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# AN EXTENSION OF THE $a$–SEMI–NORM ON $C^*$–ALGEBRAS

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*Abstract.* In this paper, we introduce the $a_{(\alpha,\beta)}$-semi-norm of elements in $C^*$-algebras, which generalizes the $a$-numerical radius $v_a(x)$ and semi-norm $\|x\|_a$. We study some of the fundamental properties of the $a_{(\alpha,\beta)}$-semi-norm and also get some results of its upper and lower bounds. Further, as applications of our results, we obtain some bounds on the $a$-numerical radius, which improves on some important inequalities that exist.

## 1. Introduction

Let $\mathfrak{A}$ be a $C^*$-algebra with unit denoted by $\mathbf{1}$ and $a \in \mathfrak{A}$ be a positive element. Let $\mathfrak{A}^*$ denote the topological dual space of $\mathfrak{A}$. A linear functional $f \in \mathfrak{A}^*$ is said to be positive, and write $f \geqslant 0$, if $f(x^*x) \geqslant 0$ for all $x \in \mathfrak{A}$. The set of states on $\mathfrak{A}$ is defined by

$$S(\mathfrak{A}) = \{f \in \mathfrak{A}^* : f \geqslant 0 \text{ and } f(\mathbf{1}) = \|f\| = 1\}.$$

Recall that a positive linear functional $f$ on $\mathfrak{A}$ is said to be pure if for every positive functional $g$ on $\mathfrak{A}$ satisfying $g(x^*x) \leqslant f(x^*x)$ for all $x \in \mathfrak{A}$, there exists a scalar $0 \leqslant \mu \leqslant 1$ such that $g = \mu f$. The set of pure states on $\mathfrak{A}$ is denoted by $P(\mathfrak{A})$. The algebraic numerical range of $x \in \mathfrak{A}$ is defined by

$$V(x) = \{f(x) : f \in S(\mathfrak{A})\}.$$

It is a nonempty compact convex subset of $\mathbb{C}$ and its maximum modulus is the so-called algebraic numerical radius of $x \in \mathfrak{A}$ defined by

$$v(x) = \sup\{|\lambda| : \lambda \in V(x)\}.$$

For more on the fundamental properties of $C^*$-algebras, see [9, 11, 12] and references therein.

When $\mathfrak{A} = \mathbb{B}(\mathscr{H})$ is the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$, the spatial numerical range of $T \in \mathbb{B}(\mathscr{H})$ is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

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The spatial numerical radius of $T$ is defined by

$$\omega(T) = \sup\{|z| : z \in W(T)\},$$

and coincide with $v(T)$ since the closure of $W(T)$ is nothing but $V(T)$.

Numerical range and numerical radius play an important role in the fields of analysis, operator theory and mathematical physics. In recent years, there are many generalizations about classical numerical range and numerical radius, and many mathematicians have paid considerable attention to the study of these concepts; see [1, 3, 6, 13, 14, 15, 16, 17] and the references therein.

In 2021, Bourhim and Mabrouk [7] introduced the $a$-numerical range and $a$-numerical radius of elements in $C^*$-algebras. They established some persistence properties of these concepts. In addition, they investigated the relationship between the algebraic and spatial $a$-numerical range of bounded linear operators on a complex Hilbert space $\mathscr{H}$. Furthermore, the authors in [2, 10] continued to study the $a$-numerical range and $a$-numerical radius.

Set $S_a(\mathfrak{A}) = \left\{ \frac{f}{f(a)} : f \in S(\mathfrak{A}), f(a) \neq 0 \right\}$. By [7, Proposition 2.3], $S_a(\mathfrak{A})$ is a non empty, convex and closed subset of the topological dual space of $\mathfrak{A}$, but it is compact if and only if $a$ is invertible in $\mathfrak{A}$. We say that a linear functional $f \in S_a(\mathfrak{A})$ is $a$-pure if $f$ has the property that whenever $g$ is a positive linear functional on $\mathfrak{A}$ such that $g \leqslant f$, then there is a scalar $0 \leqslant \mu \leqslant 1$ such that $g = \mu f$. Clearly, $f \in S_{\mathbf{1}}(\mathfrak{A}) = S(\mathfrak{A})$ is $\mathbf{1}$-pure if it is a pure state in the usual sense. The set of $a$-pure states on $\mathfrak{A}$ is denoted by $P_a(\mathfrak{A})$. Let $\|x\|_a = \sup\{\sqrt{f(x^*ax)} : f \in S_a(\mathfrak{A})\}$ for any element $x \in \mathfrak{A}$. Observe that $\|\cdot\|_{\mathbf{1}} = \|\cdot\|$ and $\|x\|_a = 0$ if and only if $ax = 0$. By [7, Example 3.2], for some $x \in \mathfrak{A}$, it may happen that $\|x\|_a = \infty$. We set $\mathfrak{A}^a = \{x \in \mathfrak{A} : \|x\|_a < \infty\}$. By [7, Proposition 3.3], $\|\cdot\|_a$ is a seminorm on $\mathfrak{A}^a$ and $\|xy\|_a \leqslant \|x\|_a\|y\|_a$ holds for any $x, y \in \mathfrak{A}^a$. Thus $\mathfrak{A}^a$ is a subalgebra of $\mathfrak{A}$ not necessarily closed. For $x \in \mathfrak{A}$, an element $x^{\sharp_a} \in \mathfrak{A}$ is called an $a$-adjoint of $x$ if $ax^{\sharp_a} = x^*a$. Generally, the existence of an $a$-adjoint element is not guaranteed. Following [3], the set of all elements in $\mathfrak{A}$ which admit $a$-adjoints is denoted by $\mathfrak{A}_a$. Note that $\mathfrak{A}_a$ is a subalgebra of $\mathfrak{A}^a$ which is neither closed nor dense in $\mathfrak{A}$. Moreover, $\mathfrak{A}_a = \mathfrak{A}$ if $\mathfrak{A}$ is commutative but in general $\mathfrak{A}_a \neq \mathfrak{A}$. Also, neither the existence nor the uniqueness of $a$-adjoint elements is guaranteed. An element $x \in \mathfrak{A}_a$ is said to be $a$-self-adjoint if $ax$ is self-adjoint, i.e., $ax = x^*a$. Observe that if $x$ is $a$-self-adjoint, then $x \in \mathfrak{A}_a$. However it does not hold, in general, that $x = x^{\sharp_a}$. And we say that $x$ is $a$-positive if $ax$ is positive. In addition, $|x|_a^2 = x^{\sharp_a}x$, $|x^{\sharp_a}|_a^2 = xx^{\sharp_a}$ are $a$-self-adjoint and $a$-positive. So we have [7, Corollary 4.9]

$$\|x\|_a^2 = \|xx^{\sharp_a}\|_a = \|x^{\sharp_a}x\|_a = \|x^{\sharp_a}\|_a^2. \tag{1}$$

