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LOGARITHMIC CONVEXITY AND INEQUALITIES FOR INTEGRALS OF GEOMETRICALLY CONVEX FUNCTIONS

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Abstract. In the paper, the authors present logarithmic convexity and inequalities for integrals of geometrically convex functions. These results refine and generalize the celebrated Hermite–Hadamard's integral inequality.

1. Introduction

In this paper, we use I to denote an interval on $\mathbb{R}$ and use I° to denote the interior of the interval I.

We firstly list several definitions and some known results.

DEFINITION 1.1. ([5, p. 15, Definition 1] and [6, p. 1, Definition 1.1.1]) A function $f: I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex on an interval I if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

DEFINITION 1.2. ([6, p. 21, Definition 1.3.1] and [7, p. 48]) A function $f: I \subseteq \mathbb{R} \to \mathbb{R}_+ = (0, \infty)$ is said to be logarithmically convex if

$$f(\lambda x + (1-\lambda)y) \leqslant [f(x)]^\lambda [f(y)]^{1-\lambda}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

DEFINITION 1.3. ([1, 4]) A function $f: I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be geometrically convex if

$$f(x^\lambda y^{1-\lambda}) \leqslant [f(x)]^\lambda [f(y)]^{1-\lambda}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

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In [2, 3], Dragomir et al. established the following integral inequalities of the Hermite–Hadamard type for the integral mean of convex functions.

THEOREM 1.1. ([2, Theorem 1]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I° and $a, b \in I^\circ$ with $a < b$. Then the function $H : [0,1] \to \mathbb{R}$ defined by*

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) \mathrm{d}x \tag{1.1}$$

is convex on $[0,1]$ and increasing on $[0,1]$ with

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

THEOREM 1.2. ([2, Theorem 2]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I° and $a, b \in I^\circ$ with $a < b$. Then the function $F : [0,1] \to \mathbb{R}$ defined by*

$$F(t) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx + (1-t)y)\,\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

is convex on $[0,1]$, symmetric with respect to $\frac{1}{2}$, decreasing on $\left[0, \frac{1}{2}\right]$, and increasing on $\left[\frac{1}{2}, 1\right]$, with

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x,$$
$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right)\mathrm{d}x\,\mathrm{d}y,$$
$$f\left(\frac{a+b}{2}\right) \leqslant F\left(\frac{1}{2}\right),$$

and $H(t) \leqslant F(t)$ for $t \in [0,1]$, where $H(t)$ is defined by (1.1).

THEOREM 1.3. ([3, Theorem 2]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I° and $a, b \in I^\circ$ with $a < b$. Then the function $G : [0,1] \to \mathbb{R}$ defined by*

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right] \tag{1.3}$$

is convex and increasing on $[0,1]$ *with*

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2},$$

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right),$$

$$H(t) - f\left(\frac{a+b}{2}\right) \leqslant G(t) - H(t),$$

and $H(t) \leqslant G(t)$ *for* $t \in [0,1]$, *where* $H(t)$ *is defined by* (1.1).

THEOREM 1.4. ([3, Theorem 3]) *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on* I° *and* $a,b \in I^\circ$ *with* $a < b$. *Then the functions* $H(t)$ *defined by* (1.1) *and* $L : [0,1] \to \mathbb{R}$ *defined by*

$$L(t) = \frac{1}{2(b-a)} \int_a^b [f(ta+(1-t)x) + f(tb+(1-t)x)]\,\mathrm{d}x$$

are convex on $[0,1]$ *and satisfy the inequalities*

$$G(t) \leqslant L(t) \leqslant \frac{1-t}{b-a}\int_a^b f(x)\,\mathrm{d}x + t\frac{f(a)+f(b)}{2} \leqslant \frac{f(a)+f(b)}{2}$$

and

$$\max\left\{H(1-t), \frac{H(t)+H(1-t)}{2}\right\} \leqslant L(t)$$

for $t \in [0,1]$ *with*

$$\sup_{t\in[0,1]} L(t) = L(1) = \frac{f(a)+f(b)}{2},$$

where $G(t)$ *is defined by* (1.3).

