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CONVEXITY AND SOME NEW POWER MEAN INEQUALITIES IN CONTINUOUS FORMS WITH ILLUSTRATIONS/APPLICATIONS RELATED TO HEAT CONDUCTION IN COMPOSITE MATERIALS

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Abstract. The focus in this paper is on convexity (see [17]) related to continuous inequalities as described in the new and only book [19] on the subject. In particular, we complement the results in these books by proving some new continuous power mean inequalities in refined forms. In particular, some new applications related to heat conduction in composite materials are pointed out.

1. Introduction

It is well-known that Convexity implies more or less directly most of the classical inequalities (see e.g. the Lecture Notes [23], the paper [26] and the new well cited book [17]. But this fact can also partly be seen in many other publications, e.g. in the more than 50 monographs (including the first most important one [10] with main focus on Inequalities. In this paper our main reference is the new book [19], which is the first and so far only book with main focus on so called continuous forms of classical inequalities. Concerning the main idea of these continuous inequalities we refer to Preface and Chapter 1 of the book [19], see also [12].

Inequalities is a very fascinating and creative research area in itself, but also, and most important, it is very important for the development of other areas of mathematics, such as Fourier Analysis, Interpolation Theory, Approximation Theory, Harmonic Analysis, Function Spaces and even Engineering Mathematics and its applications. Concerning applications of continuous inequalities we pronounce that in an Appendix of the book [19] it was pointed out that a continuous form of the Hölder inequality was crucial for the development of a real interpolation theory between infinite many, even scales of, Banach spaces (see [18] and [7]). Note that classical interpolation theory, as described in several books on the subject, only concerns interpolation between two, or at most finite, many Banach spaces. Also some applications concerning Hardy type

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operators were pointed out (see e.g. the book [14], especially Chapter 2 and Section 7.5).

In this paper we state and prove some new contributions in this fascinating new inequality area. For example, we prove some new power mean inequalities in refined continuous forms. And we also present some new applications of our results connected to material science and so called homogenization theory. There are also several monographs in these areas but as far as we know these applications are new also there.

The paper is organized as follows: For the readers convenience we use Section 2 to illustrate the usefulness of some old and new ideas and facts described above in the case of Hölder type inequalities. In Section 3 we discuss power mean inequalities and in particular state and discuss a continuous power mean inequality (see Theorem 3.1). Partly inspired not only of the book [19] but also by some new results on refined discrete power mean inequalities in [5], in Section 3 we also derive some new refinements of Theorem 3.1 by using the concept of superquadratic functions (a variant of convex functions), see Theorems 3.9 and 3.13.

In Section 4 we present a new application (in addition to the interpolation and Hardy type operator applications mentioned above) of continuous inequalities. This application is illustrated in the case of heat conduction in composite materials and homogenization theory (see e.g. the books [8] and [25]). However, for researchers in these areas it is obvious that the same ideas can be used also to solve other problems connected to composite materials, e.g. to estimate elasticity and effective strength properties, see Remark 5.4. Finally, Section 5 is reserved for final remarks, results and open questions of interest both in mathematics and engineering sciences.

Conventions: In this paper (Ω,μ) denotes a σ-finite measure space, f and g are measurable functions on (Ω,μ) and $p>1$.

2. Hölder's inequality as an example how convexity can be used to prove inequalities even in the continuous case

As a simple introductory example of the fact that convexity implies classical inequalities we show that even the fact that the exp function is convex indeed more or less immediately implies Hölder's inequality:

$$\int_\Omega f(x)g(x)d\mu(x) \leqslant \left(\int_\Omega f^p(x)d\mu(x)\right)^{1/p}\left(\int_\Omega g^q(x)d\mu(x)\right)^{1/q}, \tag{2.1}$$
$$\frac{1}{p}+\frac{1}{q}=1, \quad p>1.$$

Indeed, we note first that the homogeneity implies that it is sufficient to prove (2.1) with

$$\left(\int_\Omega f^p(x)d\mu(x)\right)^{1/p}=\left(\int_\Omega g^q(x)d\mu(x)\right)^{1/q}=1$$

By using that $y=e^{\ln y}$, convexity of the exp function and standard rules for integrals and logarithms we have that

$$\begin{aligned}\int_\Omega f(x)g(x)d\mu(x) &= \int_\Omega e^{\ln f(x)g(x)}d\mu(x)\\ &= \int_\Omega e^{1/p\ln f^p(x)+1/q\ln g^q(x)}d\mu(x)\\ &\leqslant \int_\Omega\left(\frac{1}{p}e^{\ln f^p(x)}+\frac{1}{q}e^{\ln f^q(x)}\right)d\mu(x)\\ &= \frac{1}{p}\int_\Omega f^p(x)d\mu(x)+\frac{1}{q}\int_\Omega g^q(x)d\mu(x)\\ &= \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

