since this holds for each $i\in\{1,n,n+1\}$, we conclude

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu} \gtrsim \max_{i\in\{1,n,n+1\}} \left\{ \limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}} \right\}.$$

For $0<\theta<1$, define the operator $K_\theta f(z) = f_\theta(z) := f(\theta z)$ for $f\in\mathscr{S}^p$ and $z\in\mathbb{D}$. Then K_θ is compact on $\mathscr{S}^p$ with $\|K_\theta\|\leqslant 1$, and $f_\theta \to f$ uniformly on compact subsets of $\mathbb{D}$ as $\theta\to 1$. Choose a sequence $\{\theta_j\}_{j\in\mathbb{N}}\subset(0,1)$ such that $\theta_j\to 1$ as $j\to\infty$. For each j, the operator $T^n_{u,v,\varphi}K_{\theta_j}:\mathscr{S}^p\to\mathscr{B}_\mu$ is compact, and hence

$$\begin{aligned}\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu} &\leqslant \limsup_{j\to\infty}\|T^n_{u,v,\varphi}-T^n_{u,v,\varphi}K_{\theta_j}\|_{\mathscr{S}^p\to\mathscr{B}_\mu}\\ &= \limsup_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}\|(T^n_{u,v,\varphi}-T^n_{u,v,\varphi}K_{\theta_j})f\|_{\mathscr{B}_\mu} \end{aligned}\tag{27}$$

For any $f\in\mathscr{S}^p$ satisfying $\|f\|_{\mathscr{S}^p}\leqslant 1$, we estimate

$$\begin{aligned}&\|(T^n_{u,v,\varphi}-T^n_{u,v,\varphi}K_{\theta_j})f\|_{\mathscr{B}_\mu}\\ =&|(T^n_{u,v,\varphi}f-T^n_{u,v,\varphi}f_{\theta_j})(0)|+\sup_{z\in\mathbb{D}}\mu(z)|(T^n_{u,v,\varphi}f-T^n_{u,v,\varphi}f_{\theta_j})'(z)|\\ \leqslant&\underbrace{|u(0)||(f-f_{\theta_j})(\varphi(0))|+|v(0)||(f-f_{\theta_j})^{(n)}(\varphi(0))|}_{A_1}+\underbrace{\sup_{z\in\mathbb{D}}\mu(z)|u'(z)||(f-f_{\theta_j})(\varphi(z))|}_{A_2}\\ &+\underbrace{\sup_{|\varphi(z)|\leqslant\theta_N}\mu(z)\sum_{i\in\{1,n,n+1\}}|G_i(z)||(f-f_{\theta_j})^{(i)}(\varphi(z))|}_{A_3}\\ &+\underbrace{\sup_{|\varphi(z)|>\theta_N}\mu(z)\sum_{i\in\{1,n,n+1\}}|G_i(z)||(f-f_{\theta_j})^{(i)}(\varphi(z))|}_{A_4},\end{aligned}\tag{28}$$

where $N\in\mathbb{N}$ such that $\theta_j\geqslant\frac{2}{3}$ for all $j\geqslant N$. Furthermore, we have $(f-f_{\theta_j})^{(t)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$ for any nonnegative integer t. By

Theorem 3, the boundedness of $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ implies $\sup_{z\in\mathbb{D}}\mu(z)|G_i(z)|<\infty$ for each $i\in\{1,n,n+1\}$. Hence

$$\lim_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}A_1=\lim_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}A_3=0. \tag{29}$$

Also, by Lemma 4,

$$\lim_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}A_2\leqslant\|u\|_{\mathscr{B}_\mu}\lim_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}\sup_{z\in\mathbb{D}}|(f-f_{\theta_j})(z)|=0. \tag{30}$$

For A_4, using Lemma 1, we obtain

$$\begin{aligned}A_4&\leqslant\sum_{i\in\{1,n,n+1\}}\sup_{|\varphi(z)|>\theta_N}\mu(z)|G_i(z)f^{(i)}(\varphi(z))|\\&\quad+\sum_{i\in\{1,n,n+1\}}\sup_{|\varphi(z)|>\theta_N}\mu(z)|G_i(z)\theta_j^i f^{(i)}(\theta_j\varphi(z))|\\&\lesssim\sum_{i\in\{1,n,n+1\}}\sup_{|\varphi(z)|>\theta_N}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\\&\quad+\sum_{i\in\{1,n,n+1\}}\sup_{|\varphi(z)|>\theta_N}\frac{\mu(z)|G_i(z)|}{(1-\theta_j^2|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.\end{aligned} \tag{31}$$

Letting $N\to\infty$ and then $j\to\infty$ in (31), we get

$$\lim_{j\to\infty}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}A_4\lesssim\sum_{i\in\{1,n,n+1\}}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}. \tag{32}$$

In view of (27)–(30), (32) and by taking the supremum over $\|f\|_{\mathscr{S}^p}\leqslant 1$, we obtain

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu}\lesssim\sum_{i\in\{1,n,n+1\}}\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}.$$

It follows that

$$\|T^n_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu}\lesssim\max_{i\in\{1,n,n+1\}}\left\{\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\right\}.$$

Combining the above lower and upper bound estimates, we conclude the proof. □

By a similar argument applied to the case $n=1$ and in view of Theorem 4, we arrive at the following result.