Let $V_a(x)$ and $v_a(x)$ denote the $a$-numerical range and the $a$-numerical radius of $x \in \mathfrak{A}$, respectively, defined by

$$V_a(x) = \{f(ax) : f \in S_a(\mathfrak{A})\},$$

and

$$v_a(x) = \sup\{|\lambda| : \lambda \in V_a(x)\}.$$

Let $m_a(x)$ and $c_a(x)$ denote the $a$-minimum modulus and the $a$-numerical radius Crawford number of $x \in \mathfrak{A}$, respectively, defined by

$$m_a(x) = \inf\{\sqrt{f(x^*ax)} : f \in S_a(\mathfrak{A})\},$$

and

$$c_a(x) = \inf\{|f(ax)| : f \in S_a(\mathfrak{A})\}.$$

Observe that the $a$-numerical radius of an element $x \in \mathfrak{A}_a$ is equivalent to the $a$-semi-norm of $x \in \mathfrak{A}_a$, see [7, Proposition 3.3 and Corollary 4.10]. More precisely, we have

$$\frac{1}{2}\|x\|_a \leqslant v_a(x) \leqslant \|x\|_a \tag{2}$$

for all $x \in \mathfrak{A}_a$. When $ax \neq 0$ and $ax^2 = 0$, the inequality (2) on the left becomes an equality, and when $ax = x^*a$, the inequality (2) on the right becomes an equality; see [7, 10]. There is also a fundamental inequality about $v_a(x)$, (see [7, Theorem 4.4]),

$$v_a(x^n) \leqslant v_a^n(x), \tag{3}$$

for any $x \in \mathfrak{A}$ and $n \geqslant 1$. Any element $x \in \mathfrak{A}_a$ can be presented by the Cartesian decomposition as $x = \mathfrak{R}_a(x) + i\mathfrak{I}_a(x)$, where $\mathfrak{R}_a(x) = \frac{1}{2}\left(x + x^{\sharp_a}\right)$ and $\mathfrak{I}_a(x) = \frac{1}{2i}\left(x - x^{\sharp_a}\right)$ are the $a$-real and $a$-imaginary parts of $x$, respectively. Note that $\mathfrak{R}_a(x)$ and $\mathfrak{I}_a(x)$ are $a$-self-adjoint. An important identity for the $a$-numerical radius was given in [7] as follows:

$$v_a(x) = \sup_{\theta \in \mathbb{R}} \left\|\mathfrak{R}_a\left(e^{i\theta}x\right)\right\|_a \tag{4}$$

for all $x \in \mathfrak{A}$. Unlike the classical algebraic numerical range, the $a$-numerical range $V_a(x)$ of $x \in \mathfrak{A}$ may be unbounded. Initially, these concepts were introduced in [7] as generalizations of the $A$-numerical range and $A$-numerical radius of the operator $T \in \mathbb{B}(\mathscr{H})$, defined by

$$W_A(T) = \{\langle ATx, x\rangle : x \in \mathscr{H}, \|x\|_A = 1\},$$

$$\omega_A(T) = \sup\{|z| : z \in W_A(T)\},$$

where $A$ is a positive operator on $\mathscr{H}$ and $\|x\|_A = \sqrt{\langle Ax, x\rangle}$ for all $x \in \mathscr{H}$. Clearly, when $A$ is the identity operator on $\mathscr{H}$, then $W_A(T) = W(T)$ and $\omega_A(T) = \omega(T)$ for all $T \in \mathbb{B}(\mathscr{H})$. Recently, the concept of the $a$-spectral radius of $x \in \mathfrak{A}_a$ has been introduced in [2] as follows:

$$r_a(x) = \lim_{n \to \infty} \|x^n\|_a^{1/n}.$$

Note that an element $x \in \mathfrak{A}^a$ is $a$-normaloid if $r_a(x) = \|x\|_a$. It was shown in [2] that $x$ is $a$-normaloid if and only if $v_a(x) = \|x\|_a$.

Based on the inspiration of [4, 5, 14], for each $\alpha,\beta \geqslant 0$, such that $(\alpha,\beta) \neq (0,0)$, we introduce a new $a_{(\alpha,\beta)}$-semi-norm on $C^*$-algebra, defined as follows:

$$\|x\|_{a_{(\alpha,\beta)}} = \sup\left\{ \sqrt{\alpha\,|f(ax)|^2 + \beta f(x^*ax)} : f \in S_a(\mathfrak{A}) \right\}.$$

Clearly, if $(\alpha,\beta) = (1,0)$, then $\|x\|_{a_{(\alpha,\beta)}} = v_a(x)$ and if $(\alpha,\beta) = (0,1)$, then $\|x\|_{a_{(\alpha,\beta)}} = \|x\|_a$.

Note that in the definition of the $a_{(\alpha,\beta)}$-semi-norm, we can assume that $\alpha^2+\beta^2 = 1$. Indeed, for any $x \in \mathfrak{A}$, we have

$$\begin{aligned}
&\|x\|_{a_{(\alpha,\beta)}}\\
&= (\alpha^2+\beta^2)^{\frac{1}{4}} \sup\left\{ \sqrt{\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\,|f(ax)|^2 + \frac{\beta}{\sqrt{\alpha^2+\beta^2}} f(x^*ax)} : f \in S_a(\mathfrak{A}) \right\}\\
&= (\alpha^2+\beta^2)^{\frac{1}{4}} \sup\left\{ \sqrt{\cos(\theta)\,|f(ax)|^2 + \sin(\theta) f(x^*ax)} : f \in S_a(\mathfrak{A}) \right\}\\
&= (\alpha^2+\beta^2)^{\frac{1}{4}}\, \|x\|_{a_{(\cos(\theta),\,\sin(\theta))}},
\end{aligned}$$

where

$$\cos(\theta) = \tfrac{\alpha}{\sqrt{\alpha^2+\beta^2}} \quad \text{and} \quad \sin(\theta) = \tfrac{\beta}{\sqrt{\alpha^2+\beta^2}}$$

for some $\theta \in [0, \frac{\pi}{2}]$.

In this paper, we show that the $a_{(\alpha,\beta)}$-semi-norm is equivalent to the $a$-numerical radius and the usual semi-norm. We obtain several basic properties such as upper and lower bounds for the $a_{(\alpha,\beta)}$-semi-norm. By these inequalities, we give out some bounds on the $a$-numerical radius, which improve and generalize the existing related results.

## 2. Main results

In this section, we consider the fundamental properties of $a_{(\alpha,\beta)}$-semi-norm, give the various bounds of the $a_{(\alpha,\beta)}$-semi-norm, and obtain a new upper bound on the $a$-numerical radius in $C^*$-algebra. The first result is as follows.