Several similar integral inequalities of the Hermite–Hadamard type were also acquired in the papers [8, 9, 10, 11] and related references therein.

Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a geometrically convex function on I° and $a,b \in I^\circ$ with $a < b$. For all $t \in [0,1]$, define

$$H_G(t) = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x} f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right)\mathrm{d}x, \tag{1.4}$$

$$H_B(t) = \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\left[f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right) f\left(x^{1-t}\left(\sqrt{ab}\right)^t\right)\right]^{1/2}\mathrm{d}x, \tag{1.5}$$

$$F_G(t) = \frac{1}{(\ln b - \ln a)^2}\int_a^b\int_a^b \frac{1}{xy} f\left(x^t y^{1-t}\right)\mathrm{d}x\,\mathrm{d}y, \tag{1.6}$$

$$F_B(t) = \frac{1}{(\ln b - \ln a)^2}\int_a^b\int_a^b \frac{1}{xy}\left[f\left(x^t y^{1-t}\right) f\left(x^{1-t} y^t\right)\right]^{1/2}\mathrm{d}x\,\mathrm{d}y. \tag{1.7}$$

In this paper, motivated by the above results recited from [2, 3], we will present the logarithmic convexity and establish several new integral inequalities of the Hermite–Hadamard type for the functions $H_G(t)$, $H_B(t)$, $F_G(t)$, and $F_B(t)$.

2. Main results

Now we start out to present the logarithmic convexity and establish some new integral inequalities of the Hermite–Hadamard type for the four functions $H_G(t)$, $H_B(t)$, $F_G(t)$, and $F_B(t)$.

THEOREM 2.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a geometrically convex function on I° and $a,b \in I^\circ$ with $a<b$. Then the function $H_G(t)$ defined by (1.4) is logarithmically convex and increasing on $[0,1]$ and satisfies*

$$f\big(\sqrt{ab}\big) \leqslant H_G(t) \leqslant \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \leqslant L(f(a),f(b)),$$

with

$$\inf_{t\in[0,1]} H_G(t) = H_G(0) = f\big(\sqrt{ab}\big)$$

and

$$\sup_{t\in[0,1]} H_G(t) = H_G(1) = \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x,$$

where $L(u,v)$ is the logarithmic mean

$$L(u,v) = \begin{cases} \frac{v-u}{\ln v - \ln u}, & u \neq v \\ u, & u = v \end{cases} \tag{2.1}$$

for $u,v \in \mathbb{R}_+$.

Proof. For all $\lambda, t_1, t_2 \in [0,1]$, by the geometrically convexity of f and the Hölder integral inequality, we have

$$\begin{aligned}
&H_G(\lambda t_1 + (1-\lambda)t_2)\\
&= \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x} f\Big(\big[x^{t_1}\big(\sqrt{ab}\big)^{1-t_1}\big]^\lambda \big[x^{t_2}\big(\sqrt{ab}\big)^{1-t_2}\big]^{1-\lambda}\Big)\,\mathrm{d}x\\
&\leqslant \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\Big[f\Big(x^{t_1}\big(\sqrt{ab}\big)^{1-t_1}\Big)\Big]^\lambda \Big[f\Big(x^{t_2}\big(\sqrt{ab}\big)^{1-t_2}\Big)\Big]^{1-\lambda}\,\mathrm{d}x\\
&\leqslant \frac{1}{\ln b - \ln a}\left[\int_a^b \frac{1}{x} f\Big(x^{t_1}\big(\sqrt{ab}\big)^{1-t_1}\Big)\,\mathrm{d}x\right]^\lambda \left[\int_a^b \frac{1}{x} f\Big(x^{t_2}\big(\sqrt{ab}\big)^{1-t_2}\Big)\,\mathrm{d}x\right]^{1-\lambda}\\
&= [H_B(t_1)]^\lambda [H_B(t_2)]^{1-\lambda}.
\end{aligned}$$