It is remarkable that this main fact above is also true concerning the more general continuous version of Hölder's inequality as described in [19]. We illustrate this fact by again considering Hölder's inequality (2.1). First, we note that by substitutions (2.1) can be written on the form

$$\int_\Omega (f_1(x))^a(f_2(x))^{1-a}d\mu(x) \leqslant \left(\int_\Omega f_1(x)d\mu(x)\right)^a\left(\int_\Omega f_2(x)d\mu(x)\right)^{1-a}, \quad 0<a<1, \tag{2.2}$$

with $a=1/p$, i.e. as an inequality between two standard geometric means. In (2.2) the index set contains only two numbers 1 and 2 but of course it holds in obvious form also if the index set is $1,2,\ldots,N$ (see (2.3)). But what can happen if the index set is infinite or even if the index set belongs to some general measure space? This is one basic idea behind the new theory for continuous forms of classical inequalities. Indeed the following form of Hölder's inequality (2.2) holds (see [19], Theorem 1.1 and also [12]):

THEOREM 2.1. *Let u and v be weight functions on the measure spaces (Ω,μ) and (Ψ,γ), respectively, such that $\int_\Omega u(x)d\mu(x)=1$. Let f be a positive function on $\Omega\times\Psi$ and measurable with respect to the measure $(\Omega\times\Psi,\mu\times\gamma)$. Then*

$$\int_\Psi \exp\int_\Omega \ln f(x,y)u(x)d\mu(x)v(y)d\gamma(y) \leqslant \exp\left(\int_\Omega \ln\int_\Psi f(x,y)v(y)d\gamma(y)\right)u(x)d\mu(x).$$

EXAMPLE 2.2. Let $N=2,3,\ldots,$ $\Omega=[0,1]$, $d\mu(x)=dx$, $u=1$, $p_i>1$ for $i=1,2,\ldots,N$, with $\sum_{i=1}^N 1/p_i=1$ and

$$f(x,y)=\begin{cases} f_1^{p_1}(y), & 0<x\leqslant\frac{1}{p_1},\\ f_2^{p_2}(y), & \frac{1}{p_1}<x\leqslant\frac{1}{p_1}+\frac{1}{p_2},\\ \vdots & \\ f_N^{p_N}(y), & \sum_{i=1}^{N-1}\frac{1}{p_i}<x\leqslant 1.\end{cases}$$

Then Theorem 2.1 implies that

$$\int_{\psi} \prod_{i=1}^{N} f_i(y)\nu(y)d\nu \leqslant \prod_{i=1}^{N} \left(\int_{\psi} f_i^{p_i}(y)\nu(y) \right)^{1/p_i}. \tag{2.3}$$

Next we state Jensen's inequality in a little more precise form connected to convexity.

THEOREM 2.3. *Let* (Ω,μ) *be a measure space and let* f *be a* μ*-measurable function on* (Ω,μ) *such that* $\varphi(f(x)) \subset \Omega$. *Then the inequality*

$$\varphi\left(\int_{\Omega} f(x)d\mu(x)\right) \leqslant \int_{\Omega} \varphi(f(x))d\mu(x) \tag{2.4}$$

holds for all probability measure spaces (Ω,μ), *i.e.* $\mu(\Omega)=1$, *satisfying* (2.4) *if and only if* $\varphi(x)$ *is convex.*

Proof. If φ is convex, then (2.4) holds in view of the standard form of Jensen's inequality. On the other hand, by applying (2.4) on the measure space, $(\mathbb{R},\lambda_1\delta_1+\lambda_2\delta_2)$, with $\lambda_1+\lambda_2=1$, $\lambda_1,\lambda_2\geqslant 0$, $f(1)=x_1$, $f(2)=x_2$ we find that

$$\varphi(\lambda_1 x_1+\lambda_2 x_2) \leqslant \lambda_1\varphi(x_1)+\lambda_2\varphi(x_2),$$

which means that the function φ is convex. Here, as usual, δ_1 and δ_2 denote the Dirac delta function at $x=1$ and $x=2$, respectively. $\square$

REMARK 2.4. A similar result as that in Theorem 2.3 obviously holds with (2.4) reversed and convex functions replaced by concave functions. Indeed, this fact indicates that Jensen's inequality (reversed Jensen's inequality) is more or less equivalent to the concept of convexity (concavity).

3. Old and new on power mean inequalities via convexity and superquadraticity

First we define the standard scale of discrete power means $p_\alpha, -\infty<\alpha<\infty$ as follows:

$$p_\alpha = p_\alpha(m,\lambda) = \begin{cases} (\sum_{i=1}^{\infty} m_i\lambda_i^{\alpha})^{1/2}, & \alpha \neq 0 \\ \prod_{i=1}^{\infty} \lambda_i^{m}, & \alpha = 0, \end{cases} \tag{3.1}$$

where $m=(m_1,m_2,\ldots)$, $\lambda=(\lambda_1,\lambda_2\ldots)$ with $m_i,\lambda_i\geqslant 0$ for $i=1,2,\ldots$ and $\sum_{i=1}^{\infty} m_i = 1$.