THEOREM 1. *$\|\cdot\|_{a_{(\alpha,\beta)}}$ defines a semi-norm on $\mathfrak{A}_a$. This semi-norm is equivalent to the $a$-numerical radius and the $a$-semi-norm, satisfying the following inequalities:*

*(i)* $\sqrt{\alpha+\beta}\, v_a(x) \leqslant \|x\|_{a_{(\alpha,\beta)}} \leqslant \sqrt{\alpha+4\beta}\, v_a(x)$,

*(ii)* $\max\left\{ \frac{\sqrt{\alpha+\beta}}{2}, \sqrt{\beta} \right\} \|x\|_a \leqslant \|x\|_{a_{(\alpha,\beta)}} \leqslant \sqrt{\alpha+\beta}\, \|x\|_a$

*for all $x \in \mathfrak{A}_a$.*

*Proof.* The proof follows from the inequality (2) and the definition of $a_{(\alpha,\beta)}$-semi-norm. $\square$

Obviously, when $ax \neq 0$ and $ax^2 = 0$, $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha+4\beta}\,\nu_a(x)$, and when $x$ is $a$-self-adjoint, $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha+\beta}\,\nu_a(x) = \sqrt{\alpha+\beta}\,\|x\|_a$.

In the following theorem, we characterize several equation conditions.

THEOREM 2. *Let $x \in \mathfrak{A}_a$ and $\alpha\beta \neq 0$. Then the following statements are equivalent:*

(*i*) $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2}$.

(*ii*) *There exists a sequence $(f_n)_{n\geqslant 1} \in S_a(\mathfrak{A})$ such that $\lim_{n\to\infty} f_n(x^*ax) = \|x\|_a^2$ and $\lim_{n\to\infty} |f_n(ax)| = \nu_a(x)$.*

*Proof.* $(i) \Rightarrow (ii)$ Suppose $(i)$ holds. By the definition of $\|x\|_{a_{(\alpha,\beta)}}$ we know that there exists a sequence $(f_n)_{n\geqslant 1} \in S_a(\mathfrak{A})$ such that

$$\|x\|_{a_{(\alpha,\,\beta)}} = \lim_{n\to\infty} \sqrt{\alpha\,|f_n\,(ax)|^2 + \beta f_n\,(x^*ax)}.$$

Clearly, $|f_n(ax)|$ and $f_n(x^*ax)$ are both bounded sequences of real numbers and so there exists a subsequence $(f_{n_k})_{k\geqslant 1}$ of $(f_n)_{n\geqslant 1}$ such that both $\left|f_{n_k}(ax)\right|$ and $f_{n_k}(x^*ax)$ are convergent. So we get,

$$\begin{aligned}\sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2} &= \|x\|_{a_{(\alpha,\beta)}}\\ &= \lim_{k\to\infty} \sqrt{\alpha\left|f_{n_k}\,(ax)\right|^2 + \beta f_{n_k}\,(x^*ax)}\\ &\leqslant \sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2}.\end{aligned}$$

Then we get $\lim_{k\to\infty} f_{n_k}(x^*ax) = \|x\|_a^2$ and $\lim_{k\to\infty} \left|f_{n_k}(ax)\right| = \nu_a(x)$.

$(ii) \Rightarrow (i)$ Suppose that there exists a sequence $(f_n)_{n\geqslant 1} \in S_a(\mathfrak{A})$ such that $\lim_{n\to\infty} f_n(x^*ax) = \|x\|_a^2$ and $\lim_{n\to\infty} |f_n(ax)| = \nu_a(x)$.

$$\begin{aligned}\|x\|_{a_{(\alpha,\beta)}} &= \sup \sqrt{\alpha\,|f\,(ax)|^2 + \beta f\,(x^*ax)}\\ &\geqslant \lim_{n\to\infty} \sqrt{\alpha\,|f_n\,(ax)|^2 + \beta f_n\,(x^*ax)}\\ &= \sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2}.\end{aligned}$$

Since $\|x\|_{a_{(\alpha,\beta)}} \leqslant \sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2}$, we have $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha \nu_a^2(x) + \beta\,\|x\|_a^2}$. $\square$

As a consequence of the preceding theorem, we have the following result.

COROLLARY 1. *Let $x \in \mathfrak{A}_a$ and $\alpha\beta \neq 0$. If $x$ is $a$-normaloid, then the following statements are equivalent:*

(i) $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha+\beta}\,\nu_a(x)$.

(ii) $\|x\|_{a_{(\alpha,\beta)}} = \sqrt{\alpha+\beta}\,\|x\|_a$.

(iii) *There exists a sequence $(f_n)_{n\geqslant 1} \in S_a(\mathfrak{A})$ such that $\lim\limits_{n\to\infty} f_n(x^*ax) = \|x\|_a^2$ and $\lim\limits_{n\to\infty} |f_n(ax)| = \nu_a(x)$.*

Now, in the following theorem we obtain a lower bound for the $a_{(\alpha,\beta)}$-semi-norm.

THEOREM 3. *Let $x \in \mathfrak{A}_a$. Then*

$$\|x\|_{a_{(\alpha,\beta)}}^2 \geqslant \max\Big\{\alpha\nu_a^2(x) + \beta c_a(x^{\sharp_a}x), \alpha c_a^2(x) + \beta\|x\|_a^2, 2\sqrt{\alpha\beta}\,\nu_a(x)\sqrt{c_a(x^{\sharp_a}x)}, 2\sqrt{\alpha\beta}\,c_a(x)\|x\|_a\Big\}.$$

*Proof.* Let $f \in S_a(\mathfrak{A})$. Then we get

$$\begin{aligned}\|x\|_{a_{(\alpha,\beta)}}^2 &\geqslant \alpha|f(ax)|^2 + \beta f(x^*ax)\\ &= \alpha|f(ax)|^2 + \beta f\left(ax^{\sharp_a}x\right)\\ &\geqslant \alpha|f(ax)|^2 + \beta c_a(x^{\sharp_a}x).\end{aligned}$$

Taking supremum for $f \in S_a(\mathfrak{A})$, we get that

$$\|x\|_{a_{(\alpha,\beta)}}^2 \geqslant \alpha\nu_a^2(x) + \beta c_a(x^{\sharp_a}x). \tag{5}$$

Similarly,

$$\|x\|_{a_{(\alpha,\beta)}}^2 \geqslant \alpha c_a^2(x) + \beta\|x\|_a^2. \tag{6}$$

We also have

$$\begin{aligned}\|x\|_{a_{(\alpha,\beta)}}^2 &\geqslant \alpha|f(ax)|^2 + \beta f(x^*ax)\\ &\geqslant 2\sqrt{\alpha\beta}\,|f(ax)|\sqrt{f(ax^{\sharp_a}x)}\\ &\geqslant 2\sqrt{\alpha\beta}\,|f(ax)|\sqrt{c_a(x^{\sharp_a}x)}.\end{aligned}$$

Taking supremum for $f \in S_a(\mathfrak{A})$, we get that

$$\|x\|_{a_{(\alpha,\beta)}}^2 \geqslant 2\sqrt{\alpha\beta}\,\nu_a(x)\sqrt{c_a(x^{\sharp_a}x)}. \tag{7}$$

Similarly,

$$\|x\|^2_{a_{(\alpha,\beta)}} \geqslant 2\sqrt{\alpha\beta}\, c_a(x)\, \|x\|_a. \tag{8}$$

Combining (5), (6) and (7), (8) we get our desired inequality. $\square$

Below, we give useful lemma for proving the latter theorems.

