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TWO LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING LOGARITHMIC FUNCTION

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Abstract. In the paper, using the integral representation of the Bernoulli numbers of the second kind, the authors establish the logarithmically complete monotonicity of two functions involving the logarithmic function, and show that the completely monotonic degree of another derived completely monotonic function involving the logarithmic function is equal to 1. These results resolve previously posed conjectures.

1. Preliminaries

Recall from the chapters [8, Chapter XIII] and [17, Chapter IV] and from the monograph [16] that a function $f(x)$ is said to be completely monotonic on $\mathbb{R}_+ = (0,\infty)$ if the inequalities

$$(-1)^n f^{(n)}(x) \geqslant 0, \quad n \in \mathbb{N}_0 = \{0,1,2,\dots\}$$

hold for all $x \in \mathbb{R}_+$.

Recall from the papers [1, 2, 6, 12, 13] and the monograph [16, Chapter 5] that a positive function $f(x)$ is said to be logarithmically completely monotonic on $\mathbb{R}_+$ if the inequalities

$$(-1)^n [\ln f(x)]^{(n)} \geqslant 0, \quad n \in \mathbb{N} = \{1,2,\dots\}$$

hold for all $x \in \mathbb{R}_+$.

In the papers [2, 6, 12, 14] and the monograph [16], it was proved that every logarithmically completely monotonic function on $\mathbb{R}_+$ must be a completely monotonic function on $\mathbb{R}_+$ but not conversely.

Let $f(x)$ be a completely monotonic function on $\mathbb{R}_+$ and $f(\infty) = \lim_{x\to\infty} f(x)$. Recall from [5, 9, 10] that, if the function $x^r[f(x) - f(\infty)]$ is completely monotonic on $\mathbb{R}_+$, but the function $x^{r+\varepsilon}[f(x) - f(\infty)]$ is not completely monotonic on $\mathbb{R}_+$, for some

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$r \in [0, \infty)$ and for any $\varepsilon > 0$, then we call the number r the completely monotonic degree of $f(x)$ with respect to $x \in \mathbb{R}_+$ and denote it by $\deg_{\mathrm{cm}}^x[f(x)] = r$; if the function $x^r[f(x) - f(\infty)]$ is completely monotonic on $\mathbb{R}_+$ for all $r \in \mathbb{R}$, then we say that the completely monotonic degree of $f(x)$ with respect to $x \in \mathbb{R}_+$ is ∞ and denote it by $\deg_{\mathrm{cm}}^x[f(x)] = \infty$.

The Bernstein–Widder theorem [17, p. 161, Theorem 12b] characterizes that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $\mathbb{R}_+$ is that

$$f(x) = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\mu(t), \quad x \in \mathbb{R}_+,$$

where $\mu(t)$ is non-decreasing and the above integral converges for $x \in \mathbb{R}_+$.

2. Motivation and main results

In [7, Remark 3], we conjectured that the function

$$h(x) = \begin{cases} \dfrac{(x^2-1)\ln(x+1)}{x^2\ln x}, & x \in \mathbb{R}_+ \setminus \{1\} \\ 2\ln 2, & x = 1 \end{cases} \tag{2.1}$$

is completely monotonic on $\mathbb{R}_+$. More strongly, we conjectured that $h(x)$ is a logarithmically completely monotonic function on $\mathbb{R}_+$.

In this paper, we give a confirmative resolution to the above conjectures.

Our main results are stated as follows.

THEOREM 1. *Let*

$$g(x) = \begin{cases} \dfrac{x\ln x}{x-1}, & x \in \mathbb{R}_+ \setminus \{1\}; \\ 1, & x = 1. \end{cases}$$

The reciprocal $\frac{1}{g(x)}$ *of the function* $g(x)$ *is logarithmically completely monotonic on* $\mathbb{R}_+$, *equivalently, the function* $\frac{g'(x)}{g(x)}$ *is completely monotonic on* $\mathbb{R}_+$, *and the completely monotonic degree of* $\frac{g'(x)}{g(x)}$ *is* 1, *i.e.,*

$$\deg_{\mathrm{cm}}^x\left[\frac{g'(x)}{g(x)}\right] = 1.$$

The function $h(x)$ *defined by* (2.1) *is logarithmically completely monotonic on* $\mathbb{R}_+$. *Consequently, it is also a completely monotonic function on* $\mathbb{R}_+$.