Therefore, the function $H_G(t)$ is logarithmically convex on $[0,1]$. As a result, the function $H_G(t)$ is convex on $[0,1]$. Furthermore, we obtain

$$\begin{aligned} H_G(t) &= \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x} f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right)\mathrm{d}x \\ &= \frac{1}{\ln b - \ln a}\int_a^b f\circ g\left(t\ln x + (1-t)\frac{\ln a + \ln b}{2}\right)\mathrm{d}\ln x, \end{aligned}$$

where $g(u) = \mathrm{e}^u$ for $u \in [\ln a, \ln b]$ and $f\circ g$ is a convex function on $[\ln a, \ln b]$. Utilizing the convexity of $f\circ g$ on $[\ln a, \ln b]$ and Theorem 1.1, we see that the function $H_G(t)$ is increasing on $[0,1]$. Consequently, we gain

$$\inf_{t\in[0,1]} H_G(t) = H_G(0) = \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(\sqrt{ab}\right)}{x}\mathrm{d}x = f\left(\sqrt{ab}\right)$$

and

$$\begin{aligned} \sup_{t\in[0,1]} H_G(t) = H_G(1) &= \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x \\ &= \int_0^1 f(a^t b^{1-t})\,\mathrm{d}t \leqslant \int_0^1 [f(a)]^t[f(b)]^{1-t}\,\mathrm{d}t = L(f(a), f(b)). \end{aligned}$$

The proof of Theorem 2.1 is complete. □

THEOREM 2.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ be a geometrically convex function on I° and $a, b \in I^\circ$ with $a < b$.*

1. *The functions $F_G(t)$ and $F_B(t)$ defined by (1.6) and (1.7), respectively, are logarithmically convex on $[0,1]$ and symmetric with respect to $\frac{1}{2}$, decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, with*

$$\begin{aligned} \sup_{t\in[0,1]} F_B(t) = F_B(0) = F_B(1) &= \left[\frac{1}{\ln b - \ln a}\int_a^b \frac{[f(x)]^{1/2}}{x}\mathrm{d}x\right]^2 \\ &\leqslant \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\mathrm{d}x = \sup_{t\in[0,1]} F_G(t) = F_G(0) = F_G(1) \end{aligned}$$

and

$$\begin{aligned} \inf_{t\in[0,1]} F_G(t) = \inf_{t\in[0,1]} F_B(t) &= F_G\left(\frac{1}{2}\right) \\ &= F_B\left(\frac{1}{2}\right) = \frac{1}{(\ln b - \ln a)^2}\int_a^b\int_a^b \frac{f\left(\sqrt{xy}\right)}{xy}\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

2. *For all* $t\in[0,1]$, *we have the inequality*

$$
\begin{aligned}
f\left(\sqrt{ab}\right) &\leqslant \frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{f\left(\sqrt{xy}\right)}{xy}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant F_B(t)\leqslant F_G(t)\leqslant \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x\leqslant L\big(f(a),f(b)\big),
\end{aligned}
$$

where $L(u,v)$ *is the logarithmic mean defined by* (2.1).

Proof. For all $\lambda,t_1,t_2\in[0,1]$, using the geometrically convexity of f and making use of the Hölder integral inequality yield

$$
\begin{aligned}
&F_G(\lambda t_1+(1-\lambda)t_2)\\
&=\frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{1}{xy} f\left(\left[x^{t_1}y^{1-t_1}\right]^{\lambda}\left[x^{t_2}y^{1-t_2}\right]^{1-\lambda}\right)\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{1}{xy}\left[f\left(x^{t_1}y^{1-t_1}\right)\right]^{\lambda}\left[f\left(x^{t_2}y^{1-t_2}\right)\right]^{1-\lambda}\mathrm{d}x\,\mathrm{d}y\\
&\leqslant \frac{1}{(\ln b-\ln a)^2}\left[\int_a^b\int_a^b \frac{1}{xy} f\left(x^{t_1}y^{1-t_1}\right)\mathrm{d}x\,\mathrm{d}y\right]^{\lambda}\left[\int_a^b\int_a^b \frac{1}{xy} f\left(x^{t_2}y^{1-t_2}\right)\mathrm{d}x\,\mathrm{d}y\right]^{1-\lambda}\\
&=\left[F_G(t_1)\right]^{\lambda}\left[F_G(t_2)\right]^{1-\lambda}.
\end{aligned}
$$