LEMMA 1. *Let $x,\ y \in \mathfrak{A}_a$. Then the following inequalities hold:*

$(i)$ *If $xy = yx$, then $v_a(xy) \leqslant 2v_a(x)v_a(y)$.*

$(ii)$ *If $(xy)^{\sharp_a} = x^{\sharp_a}y$, then $v_a(xy) \leqslant v_a(x)\,\|y\|_a$.*

*Proof.* $(i)$ When $xy = yx$, without loss of generality we assume that $v_a(x) = 1$ and $v_a(y) = 1$. Then

$$\begin{aligned}
v_a(xy) &= v_a\left(\frac{1}{4}\left((x+y)^2 - (x-y)^2\right)\right)\\
&\leqslant \frac{1}{4}\left(v_a\left((x+y)^2\right) + v_a\left((x-y)^2\right)\right)\\
&\leqslant \frac{1}{4}\left(v_a^2\,(x+y) + v_a^2\,(x-y)\right) \qquad (\text{by } (3))\\
&\leqslant \frac{1}{4}\left(\left(v_a\,(x) + v_a\,(y)\right)^2 + \left(v_a\,(x) + v_a\,(y)\right)^2\right)\\
&= 2.
\end{aligned}$$

Thus, $v_a(xy) \leqslant 2v_a(x)v_a(y)$.

$(ii)$ When $(xy)^{\sharp_a} = x^{\sharp_a}y$,

$$\begin{aligned}
v_a(xy) &= v_a\left((xy)^{\sharp_a}\right) = \frac{1}{2}v_a\left((xy)^{\sharp_a} + x^{\sharp_a}y\right) = \frac{1}{2}v_a\left(\left((xy)^{\sharp_a} + x^{\sharp_a}y\right)^{\sharp_a}\right)\\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\frac{e^{i\theta}\left((xy)^{\sharp_a} + x^{\sharp_a}y\right)^{\sharp_a} + \left(e^{i\theta}\left((xy)^{\sharp_a} + x^{\sharp_a}y\right)^{\sharp_a}\right)^{\sharp_a}}{2}\right\|_a\\
&\leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left(\left\|\frac{e^{-i\theta}x^{\sharp_a} + \left(e^{-i\theta}x^{\sharp_a}\right)^{\sharp_a}}{2}\left(y^{\sharp_a}\right)^{\sharp_a}\right\|_a + \left\|y^{\sharp_a}\frac{e^{-i\theta}x^{\sharp_a} + \left(e^{-i\theta}x^{\sharp_a}\right)^{\sharp_a}}{2}\right\|_a\right)\\
&\leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\frac{e^{-i\theta}x^{\sharp_a} + \left(e^{-i\theta}x^{\sharp_a}\right)^{\sharp_a}}{2}\right\|_a\left(\left\|\left(y^{\sharp_a}\right)^{\sharp_a}\right\|_a + \left\|y^{\sharp_a}\right\|_a\right)\\
&= \frac{1}{2}v_a\left(x^{\sharp_a}\right)\left(\|y\|_a + \|y\|_a\right)\\
&= v_a(x)\,\|y\|_a.
\end{aligned}$$

Then $v_a(xy) \leqslant v_a(x)\,\|x\|_a$. $\square$

Next, in the following theorems we obtain upper bounds for the $a_{(\alpha,\beta)}$-semi-norm of the product of two elements.

THEOREM 4. *Let $x,\ y \in \mathfrak{A}_a$ and $\beta \neq 0$. Then we have the following inequality:*

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \min\left\{\frac{2}{\sqrt{\beta}}, \frac{\sqrt{\alpha+\beta}}{\beta}, \frac{4}{\sqrt{\alpha+\beta}}\right\} \|x\|_{a_{(\alpha,\beta)}} \|y\|_{a_{(\alpha,\beta)}}.$$

*Proof.* Let $f \in S_a(\mathfrak{A})$. Then by Theorem 1 we have

$$\begin{aligned}
\|xy\|^2_{a_{(\alpha,\beta)}} &= \sup_{f\in S_a(\mathfrak{A})} \left\{\alpha\,|f(axy)|^2 + \beta f((xy)^* a(xy))\right\} \\
&\leqslant \alpha v_a^2(xy) + \beta\,\|xy\|_a^2 \\
&\leqslant (\alpha+\beta)\,\|x\|_a^2\,\|y\|_a^2 \\
&\leqslant 4(\alpha+\beta) v_a^2(x)\,\|y\|_a^2 \\
&\leqslant 4\frac{\alpha+\beta}{\beta} v_a^2(x)\,\|y\|^2_{a_{(\alpha,\beta)}} \\
&\leqslant \frac{4}{\beta}\,\|x\|^2_{a_{(\alpha,\beta)}}\,\|y\|^2_{a_{(\alpha,\beta)}}.
\end{aligned} \tag{9}$$

Therefore,

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \frac{2}{\sqrt{\beta}}\,\|x\|_{a_{(\alpha,\beta)}}\,\|y\|_{a_{(\alpha,\beta)}}. \tag{10}$$

By using $\sqrt{\beta}\,\|x\|_a \leqslant \|x\|_{a_{(\alpha,\beta)}}$ and $\sqrt{\beta}\,\|y\|_a \leqslant \|y\|_{a_{(\alpha,\beta)}}$ in (9), we have

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \frac{\sqrt{\alpha+\beta}}{\beta}\,\|x\|_{a_{(\alpha,\beta)}}\,\|y\|_{a_{(\alpha,\beta)}}. \tag{11}$$

By using (2) and $\sqrt{\alpha+\beta}\,v_a(x) \leqslant \|x\|_{a_{(\alpha,\beta)}}$, $\sqrt{\alpha+\beta}\,v_a(y) \leqslant \|y\|_{a_{(\alpha,\beta)}}$ in (9), we have

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \frac{4}{\sqrt{\alpha+\beta}}\,\|x\|_{a_{(\alpha,\beta)}}\,\|y\|_{a_{(\alpha,\beta)}}. \tag{12}$$