3. Proof of main results

We now start out to give nice proofs of our main results. Because our proofs are elegant and intricate, so they are short.

It is easy to see that

$$(-1)^n\left[\ln\frac{1}{g(x)}\right]^{(n)}\geqslant 0,\quad n\in\mathbb{N}\quad\Longleftrightarrow\quad(-1)^n\left[\frac{g'(x)}{g(x)}\right]^{(n)}\geqslant 0,\quad n\in\mathbb{N}_0.$$

Accordingly, the logarithmically complete monotonicity of the reciprocal $\frac{1}{g(x)}$ on $\mathbb{R}_+$ is equivalent to the complete monotonicity of $\frac{g'(x)}{g(x)}$ on $\mathbb{R}_+$.

A direct computation gives

$$\begin{aligned}\frac{g'(x)}{g(x)}&=\begin{cases}\dfrac{1}{x(x-1)}\left(\dfrac{x-1}{\ln x}-1\right), & x\in\mathbb{R}_+\setminus\{1\}\\ \dfrac{1}{2}, & x=1\end{cases}\\&=\frac{1}{x}\sum_{n=0}^{\infty}b_{n+1}(x-1)^n,\quad |x-1|<1,\end{aligned}\tag{3.1}$$

where the Bernoulli numbers of the second kind b_n, see [11, Remarks 7 and 14], also known [3, pp. 293–294] as the Cauchy numbers of the first kind, the Gregory coefficients, or logarithmic numbers, is generated [4] by

$$\frac{x}{\ln(1+x)}=\sum_{n=0}^{\infty}b_nx^n,\quad |x|<1,$$

with $b_0=1$, $b_1=\frac{1}{2}$, $b_2=-\frac{1}{12}$, $b_3=\frac{1}{24}$, and $b_4=-\frac{19}{720}$. In [15, Theorem 1], we established the integral representation

$$b_n=(-1)^{n+1}\int_0^{\infty}\frac{1}{[(\ln u)^2+\pi^2](1+u)^n}\,\mathrm{d}u,\quad n\in\mathbb{N}.\tag{3.2}$$

Substituting (3.2) into (3.1) gives

$$\begin{aligned}\frac{g'(x)}{g(x)}&=\frac{1}{x}\int_0^{\infty}\frac{1}{(\ln u)^2+\pi^2}\sum_{n=0}^{\infty}\frac{(1-x)^n}{(1+u)^{n+1}}\,\mathrm{d}u,\quad |x-1|<1\\&=\frac{1}{x}\int_0^{\infty}\frac{1}{(\ln u)^2+\pi^2}\frac{1}{u+x}\,\mathrm{d}u,\quad x\in\mathbb{R}_+,\end{aligned}$$

where we used the uniqueness theorem in complex analysis. Since the functions $\frac{1}{x}$ and $\int_0^{\infty}\frac{1}{(\ln u)^2+\pi^2}\frac{1}{u+x}\,\mathrm{d}u$ are completely monotonic of $x\in\mathbb{R}_+$ and the product of finitely many completely monotonic functions is still a completely monotonic function, the function $\frac{g'(x)}{g(x)}$ is completely monotonic on $\mathbb{R}_+$.

A straightforward computation yields $\lim_{x\to\infty}\frac{g'(x)}{g(x)}=0$. If the function $x^{\alpha}\frac{g'(x)}{g(x)}$ were completely monotonic on $\mathbb{R}_+$, then

$$\left[x^{\alpha}\frac{g'(x)}{g(x)}\right]'=x^{\alpha-1}\left\{\alpha\frac{g'(x)}{g(x)}+x\left[\frac{g'(x)}{g(x)}\right]'\right\}\leqslant 0,$$

that is,

$$\alpha\leqslant -x\left[\frac{g'(x)}{g(x)}\right]'\bigg/\frac{g'(x)}{g(x)}=\frac{\frac{1-2x}{(x-1)^2}+\frac{1}{\ln^2 x}+\frac{1}{\ln x}}{\frac{1}{\ln x}-\frac{1}{x-1}}\to 1,\quad x\to 0^+.$$

On the other hand, the function

$$x\frac{g'(x)}{g(x)}=\int_0^{\infty}\frac{1}{(\ln u)^2+\pi^2}\frac{1}{u+x}\,\mathrm{d}u \tag{3.3}$$

is clearly completely monotonic on $\mathbb{R}_+$. Consequently, the completely monotonic degree of $\frac{g'(x)}{g(x)}$ is 1.