Similarly, we have

$$
F_B(\lambda t_1+(1-\lambda)t_2)\leqslant [F_B(t_1)]^{\lambda}[F_B(t_2)]^{1-\lambda}.
$$

Accordingly, the function $F_G(t)$ and $F_B(t)$ are logarithmically convex on $[0,1]$.

It is straightforward to show that

$$
F_G\left(\frac{1}{2}+t\right)=F_G\left(\frac{1}{2}-t\right)\quad\text{and}\quad F_B\left(\frac{1}{2}+t\right)=F_B\left(\frac{1}{2}-t\right)
$$

for all $t\in\left[0,\frac{1}{2}\right]$.

Employing the geometrically convexity of f yields

$$
\begin{aligned}
F_G(t)&=\frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{f\left(x^ty^{1-t}\right)+f\left(x^{1-t}y^t\right)}{2xy}\,\mathrm{d}x\,\mathrm{d}y\\
&\geqslant \frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{1}{xy}\left[f\left(x^ty^{1-t}\right)f\left(x^{1-t}y^t\right)\right]^{1/2}\mathrm{d}x\,\mathrm{d}y\\
&=F_B(t)\\
&\geqslant \frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b \frac{f\left(\sqrt{xy}\right)}{xy}\,\mathrm{d}x\,\mathrm{d}y\\
&=F_G\left(\frac{1}{2}\right)=F_B\left(\frac{1}{2}\right).
\end{aligned}
$$

Thus, the functions $F_G(t)$ and $F_B(t)$ are symmetric with respect to $\frac{1}{2}$, decreasing on $\left[0,\frac{1}{2}\right]$, and increasing on $\left[\frac{1}{2},1\right]$.

By the geometrically convexity of f on $[a,b]$, we acquire

$$\begin{aligned} f\left(\sqrt{ab}\right) &= f\left(\left[\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{\mu}b^{1-\mu}\right)\right]^{1/4}\left[\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\mu}b^{\mu}\right)\right]^{1/4}\right) \\ &\leqslant \left[f\left(\left[\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{\mu}b^{1-\mu}\right)\right]^{1/2}\right)\right]^{1/2}\left[f\left(\left[\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\mu}b^{\mu}\right)\right]^{1/2}\right)\right]^{1/2} \end{aligned} \tag{2.2}$$

for all $\lambda,\mu\in[0,1]$.

For all $\lambda,\mu\in[0,1]$, taking $x=a^{\lambda}b^{1-\lambda}$, $y=a^{\mu}b^{1-\mu}$ and $x=a^{1-\lambda}b^{\lambda}$, $y=a^{1-\mu}b^{\mu}$, integrating both sides of (2.2) over $(\lambda,\mu)\in[0,1]\times[0,1]$, and utilizing the Cauchy inequality figure out

$$\begin{aligned} f\left(\sqrt{ab}\right) &\leqslant \left[\int_0^1\int_0^1 f\left(\left[\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{\mu}b^{1-\mu}\right)\right]^{1/2}\right)\mathrm{d}\lambda\,\mathrm{d}\mu\right]^{1/2} \\ &\quad\times\left[\int_0^1\int_0^1 f\left(\left[\left(a^{1-\lambda}b^{\lambda}\right)\left(a^{1-\mu}b^{\mu}\right)\right]^{1/2}\right)\mathrm{d}\lambda\,\mathrm{d}\mu\right]^{1/2} \\ &= \frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b\frac{f\left(\sqrt{xy}\right)}{xy}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{[f(x)]^{1/2}}{x}\,\mathrm{d}x\leqslant\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\right]^{1/2},$$

and

$$\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}x\leqslant L(f(a),f(b)).$$