Combining (10), (11) and (12) we get our desired inequality. $\square$

THEOREM 5. *Let $x,\ y \in \mathfrak{A}_a$ and $\beta \neq 0$. Then the following inequalities hold:*

$(i)$ *If $xy = yx$, then*

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \min\left\{\sqrt{\frac{4\alpha}{(\alpha+\beta)^2} + \frac{1}{\beta}}, \frac{\sqrt{4\alpha+16\beta}}{\alpha+\beta}\right\} \|x\|_{a_{(\alpha,\beta)}}\,\|y\|_{a_{(\alpha,\beta)}}.$$

*(ii) If $(xy)^{\sharp_a} = x^{\sharp_a}y$, then*

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \sqrt{\frac{\alpha}{\alpha+\beta}+1}\min\left\{\frac{2}{\sqrt{\alpha+\beta}},\frac{1}{\sqrt{\beta}}\right\}\|x\|_{a_{(\alpha,\beta)}}\|y\|_{a_{(\alpha,\beta)}}.$$

*Proof.* $(i)$ Let $f \in S_a(\mathfrak{A})$. Then we have

$$\begin{aligned}
\|xy\|^2_{a_{(\alpha,\beta)}} &= \sup_{f\in S_a(\mathfrak{A})}\left\{\alpha\,|f(axy)|^2+\beta f((xy)^*a(xy))\right\}\\
&\leqslant \alpha v_a^2(xy)+\beta\,\|xy\|_a^2\\
&\leqslant 4\alpha v_a^2(x)v_a^2(y)+\beta\,\|x\|_a^2\,\|y\|_a^2 \qquad \text{(by Lemma 1 } (i))\\
&\leqslant \left(\frac{4\alpha}{(\alpha+\beta)^2}+\min\left\{\frac{2}{\sqrt{\alpha+\beta}},\frac{1}{\sqrt{\beta}}\right\}^4\beta\right)\|x\|^2_{a_{(\alpha,\beta)}}\|y\|^2_{a_{(\alpha,\beta)}}\\
&\qquad \text{(by Theorem 1)}\\
&= \min\left\{\frac{4\alpha}{(\alpha+\beta)^2}+\frac{1}{\beta},\frac{4\alpha+16\beta}{(\alpha+\beta)^2}\right\}\|x\|^2_{a_{(\alpha,\beta)}}\|y\|^2_{a_{(\alpha,\beta)}}.
\end{aligned}$$

Thus,

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \min\left\{\sqrt{\frac{4\alpha}{(\alpha+\beta)^2}+\frac{1}{\beta}},\frac{\sqrt{4\alpha+16\beta}}{\alpha+\beta}\right\}\|x\|_{a_{(\alpha,\beta)}}\|y\|_{a_{(\alpha,\beta)}}.$$

$(ii)$ Let $f \in S_a(\mathfrak{A})$. Then we have

$$\begin{aligned}
\|xy\|^2_{a_{(\alpha,\beta)}} &= \sup_{f\in S_a(\mathfrak{A})}\left\{\alpha\,|f(axy)|^2+\beta f((xy)^*a(xy))\right\}\\
&\leqslant \alpha v_a^2(xy)+\beta\,\|xy\|_a^2\\
&\leqslant \alpha v_a^2(x)\,\|y\|_a^2+\beta\,\|x\|_a^2\,\|y\|_a^2 \qquad \text{(by Lemma 1 } (ii))\\
&\leqslant \frac{\alpha}{\alpha+\beta}\|x\|^2_{a_{(\alpha,\beta)}}\|y\|_a^2+\|x\|^2_{a_{(\alpha,\beta)}}\|y\|_a^2 \qquad \text{(by Theorem 1)}\\
&= \left(\frac{\alpha}{\alpha+\beta}+1\right)\|x\|^2_{a_{(\alpha,\beta)}}\|y\|_a^2\\
&\leqslant \left(\frac{\alpha}{\alpha+\beta}+1\right)\min\left\{\frac{2}{\sqrt{\alpha+\beta}},\frac{1}{\sqrt{\beta}}\right\}^2\|x\|^2_{a_{(\alpha,\beta)}}\|y\|^2_{a_{(\alpha,\beta)}}.
\end{aligned}$$

Thus,

$$\|xy\|_{a_{(\alpha,\beta)}} \leqslant \sqrt{\frac{\alpha}{\alpha+\beta}+1}\min\left\{\frac{2}{\sqrt{\alpha+\beta}},\frac{1}{\sqrt{\beta}}\right\}\|x\|_{a_{(\alpha,\beta)}}\|y\|_{a_{(\alpha,\beta)}}. \qquad \square$$

Now, let us use the Cartesian decomposition to get some inequalities.

THEOREM 6. *Let $x \in \mathfrak{A}_a$. Then the following inequalities hold:*

$(i)$ $\|x\|^2_{a_{(\alpha,\beta)}} \geqslant \max\left\{\frac{1}{2}\left\|\frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a, \frac{1}{3}\left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a\right\}.$

$(ii)$ $\|x\|^2_{a_{(\alpha,\beta)}} \leqslant \left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a.$

*Proof.* $(i)$ It is clearly that $\left(\Re_a\left(x^{\sharp_a}\right)\right)^{\sharp_a} = \Re_a\left(x^{\sharp_a}\right)$, $\left(\Im_a\left(x^{\sharp_a}\right)\right)^{\sharp_a} = \Im_a\left(x^{\sharp_a}\right)$, and

$$\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right) = \frac{x^{\sharp_a}\left(x^{\sharp_a}\right)^{\sharp_a} + \left(x^{\sharp_a}\right)^{\sharp_a}x^{\sharp_a}}{2} = \left(\frac{x^{\sharp_a}x + xx^{\sharp_a}}{2}\right)^{\sharp_a}.$$

By [10], we have

$$v_a^2(x) = v_a^2\left(x^{\sharp_a}\right) \geqslant \max\left\{\left\|\Re_a\left(x^{\sharp_a}\right)\right\|_a^2, \left\|\Im_a\left(x^{\sharp_a}\right)\right\|_a^2\right\}.$$

Therefore,

$$\begin{aligned}\sup_{f\in S_a(\mathfrak{A})}|f(ax)|^2 &\geqslant \max\left\{\sup_{f\in S_a(\mathfrak{A})} f\left(a\Re_a^2\left(x^{\sharp_a}\right)\right), \sup_{f\in S_a(\mathfrak{A})} f\left(a\Im_a^2\left(x^{\sharp_a}\right)\right)\right\}\\ &\geqslant \sup_{f\in S_a(\mathfrak{A})} f\left(\frac{a}{2}\left(\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right)\right)\right).\end{aligned}$$

Thus,

$$\|x\|^2_{a_{(\alpha,\beta)}} \geqslant \sup_{f\in S_a(\mathfrak{A})} \alpha|f(ax)|^2 \geqslant \sup_{f\in S_a(\mathfrak{A})} f\left(\frac{\alpha}{2}a\left(\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right)\right)\right). \tag{13}$$

