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AN ELEMENTARY PROOF OF THE CONVEXITY OF $\log(\Gamma(x)\Gamma(1/x))$

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Abstract. We present a new elementary proof that the function $\Gamma(x)\Gamma(1/x)$ is log-convex for $x > 0$. The proof is self-contained and avoids the technical estimates of polygamma functions found in existing literature.

1. Introduction

The Euler Gamma function, defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, is one of the most fundamental special functions in mathematics. The digamma function, denoted by $\psi(x)$ is defined as the logarithmic derivative of the Gamma function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1}$$

For $x > 0$, the function admits the following series representation:

$$\psi(x) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n+1}\right), \tag{2}$$

where γ denotes the Euler-Mascheroni constant. Furthermore, the derivative of the digamma function, known as the trigamma function, is given by the series:

$$\psi'(x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2}, \quad x > 0. \tag{3}$$

Inequalities involving the sum and product of $\Gamma(x)$ and $\Gamma(1/x)$ were first established by Gautschi [4] in 1974. Numerous refinements and extensions of these inequalities have appeared in the literature [1, 2, 3, 6].

More recently, Jameson [5] proved that the function $L(x) = \log\Gamma(x) + \log\Gamma(1/x)$ is convex on $(0,\infty)$, which directly implies the log-convexity of the product $\Gamma(x)\Gamma(1/x)$. While Jameson's proof provides a rigorous foundation, it relies on detailed estimates of the polygamma functions – specifically, bounding the second derivative of the log-gamma function.

The purpose of this paper is to provide a strictly elementary proof of the convexity of $\log\Gamma(x) + \log\Gamma(1/x)$. By utilizing series expansions, we demonstrate that this result can be obtained without polygamma bounds.

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2. Main result

In the following theorem, we state and prove the main result.

THEOREM 1. *Let $f:(0,\infty)\to\mathbb{R}$ be the function defined by*

$$f(x)=\log\Gamma(x)+\log\Gamma\left(\frac{1}{x}\right).$$

Then $f(x)$ is strictly convex on the interval $(0,\infty)$.

Proof. The first and second derivative of $f(x)$ are given by

$$f'(x)=\psi(x)-\frac{1}{x^2}\psi\left(\frac{1}{x}\right),$$

and

$$f''(x)=\psi'(x)+\frac{2}{x^3}\psi\left(\frac{1}{x}\right)+\frac{1}{x^4}\psi'\left(\frac{1}{x}\right). \tag{4}$$

Now, using equations (2) and (3), equation (4) is equal to

$$f''(x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}-\frac{2\gamma}{x^3}+\frac{2}{x^3}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{x}{1+nx}\right)+\frac{1}{x^2}\sum_{n=0}^{\infty}\frac{1}{(1+nx)^2}. \tag{5}$$

Further simplifying equation (5) into a common summation, we obtain

$$f''(x)=\frac{2(1-\gamma)}{x^3}+\sum_{n=1}^{\infty}\left(\frac{1}{(n+x)^2}+\frac{2}{x^3(n+1)}-\frac{2}{x^2(nx+1)}+\frac{1}{x^2(nx+1)^2}\right). \tag{6}$$

Further simplifying equation (6), we obtain

$$f''(x)=\frac{2(1-\gamma)}{x^3}+\sum_{n=1}^{\infty}\frac{P_n(x)}{x^3(n+1)(n+x)^2(nx+1)^2}, \tag{7}$$

where $P_n(x)=A_n(x)+B_n(x)+C_n(x)+2x^2$, with

$$\begin{aligned}A_n(x)&=n^3x(x^4-2x+3),\\ B_n(x)&=n^2(x^5+2x^4-4x^3+6x^2-x+2),\\ C_n(x)&=nx(4x^2-2x+4).\end{aligned}$$

Since $1-\gamma\approx 0.422>0$, it suffices to show that $P_n(x)>0$ for $x>0$. Next, we claim that $A_n(x)>0$, $B_n(x)>0$, and $C_n(x)>0$ on $(0,\infty)$.

Claim I. $C_n(x)>0$. Observe that $4x^2-2x+4=4[(x-\frac{1}{4})^2+\frac{15}{16}]$, which is strictly positive for all $x\in\mathbb{R}$. Thus, $C_n(x)>0$ for $x>0$.

Claim II. We show that $A_n(x) > 0$ for all $x > 0$. Grouping the terms of $A_n(x)$ as follows:

$$\begin{aligned} A_n(x) &= n^3 x(x^4 - 2x + 3) \\ &= n^3 x(x^4 - 2x^2 + 2x^2 - 2x + 3) \\ &= n^3 x\left[(x^2-1)^2 + 2\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right] > 0 \quad \text{for all } x > 0. \end{aligned}$$

Thus, $A_n(x) > 0$ for $x > 0$.

Claim III. We show that $B_n(x) > 0$ for all $x > 0$. Grouping the terms of $B_n(x)$ as follows:

$$\begin{aligned} \frac{1}{n^2} B_n(x) &= x^5 + x^4 + (x^4 - 4x^3 + 4x^2) + 2x^2 - x + 2 \\ &= x^4(x+1) + x^2(x-2)^2 + 2\left[\left(x - \frac{1}{4}\right)^2 + \frac{15}{16}\right] > 0 \quad \text{for all } x > 0. \end{aligned}$$

Since $1 - \gamma > 0$ and all coefficients A_n, B_n, C_n are positive, it follows that $f''(x) > 0$ for all $x > 0$. Thus, $f(x)$ is strictly convex. $\square$

COROLLARY 1. *The product of the Gamma functions,* $\Gamma(x)\Gamma(1/x)$*, is log-convex and thus strictly convex for all* $x > 0$.

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