The proof of Theorem 2.2 is complete. □

COROLLARY 2.1. *Let the function $F(t)$ be defined by (1.2). Under the conditions of Theorem 2.2,*

1. *if f is increasing on $[a,b]$, then, for all $t\in[0,1]$,*

$$\begin{aligned} &\frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b f\left(\sqrt{xy}\right)\mathrm{d}x\,\mathrm{d}y \\ &\qquad\leqslant F_B(t)\leqslant F_G(t)\leqslant F(t)\leqslant\frac{1}{\ln b-\ln a}\int_a^b f(x)\,\mathrm{d}x; \end{aligned}$$

2. *if f is decreasing on $[a,b]$, then, for all $t\in[0,1]$,*

$$\frac{1}{(\ln b-\ln a)^2}\int_a^b\int_a^b f\left(\sqrt{xy}\right)\mathrm{d}x\,\mathrm{d}y\leqslant F(t)\leqslant F_G(t)\leqslant\frac{1}{\ln b-\ln a}\int_a^b f(x)\,\mathrm{d}x.$$

THEOREM 2.3. *Let $f:I\subseteq\mathbb{R}_+\to\mathbb{R}_+$ be a geometrically convex function on I°, let $a,b\in I^{\circ}$ with $a<b$, and let $H_B(t)$, $F_G(t)$, and $F_B(t)$ be the functions defined*

by (1.5), (1.6), *and* (1.7), *respectively. Then*

1. *the function* $H_B(t)$ *is logarithmically convex on* $[0,1]$, *symmetric with respect to* $\frac{1}{2}$, *decreasing on* $\left[0,\frac{1}{2}\right]$, *and increasing on* $\left[\frac{1}{2},1\right]$, *with*

$$\sup_{t\in[0,1]} H_B(t) = H_B(0) = H_B(1) = \frac{\left[f\left(\sqrt{ab}\right)\right]^{1/2}}{\ln b - \ln a}\int_a^b \frac{[f(x)]^{1/2}}{x}\,\mathrm{d}x$$

and

$$\inf_{t\in[0,1]} H_B(t) = H_B\left(\frac{1}{2}\right) = \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(\left(x\sqrt{ab}\right)^{1/2}\right)}{x}\,\mathrm{d}x;$$

2. *the inequalities*

$$f\left(\sqrt{ab}\right) \leqslant H_B\left(\frac{1}{2}\right) \leqslant H_B(t) \leqslant F_B(t) \leqslant F_G(t) \leqslant \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}x \leqslant L(f(a), f(b))$$

and

$$f\left(\sqrt{ab}\right) \leqslant H_B\left(\frac{1}{2}\right) \leqslant H_B(t) \leqslant \frac{\left[f\left(\sqrt{ab}\right)\right]^{1/2}}{\ln b - \ln a}\int_a^b \frac{[f(x)]^{1/2}}{x}\,\mathrm{d}x \leqslant \left[f\left(\sqrt{ab}\right)\right]^{1/2} L\left([f(a)]^{1/2}, [f(b)]^{1/2}\right),$$

hold for all $t \in [0,1]$, *where* $L(u,v)$ *is the logarithmic mean defined by* (2.1).