By Theorem 1, we have

$$\|x\|^2_{a_{(\alpha,\beta)}} \geqslant \sup_{f\in S_a(\mathfrak{A})} f\left(\beta a x^{\sharp_a}x\right). \tag{14}$$

Combining (13) and (14), we get

$$\begin{aligned}2\|x\|^2_{a_{(\alpha,\beta)}} &\geqslant \sup_{f\in S_a(\mathfrak{A})}\left(f\left(\frac{\alpha}{2}a\left(\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right)\right)\right) + f\left(\beta a x^{\sharp_a}x\right)\right)\\ &= \sup_{f\in S_a(\mathfrak{A})} f\left(a\left(\frac{\alpha}{2}\left(\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right)\right)\\ &= \left\|\frac{\alpha}{2}\left(\Re_a^2\left(x^{\sharp_a}\right) + \Im_a^2\left(x^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right\|_a\\ &= \left\|\frac{\alpha}{2}\left(\frac{x^{\sharp_a}x + xx^{\sharp_a}}{2}\right)^{\sharp_a} + \beta\left(x^{\sharp_a}x\right)^{\sharp_a}\right\|_a\\ &= \left\|\frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a.\end{aligned}$$

So,

$$\|x\|^2_{a_{(\alpha,\beta)}} \geqslant \frac{1}{2}\left\|\frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a. \tag{15}$$

Again, by using similar arguments as above, we have that

$$\begin{aligned} 3\|x\|^2_{a_{(\alpha,\beta)}} &\geqslant \sup_{f\in S_a(\mathfrak{A})} f\left(a\left(\alpha\left(\mathfrak{R}^2_a\left(x^{\sharp_a}\right) + \mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right)\right) \\ &= \left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a. \end{aligned}$$

Thus

$$\|x\|^2_{a_{(\alpha,\beta)}} \geqslant \frac{1}{3}\left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a. \tag{16}$$

Combining (15) and (16), we get the inequality $(i)$.

$(ii)$ Let $f \in S_a(\mathfrak{A})$. Then we have

$$\begin{aligned} |f(ax)|^2 &= \left|f\left(a\mathfrak{R}_a\left(x^{\sharp_a}\right)\right)\right|^2 + \left|f\left(a\mathfrak{I}_a\left(x^{\sharp_a}\right)\right)\right|^2 \\ &\leqslant f\left(a\mathfrak{R}^2_a\left(x^{\sharp_a}\right)\right) + f\left(a\mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right) \\ &= f\left(a\left(\mathfrak{R}^2_a\left(x^{\sharp_a}\right) + \mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} \alpha|f(ax)|^2 + \beta f(x^*ax) &\leqslant f\left(\alpha a\left(\mathfrak{R}^2_a\left(x^{\sharp_a}\right) + \mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right)\right) + f\left(\beta a x^{\sharp_a}x\right) \\ &= f\left(a\left(\alpha\left(\mathfrak{R}^2_a\left(x^{\sharp_a}\right) + \mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right)\right) \\ &\leqslant \left\|\alpha\left(\mathfrak{R}^2_a\left(x^{\sharp_a}\right) + \mathfrak{I}^2_a\left(x^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right\|_a \\ &= \left\|\alpha\left(\frac{x^{\sharp_a}x + xx^{\sharp_a}}{2}\right)^{\sharp_a} + \beta\left(x^{\sharp_a}x\right)^{\sharp_a}\right\|_a \\ &= \left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a. \end{aligned}$$

Taking supremum for $f \in S_a(\mathfrak{A})$, we get the inequality $(ii)$. $\square$

The following upper bound for the $a$-numerical radius follows easily from Theorem 6 together with the Theorem 1.

COROLLARY 2. *Let $x \in \mathfrak{A}_a$. Then*

$$v^2_a(x) \leqslant \inf_{\alpha,\beta} \frac{\left\|\frac{\alpha}{2}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a}{\alpha+\beta}.$$

REMARK 1. In [10, Corollary 3.5], the author concluded

$$v_a^2(x) \leqslant \frac{1}{2}\left\|x^{\sharp_a}x + xx^{\sharp_a}\right\|_a.$$

If taking $\alpha = 1$ and $\beta = 0$ in Corollary 2, it is clear that Corollary 2 is an improvement of [10, Corollary 3.5].

Motivated by the Buzano's inequality [8], we get the following lemma.

LEMMA 2. *Let $x,\ y \in \mathfrak{A}_a,\ f \in S_a(\mathfrak{A})$. Then*

$$|f(y^*a)f(ax)| \leqslant \frac{1}{2}\left(|f(y^*ax)| + \sqrt{f(x^*ax)}\sqrt{f(y^*ay)}\right).$$

*Proof.* It is known that for all $a,\ b,\ c,\ d \in \mathbb{R}$, we have $(ac-bd)^2 \geqslant (a^2-b^2)(c^2-d^2)$. By using this inequality and the Cauchy Schwarz inequality, we can get

$$\begin{aligned}
&|f(y^*ax) - f(y^*a)f(ax)|^2 \\
&= \left|f\left[(y-f(ay))^* a(x-f(ax))\right]\right|^2 \\
&\leqslant f\left[(x-f(ax))^* a(x-f(ax))\right] f\left[(y-f(ay))^* a(y-f(ay))\right] \\
&= \left(f(x^*ax) - |f(ax)|^2\right)\left(f(y^*ay) - |f(ay)|^2\right) \\
&\leqslant \left(\sqrt{f(x^*ax)}\sqrt{f(y^*ay)} - |f(ax)|\,|f(ay)|\right)^2.
\end{aligned}$$

Clearly, $|f(ax)| \leqslant \sqrt{f(x^*ax)}$ and $|f(ay)| \leqslant \sqrt{f(y^*ay)}$, then

$$\left(\sqrt{f(x^*ax)}\sqrt{f(y^*ay)} - |f(ax)|\,|f(ay)|\right) \geqslant 0.$$

So,

$$|f(y^*ax) - f(y^*a)f(ax)| \leqslant \sqrt{f(x^*ax)}\sqrt{f(y^*ay)} - |f(ax)|\,|f(ay)|.$$

Then

$$|f(y^*ax) - f(y^*a)f(ax)| \leqslant \sqrt{f(x^*ax)}\sqrt{f(y^*ay)} - |f(ax)|\,|f(y^*a)|.$$

Thus,

$$2|f(y^*a)f(ax)| \leqslant |f(y^*ax)| + \sqrt{f(x^*ax)}\sqrt{f(y^*ay)}.$$

This completes the proof of the lemma. $\square$

THEOREM 7. *Let $x \in \mathfrak{A}_a$. Then*

$$\|x\|^2_{a_{(\alpha,\beta)}} \leqslant \frac{\alpha}{2}v_a(x^2) + \left\|\frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x\right\|_a.$$