THEOREM 8. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight such that $T_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded. Then*

$$\|T_{u,v,\varphi}\|_{e,\mathscr{S}^p\to\mathscr{B}_\mu}\approx\max\left\{\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}},\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}\right\}.$$

From Theorems 3, 4, 7 and 8, together with the fact that $\|T\|_{e,X\to Y}=0$ if and only if T is compact, we obtain the following compactness criteria for $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$.

COROLLARY 2. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $n\in\mathbb{N}$ with $n>1$ and μ be a radial weight. Then the operator $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is compact if and only if the following two conditions hold:*

(i) $u\in\mathscr{B}_\mu$ *and* $\sup_{z\in\mathbb{D}}\mu(z)|G_i(z)|<\infty$ *for* $i\in\{1,n,n+1\}$.

(ii) $\limsup_{|\varphi(z)|\to 1}\dfrac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}=0$ *for* $i\in\{1,n,n+1\}$.

Proof. By Theorems 3 and 7, it suffices to show that conditions (i) and (ii) together imply the boundedness of operator $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$. Suppose that conditions (i) and (ii) hold. For an arbitrary $\varepsilon>0$, condition (ii) guarantees the existence of some constant $r\in(0,1)$ such that

$$\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}<\varepsilon,$$

for all $z\in\mathbb{D}$ with $|\varphi(z)|>r$ and each $i\in\{1,n,n+1\}$. For the case $|\varphi(z)|\leqslant r$, condition (i) implies that

$$\sup_{|\varphi(z)|\leqslant r}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\lesssim\sup_{|\varphi(z)|\leqslant r}\mu(z)|G_i(z)|<\infty,$$

for each $i\in\{1,n,n+1\}$. Therefore,

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}<\infty,\quad i\in\{1,n,n+1\}.$$

Theorem 3 now implies that $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is bounded, which completes the proof. □

COROLLARY 3. *Let $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ and μ be a radial weight. Then the operator $T_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ is compact if and only if the following two conditions hold:*

(i) $u\in\mathscr{B}_\mu$, $\sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'(z)+v'(z)|<\infty$, $\sup_{z\in\mathbb{D}}\mu(z)|v(z)\varphi'(z)|<\infty$.

(ii) $\limsup_{|\varphi(z)|\to 1}\dfrac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}=\limsup_{|\varphi(z)|\to 1}\dfrac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}=0.$

Finally, we provide compactness characterizations for $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$.

THEOREM 9. *Let* $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, $n\in\mathbb{N}$ *with* $n>1$ *and* μ *be a radial weight. Then the operator* $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ *is compact if and only if* $u\in\mathscr{B}_{\mu,0}$ *and for each* $i\in\{1,n,n+1\}$,

$$\limsup_{|z|\to 1}\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}=0. \tag{33}$$

Proof. Suppose that $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ is compact. Then it is also compact as an operator into $\mathscr{B}_\mu$, and bounded as an operator into $\mathscr{B}_{\mu,0}$. By Theorem 5, the boundedness of $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ yields $u\in\mathscr{B}_{\mu,0}$. For any $\varepsilon>0$, by the compactness of $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_\mu$ and Corollary 2, there exists $\delta\in(0,1)$ such that for all $z\in\mathbb{D}$ with $|\varphi(z)|>\delta$ and each $i\in\{1,n,n+1\}$,

$$\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}<\varepsilon, \tag{34}$$

Moreover, by Theorem 5 we have $\lim_{|z|\to 1}\mu(z)|G_i(z)|=0$ for each $i\in\{1,n,n+1\}$, there exists $\eta\in(0,1)$ such that

$$\mu(z)|G_i(z)|<\varepsilon, \tag{35}$$

for $|z|>\eta$. Now consider any $z\in\mathbb{D}$ with $|z|>\eta$. If $|\varphi(z)|>\delta$ then (34) directly implies the desired estimate. If $|\varphi(z)|\leqslant\delta$, by (35) we obtain

$$\frac{\mu(z)|G_i(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+i-1}}\leqslant\frac{\varepsilon}{(1-\delta^2)^{\frac{1}{p}+i-1}}\lesssim\varepsilon, \tag{36}$$

for each $i\in\{1,n,n+1\}$. From (34) and (36), together with the arbitrariness of ε, we conclude that (33) holds.

Conversely, assume that $u\in\mathscr{B}_{\mu,0}$ and and (33) is satisfied. From Theorems 3 and 5, the boundedness of $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ follows immediately. Taking the supremum in (20) over all $f\in\mathscr{S}^p$ such that $\|f\|_{\mathscr{S}^p}\leqslant 1$ and letting $|z|\to 1$, we get

$$\lim_{|z|\to 1}\sup_{\|f\|_{\mathscr{S}^p}\leqslant 1}\mu(z)|(T^n_{u,v,\varphi}f)'(z)|=0.$$

By Lemma 5, this implies that $T^n_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ is compact. $\square$

The following theorem can be proved similar to Theorem 9. We omit the proof.

THEOREM 10. *Let* $1<p<\infty$, $u,v\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$ *and* μ *be a radial weight. Then the operator* $T_{u,v,\varphi}:\mathscr{S}^p\to\mathscr{B}_{\mu,0}$ *is compact if and only if* $u\in\mathscr{B}_{\mu,0}$ *and*

$$\limsup_{|z|\to 1}\frac{\mu(z)|u(z)\varphi'(z)+v'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}}}=\limsup_{|z|\to 1}\frac{\mu(z)|v(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1}{p}+1}}=0.$$

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(Received March 7, 2026)

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