Proof. For all $\lambda, t_1, t_2 \in [0,1]$, using the geometrically convexity of f and the Hölder integral inequality gives

$$\begin{aligned}
&H_B(\lambda t_1 + (1-\lambda)t_2) \\
&= \frac{1}{b-a}\int_a^b \frac{1}{x}\left[f\left(\left[x^{t_1}\left(\sqrt{ab}\right)^{1-t_1}\right]^{\lambda}\left[x^{t_2}\left(\sqrt{ab}\right)^{1-t_2}\right]^{1-\lambda}\right)\right. \\
&\quad \left.\times f\left(\left[x^{1-t_1}\left(\sqrt{ab}\right)^{t_1}\right]^{\lambda}\left[x^{1-t_2}\left(\sqrt{ab}\right)^{t_2}\right]^{1-\lambda}\right)\right]^{1/2}\mathrm{d}x \\
&\leqslant \frac{1}{b-a}\int_a^b \frac{1}{x}\left[f\left(x^{t_1}\left(\sqrt{ab}\right)^{1-t_1}\right)f\left(x^{1-t_1}\left(\sqrt{ab}\right)^{t_1}\right)\right]^{\lambda/2} \\
&\quad \times \left[f\left(x^{t_2}\left(\sqrt{ab}\right)^{1-t_2}\right)f\left(x^{1-t_2}\left(\sqrt{ab}\right)^{t_2}\right)\right]^{(1-\lambda)/2}\mathrm{d}x \\
&\leqslant \frac{1}{b-a}\left\{\int_a^b \frac{1}{x}\left[f\left(x^{t_1}\left(\sqrt{ab}\right)^{1-t_1}\right)f\left(x^{1-t_1}\left(\sqrt{ab}\right)^{t_1}\right)\right]^{1/2}\mathrm{d}x\right\}^{\lambda} \\
&\quad \times \left\{\int_a^b \frac{1}{x}\left[f\left(x^{t_2}\left(\sqrt{ab}\right)^{1-t_2}\right)f\left(x^{1-t_2}\left(\sqrt{ab}\right)^{t_2}\right)\right]^{1/2}\mathrm{d}x\right\}^{1-\lambda} \\
&= \left[H_B(t_1)\right]^{\lambda}\left[H_B(t_2)\right]^{1-\lambda}.
\end{aligned}$$

Hence, the function $H_B(t)$ is logarithmically convex on $[0,1]$.

For all $t \in \left[0,\frac{1}{2}\right]$, it is easy to observe that $H_B\left(\frac{1}{2}+t\right) = H_B\left(\frac{1}{2}-t\right)$ and by the geometrically convexity of f, we have

$$\begin{aligned}
H_B(t) &= \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\left[f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right)f\left(x^{1-t}\left(\sqrt{ab}\right)^t\right)\right]^{1/2}\mathrm{d}x \\
&\geqslant \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x} f\left(\left[x\left(\sqrt{ab}\right)\right]^{1/2}\right)\mathrm{d}x \\
&= H_B\left(\frac{1}{2}\right).
\end{aligned}$$

Therefore, the function $H_B(t)$ is symmetric with respect to $\frac{1}{2}$, decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$.

For all $\lambda \in [0,1]$ and $x \in [a,b]$, it is clear that $ab = a^\lambda b^{1-\lambda}a^{1-\lambda}b^\lambda$ for $\lambda \in [0,1]$. By the geometrically convexity of f, we can write

$$\begin{aligned}
& f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right)f\left(x^{1-t}\left(\sqrt{ab}\right)^t\right) \\
&= f\left(\left(\left(x^t\left(a^\lambda b^{1-\lambda}\right)^{1-t}\right)\left(x^t\left(a^{1-\lambda}b^\lambda\right)^{1-t}\right)\right)^{1/2}\right) \\
&\quad \times f\left(\left(\left(x^{1-t}\left(a^\lambda b^{1-\lambda}\right)^t\right)\left(x^{1-t}\left(a^{1-\lambda}b^\lambda\right)^t\right)\right)^{1/2}\right) \\
&\leqslant \left[f\left(x^t(a^\lambda b^{1-\lambda})^{1-t}\right)f\left(x^{1-t}(a^\lambda b^{1-\lambda})^t\right)\right. \\
&\quad \left.\times f\left(x^t(a^{1-\lambda}b^\lambda)^{1-t}\right)f\left(x^{1-t}(a^{1-\lambda}b^\lambda)^t\right)\right]^{1/2}.
\end{aligned}$$