*Proof.* Let $f \in S_a(\mathfrak{A})$. Then we have

$$\begin{aligned}
|f(ax)|^2 &= \left| f\left(\left(x^{\sharp_a}\right)^* a\right) f(ax) \right| \\
&\leqslant \frac{1}{2}\left( \left| f\left(\left(x^{\sharp_a}\right)^* ax\right)\right| + \sqrt{f(x^*ax)}\sqrt{f\left((x^{\sharp_a})^* ax^{\sharp_a}\right)} \right) \qquad \text{(by Lemma 2)} \\
&= \frac{1}{2}\left|f\left(ax^2\right)\right| + \frac{1}{2}\sqrt{f(ax^{\sharp_a}x) f(axx^{\sharp_a})} \\
&\leqslant \frac{1}{2}\left|f\left(ax^2\right)\right| + \frac{1}{4}\left( f\left(ax^{\sharp_a}x\right) + f\left(axx^{\sharp_a}\right)\right) \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\left|f\left(ax^2\right)\right| + \frac{1}{4} f\left(a\left(x^{\sharp_a}x + xx^{\sharp_a}\right)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\alpha|f(ax)|^2 + \beta f(x^*ax) &\leqslant \frac{\alpha}{2}\left|f\left(ax^2\right)\right| + \frac{\alpha}{4} f\left(a\left(x^{\sharp_a}x + xx^{\sharp_a}\right)\right) + \beta f\left(ax^{\sharp_a}x\right) \\
&= \frac{\alpha}{2}\left|f\left(ax^2\right)\right| + f\left(a\left(\frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right)\right) + \beta x^{\sharp_a}x\right) \\
&\leqslant \frac{\alpha}{2} v_a(x^2) + \left\| \frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x \right\|_a.
\end{aligned}$$

Taking supremum for $f \in S_a(\mathfrak{A})$, we get

$$\|x\|^2_{a_{(\alpha,\beta)}} \leqslant \frac{\alpha}{2} v_a(x^2) + \left\| \frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x \right\|_a. \qquad \square$$

The following upper bound for the $a$-numerical radius follows easily from Theorem 7 together with the Theorem 1.

COROLLARY 3. *Let $x \in \mathfrak{A}_a$. Then*

$$v_a^2(x) \leqslant \inf_{\alpha,\beta} \left\{ \frac{\frac{\alpha}{2} v_a(x^2) + \left\| \frac{\alpha}{4}\left(x^{\sharp_a}x + xx^{\sharp_a}\right) + \beta x^{\sharp_a}x \right\|_a}{\alpha+\beta} \right\}.$$

REMARK 2. In [10, Theorem 3.3], the author concluded

$$v_a^2(x) \leqslant \frac{1}{2} v_a(x^2) + \frac{1}{4}\left\| x^{\sharp_a}x + xx^{\sharp_a} \right\|_a.$$

If taking $\alpha = 1$ and $\beta = 0$ in Corollary 3, it is clear that Corollary 3 is an improvement of [10, Theorem 3.3].

LEMMA 3. *Let $x,\ y \in \mathfrak{A}_a,\ f \in S_a(\mathfrak{A})$. Then*

$$|f(y^*a) f(ax)|^2 \leqslant \frac{3}{4} f(x^*ax) f(y^*ay) + \frac{1}{4}\sqrt{f(x^*ax) f(y^*ay)}\, |f(y^*ax)|.$$

*Proof.* By the Schwarz inequality, we have

$$\begin{aligned} 2\left|f\left(y^{*}ax\right)\right|^{2} &= \left|f\left(y^{*}ax\right)\right|^{2}+\left|f\left(y^{*}ax\right)\right|^{2} \\ &\leqslant f\left(x^{*}ax\right)f\left(y^{*}ay\right)+\sqrt{f\left(x^{*}ax\right)f\left(y^{*}ay\right)}\left|f\left(y^{*}ax\right)\right|. \end{aligned}$$

Therefore,

$$\left|f\left(y^{*}ax\right)\right|^{2} \leqslant \frac{1}{2}\left(f\left(x^{*}ax\right)f\left(y^{*}ay\right)+\sqrt{f\left(x^{*}ax\right)f\left(y^{*}ay\right)}\left|f\left(y^{*}ax\right)\right|\right). \tag{17}$$

By Lemma 2, the convexity of $f(t)=t^{2}$ and the inequality (17), we obtain

$$\begin{aligned} &\left|f\left(y^{*}a\right)f\left(ax\right)\right|^{2} \\ &\leqslant \left(\frac{\left|f\left(y^{*}ax\right)\right|+\sqrt{f\left(x^{*}ax\right)}\sqrt{f\left(y^{*}ay\right)}}{2}\right)^{2} \\ &\leqslant \frac{\left|f\left(y^{*}ax\right)\right|^{2}+f\left(x^{*}ax\right)f\left(y^{*}ay\right)}{2} \\ &\leqslant \frac{1}{2}\left(\frac{1}{2}\left(f\left(x^{*}ax\right)f\left(y^{*}ay\right)+\sqrt{f\left(x^{*}ax\right)f\left(y^{*}ay\right)}\left|f\left(y^{*}ax\right)\right|\right)+f\left(x^{*}ax\right)f\left(y^{*}ay\right)\right) \\ &= \frac{3}{4}f\left(x^{*}ax\right)f\left(y^{*}ay\right)+\frac{1}{4}\sqrt{f\left(x^{*}ax\right)f\left(y^{*}ay\right)}\left|f\left(y^{*}ax\right)\right|. \end{aligned}$$

This completes the proof of the lemma. □

THEOREM 8. *Let $x\in\mathfrak{A}_a$. Then*

$$\|x\|_{a_{(\alpha,\beta)}}^{4} \leqslant \left\|\frac{3\alpha^{2}}{4}\left(|x|_a^{4}+\left|x^{\sharp_a}\right|_a^{4}\right)+2\beta^{2}|x|_a^{4}\right\|_a+\frac{\alpha^{2}}{4}\left\||x|_a^{2}+\left|x^{\sharp_a}\right|_a^{2}\right\|_a v_a(x^{2}).$$