A closer and more intrinsic observation enables us to write the function $h(x)$ as

$$h(x)=\frac{g(x+1)}{g(x)},\quad x\in\mathbb{R}_+.$$

This is the crucial step in our proof. Accordingly, we acquire

$$\ln h(x)=\ln g(x+1)-\ln g(x)$$

and

$$[\ln h(x)]'=\frac{g'(x+1)}{g(x+1)}-\frac{g'(x)}{g(x)}.$$

Since $\frac{g'(x)}{g(x)}$ is completely monotonic on $\mathbb{R}_+$, then $(-1)^n\left[\frac{g'(x)}{g(x)}\right]^{(n)}\geqslant 0$ for $n\in\mathbb{N}_0$, this means that $\left[\frac{g'(x)}{g(x)}\right]^{(2n)}$ for $n\in\mathbb{N}_0$ decreases and $\left[\frac{g'(x)}{g(x)}\right]^{(2n+1)}$ for $n\in\mathbb{N}_0$ increases on $\mathbb{R}_+$. Therefore, we arrive at

$$[\ln h(x)]^{(2n+1)}=\left[\frac{g'(x+1)}{g(x+1)}\right]^{(2n)}-\left[\frac{g'(x)}{g(x)}\right]^{(2n)}\leqslant 0$$

and

$$[\ln h(x)]^{(2n+2)}=\left[\frac{g'(x+1)}{g(x+1)}\right]^{(2n+1)}-\left[\frac{g'(x)}{g(x)}\right]^{(2n+1)}\geqslant 0$$

for $n\in\mathbb{N}_0$. Equivalently,

$$(-1)^n[\ln h(x)]^n\geqslant 0,\quad n\in\mathbb{N}.$$

Consequently, the function $h(x)$ is logarithmically completely monotonic, and then it is also completely monotonic, on $\mathbb{R}_+$.

4. Remarks

Finally, we give several remarks on two functions we discussed above.

REMARK 1. In [15, Lemma 2], the integral representation

$$\frac{1}{g(z)} = \int_0^\infty \frac{t+1}{t[(\ln t)^2+\pi^2]} \frac{\mathrm{d}t}{t+z}, \quad z \in \mathbb{C} \setminus (-\infty, 0] \tag{4.1}$$

was established. From the integral representation (4.1), we derive that the reciprocal $\frac{1}{g(x)}$ is a completely monotonic function on $\mathbb{R}_+$.

In fact, the reciprocal $\frac{1}{g(x)}$ is a Stieltjes transform, a stronger conclusion than being a logarithmically completely monotonic function on $\mathbb{R}_+$; see [14, p. 627, Eq. (1.4)]. A function $f : \mathbb{R}_+ \to [0, \infty)$ is said to be a Stieltjes transform, or a Stieltjes function, if it can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} \mathrm{d}\mu(s),$$

where $a, b \geqslant 0$ are non-negative constants and μ is a non-negative measure on $\mathbb{R}_+$ such that $\int_0^\infty \frac{1}{1+s} \mathrm{d}\mu(s) < \infty$. See [16, p. 16, Definition 2.1].

REMARK 2. The equation (3.3) means that the function $x\frac{g'(x)}{g(x)}$ is a Stieltjes transform.

REMARK 3. It is easy to see that $\lim_{x\to\infty} h(x) = 1$.

What is the completely monotonic degree of the function $h(x)$? In other words, what is the largest constant $\beta \in \mathbb{R}_0$ such that the function $x^\beta[h(x) - 1]$ is completely monotonic on $\mathbb{R}_+$?

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