Integrating the above inequalities over $(x,\lambda) \in [a,b]\times[0,1]$ and making the change of variable $y = a^\lambda b^{1-\lambda}$ for $\lambda \in [0,1]$, we arrive at

$$\begin{aligned}
H_B(t) &= \frac{1}{\ln b - \ln a}\int_a^b \frac{1}{x}\left[f\left(x^t\left(\sqrt{ab}\right)^{1-t}\right)f\left(x^{1-t}\left(\sqrt{ab}\right)^t\right)\right]^{1/2}\mathrm{d}x \\
&\leqslant \frac{1}{\ln b - \ln a}\int_0^1\int_a^b \frac{1}{x}\left[f\left(x^t(a^\lambda b^{1-\lambda})^{1-t}\right)f\left(x^{1-t}(a^\lambda b^{1-\lambda})^t\right)\right. \\
&\quad \left.\times f\left(x^t(a^{1-\lambda}b^\lambda)^{1-t}\right)f\left(x^{1-t}(a^{1-\lambda}b^\lambda)^t\right)\right]^{1/4}\mathrm{d}x\,\mathrm{d}\lambda \\
&\leqslant \frac{1}{\ln b - \ln a}\left\{\int_0^1\int_a^b \left[a^\lambda b^{1-\lambda} f\left(x^t(a^\lambda b^{1-\lambda})^{1-t}\right)f\left(x^{1-t}(a^\lambda b^{1-\lambda})^t\right)\right]^{1/2}\mathrm{d}x\,\mathrm{d}\lambda\right\}^{1/2} \\
&\quad \times\left\{\int_0^1\int_a^b \left[a^{1-\lambda}b^\lambda f\left(x^t(a^{1-\lambda}b^\lambda)^{1-t}\right)f\left(x^{1-t}(a^{1-\lambda}b^\lambda)^t\right)\right]^{1/2}\mathrm{d}x\,\mathrm{d}\lambda\right\}^{1/2} \\
&= \frac{1}{(\ln b - \ln a)^2}\int_a^b\int_a^b \frac{1}{xy}\left[f\left(x^t y^{1-t}\right)f\left(x^{1-t}y^t\right)\right]^{1/2}\mathrm{d}x\,\mathrm{d}y \\
&= F_B(t).
\end{aligned}$$

Similarly, making the change of variable and the Cauchy inequality, we acquire

$$\begin{aligned}
f\left(\sqrt{ab}\right) &= \int_0^1 f\left(\sqrt{ab}\right)\mathrm{d}\lambda \\
&= \int_0^1 f\left(\left[\left(a^{\lambda}b^{1-\lambda}\right)\left(a^{1-\lambda}b^{\lambda}\right)\right]^{1/4}(ab)^{1/4}\right)\mathrm{d}\lambda \\
&\leqslant \int_0^1 \left[f\left([a^{\lambda}b^{1-\lambda}]^{1/2}(ab)^{1/4}\right)\right]^{1/2}\left[f\left([a^{1-\lambda}b^{\lambda}]^{1/2}(ab)^{1/4}\right)\right]^{1/2}\mathrm{d}\lambda \\
&\leqslant \left[\int_0^1 f\left([a^{\lambda}b^{1-\lambda}]^{1/2}(ab)^{1/4}\right)\mathrm{d}\lambda\right]^{1/2}\left[\int_0^1 f\left([a^{1-\lambda}b^{\lambda}]^{1/2}(ab)^{1/4}\right)\mathrm{d}\lambda\right]^{1/2} \\
&= \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(\left(x\sqrt{ab}\right)^{1/2}\right)}{x}\,\mathrm{d}x.
\end{aligned}$$

Theorem 2.3 is thus proved. □

3. Conclusions

This paper defined four integral mean functions of geometrically convex functions and investigated the logarithmically convexity of these four types of integral means. It also established several related inequalities. These results refined and extended the well-known Hermite–Hadamard integral inequality.

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