*Proof.* Let $f\in S_a(\mathfrak{A})$. Then we have

$$\begin{aligned} |f(ax)|^{4} &= \left|f\left(\left(x^{\sharp_a}\right)^{*}a\right)f(ax)\right|^{2} \\ &\leqslant \frac{3}{4}f\left(x^{*}ax\right)f\left(\left(x^{\sharp_a}\right)^{*}ax^{\sharp_a}\right)+\frac{1}{4}\sqrt{f\left(x^{*}ax\right)f\left(\left(x^{\sharp_a}\right)^{*}ax^{\sharp_a}\right)}\left|f\left(\left(x^{\sharp_a}\right)^{*}ax\right)\right| \\ &\qquad \text{(by Lemma 3)} \\ &= \frac{3}{4}f\left(a|x|_a^{2}\right)f\left(a\left|x^{\sharp_a}\right|_a^{2}\right)+\frac{1}{4}\sqrt{f\left(a|x|_a^{2}\right)f\left(a\left|x^{\sharp_a}\right|_a^{2}\right)}\left|f\left(ax^{2}\right)\right| \\ &\leqslant \frac{3}{8}\left(f^{2}\left(a|x|_a^{2}\right)+f^{2}\left(a\left|x^{\sharp_a}\right|_a^{2}\right)\right)+\frac{1}{8}\left(f\left(a|x|_a^{2}\right)+f\left(a\left|x^{\sharp_a}\right|_a^{2}\right)\right)\left|f\left(ax^{2}\right)\right| \\ &\qquad \text{(by the arithmetic-geometric mean inequality)} \\ &\leqslant \frac{3}{8}\left(f\left(a|x|_a^{4}\right)+f\left(a\left|x^{\sharp_a}\right|_a^{4}\right)\right)+\frac{1}{8}\left(f\left(a|x|_a^{2}\right)+f\left(a\left|x^{\sharp_a}\right|_a^{2}\right)\right)\left|f\left(ax^{2}\right)\right| \\ &= \frac{3}{8}f\left(a\left(|x|_a^{4}+\left|x^{\sharp_a}\right|_a^{4}\right)\right)+\frac{1}{8}f\left(a\left(|x|_a^{2}+\left|x^{\sharp_a}\right|_a^{2}\right)\right)\left|f\left(ax^{2}\right)\right|. \end{aligned}$$

By the convexity of $f(t)=t^2$, we have

$$\begin{aligned}
&\left(\alpha\,|f(ax)|^2+\beta f(x^*ax)\right)^2\\
&\leqslant 2\left(\alpha^2\,|f(ax)|^4+\beta^2 f^2(x^*ax)\right)\\
&\leqslant \frac{3\alpha^2}{4}f\left(a\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)\right)+\frac{\alpha^2}{4}f\left(a\left(|x|_a^2+\left|x^{\sharp_a}\right|_a^2\right)\right)\left|f\left(ax^2\right)\right|+2\beta^2 f\left(a\left(|x|_a^4\right)\right)\\
&= f\left(a\left(\frac{3\alpha^2}{4}\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)+2\beta^2\,|x|_a^4\right)\right)+\frac{\alpha^2}{4}f\left(a\left(|x|_a^2+\left|x^{\sharp_a}\right|_a^2\right)\right)\left|f\left(ax^2\right)\right|\\
&\leqslant \left\|\frac{3\alpha^2}{4}\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)+2\beta^2\,|x|_a^4\right\|_a+\frac{\alpha^2}{4}\left\||x|_a^2+\left|x^{\sharp_a}\right|_a^2\right\|_a\nu_a(x^2).
\end{aligned}$$

Taking supremum for $f\in S_a(\mathfrak{A})$, we get

$$\|x\|_{a_{(\alpha,\beta)}}^4\leqslant \left\|\frac{3\alpha^2}{4}\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)+2\beta^2\,|x|_a^4\right\|_a+\frac{\alpha^2}{4}\left\||x|_a^2+\left|x^{\sharp_a}\right|_a^2\right\|_a\nu_a(x^2).\quad\square$$

The following upper bound for the $a$-numerical radius follows easily from Theorem 8 together with the Theorem 1.

COROLLARY 4. *Let $x\in\mathfrak{A}_a$. Then*

$$\nu_a^4(x)\leqslant\inf_{\alpha,\beta}\left\{\frac{\left\|\frac{3\alpha^2}{4}\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)+2\beta^2\,|x|_a^4\right\|_a+\frac{\alpha^2}{4}\left\||x|_a^2+\left|x^{\sharp_a}\right|_a^2\right\|_a\nu_a(x^2)}{(\alpha+\beta)^2}\right\}.$$

REMARK 3. If taking $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{2}$ in Corollary 4, we have

$$\begin{aligned}
\nu_a^4(x)&\leqslant\left\|\frac{3}{16}\left(|x|_a^4+\left|x^{\sharp_a}\right|_a^4\right)+\frac{1}{2}\,|x|_a^4\right\|_a+\frac{1}{16}\left\||x|_a^2+\left|x^{\sharp_a}\right|_a^2\right\|_a\nu_a(x^2)\\
&\leqslant\frac{7}{8}\,\|x\|_a^4+\frac{1}{8}\,\|x\|_a^2\,\nu_a^2(x)\\
&\leqslant\|x\|_a^4.
\end{aligned}$$

Therefore, it is clear that Corollary 4 is an improvement of the right side of inequality (2).

THEOREM 9. *Let $x\in\mathfrak{A}_a$. Then*

$$\|x\|_{a_{(\alpha,\beta)}}^2\leqslant\left\|\alpha xx^{\sharp_a}+\beta x^{\sharp_a}x\right\|_a.$$

*Proof.* Let $f \in S_a(\mathfrak{A})$. Then by Cauchy Schwarz inequality, we have

$$\begin{aligned}
\alpha \left| f(ax) \right|^2 + \beta f(x^* ax) &\leqslant \alpha f\left( \left( x^{\sharp_a} \right)^* a x^{\sharp_a} \right) + \beta f(x^* ax) \\
&= \alpha f\left( a x x^{\sharp_a} \right) + \beta f\left( a x^{\sharp_a} x \right) \\
&\leqslant \left\| \alpha x x^{\sharp_a} + \beta x^{\sharp_a} x \right\|_a .
\end{aligned}$$

Taking supremum for $f \in S_a(\mathfrak{A})$, we get

$$\|x\|^2_{a_{(\alpha,\beta)}} \leqslant \left\| \alpha x x^{\sharp_a} + \beta x^{\sharp_a} x \right\|_a . \qquad \square$$

The following upper bound for the $a$-numerical radius follows easily from Theorem 9 together with the Theorem 1.

COROLLARY 5. *Let $x \in \mathfrak{A}_a$. Then*

$$v_a^2(x) \leqslant \inf_{\alpha,\beta} \frac{\left\| \alpha x x^{\sharp_a} + \beta x^{\sharp_a} x \right\|_a}{\alpha + \beta}.$$

REMARK 4. In [10, Corollary 3.5], the author concluded

$$v_a^2(x) \leqslant \frac{1}{2} \left\| x^{\sharp_a} x + x x^{\sharp_a} \right\|_a .$$

If taking $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in Corollary 5, it is clear that Corollary 5 is an improvement of [10, Corollary 3.5].

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