There exists also a similar definition for the continuous case with the Lebesgue measure involved. More generally, we define the following more general scale of power means $P_\alpha=P_\alpha(\Omega,\mu,f)$, $-\infty\leqslant\alpha\leqslant\infty$ as follows:

$$P_\alpha = \begin{cases} \operatorname{ess\,inf} f(x), & \alpha=-\infty \\ (\int_\Omega f^{\alpha}(x)d\mu(x))^{1/\alpha}, & -\infty<\alpha<\infty,\ \alpha\neq 0 \\ \exp\int_\Omega \ln f(x)d\mu(x), & \alpha=0 \\ \operatorname{ess\,sup} f(x), & \alpha=\infty, \end{cases} \tag{3.2}$$

where $f(x) > 0$ for the case $\alpha \leqslant 0$.

Next we state the following *continuous* form of the well-known standard power mean inequality, i.e. that P_α defined by (3.2) is a non-decreasing function of α (see e.g. [5, 6, 22, 23, 26] and the references given there).

THEOREM 3.1. *The scale of generalized power means P_α, $-\infty \leqslant \alpha \leqslant \infty$, as defined in* (3.2)*, is non-decreasing in α.*

REMARK 3.2. There are many proofs of this Theorem in the literature. Indeed, one such standard proof is just to apply the Jensen inequality (c.f. Theorem 2.3) to the functions $\varphi_1(u) = u^p$, $p \geqslant 1$, $p < 0$, and $\varphi(u) = \exp u$ in an appropriate way.

EXAMPLE 3.3. By applying Theorem 3.1 for the case $\Omega = \mathbb{R}_+$, $d\mu = \sum_{i=1}^{\infty} m_i\delta_i$, where δ_i is the Dirac delta function at $x = i$, $f(i) = \lambda_i$, $i = 1,2,\ldots$, and $\sum_{i=1}^{\infty} m_i = 1$, we obtain the standard power scale of power means defined in (3.1). Hence, in particular, Theorem 3.1 is a genuine generalization of the standard discrete power mean inequality: This, in its turn, is a generalization of the Harmonic-Geometric-Aritmetric (HGA) inequality $H \leqslant G \leqslant A$ where $H = p_{-1}$, $G = p_0$ and $A = p_1$.

REMARK 3.4. In particular, the HGA inequality between harmonic, geometric and arithmetric means is just a special case of Theorem 3.1, see Example 3.3. Indeed, Theorem 3.1 implies the following generalized HGA inequality: $P_{-1} \leqslant P_0 \leqslant P_1$, which is of special importance in our applications in Section 4.

The main aim is to prove some new refined forms of the power mean inequality in Theorem 3.1, but first, we need some preliminaries on superquadratic/subquadratic functions.

DEFINITION 3.5. A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there exists a constant $C_x \in \mathbb{R}$ such that, for all $y \geqslant 0$

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geqslant C_x(y-x). \tag{3.3}$$

We say that φ is subquadratic if $-\varphi$ is superquadratic.

EXAMPLE 3.6. If φ is continuously differentiable with $\varphi(0) \leqslant 0$ and $\varphi'(x)$ is superadditive or $\varphi'(x)/x$ is non-decreasing, then $\varphi(x)$ is superquadratic. For example the function $\varphi(x) = x^\alpha$ is superquadratic if $\alpha \geqslant 2$, and it is subquadratic if $1 < \alpha \leqslant 2$, (see [1] or [19, p. 65]). In this case $C_x = px^{p-1}$. Moreover, if $p = 2$, we indeed have equality in (3.3) in this case.

The following variant of Jensen's inequality, or more exactly, our more precise version in Theorem 2.3, holds (see [1, Theorem 2.2]).

LEMMA 3.7. *Let* (Ω,μ) *be a probability measure space (i.e* $\mu(\Omega)=1$*). Then the inequality*

$$\varphi\left(\int_\Omega f(x)d\mu(x)\right) \leqslant \int_\Omega \varphi(f(x)d\mu(x) - \int_\Omega \varphi\left(\left|f(x)-\int_\Omega f(x)d\mu(x)\right|\right)d\mu(x), \quad (3.4)$$

holds for all non-negative μ*-measurable functions* $f(x)$ *if and only if* $\varphi(x)$ *is superquadratic. Moreover,* (3.4) *holds in the reversed direction if and only if* $\varphi(x)$ *us subquadratic.*

EXAMPLE 3.8. The following special case of (3.4) is of special interest: if $f:\mathbb{R}_+\to\mathbb{R}$ is superquadratic, then the inequality

$$f\left(\sum_{i=1}^{\infty} m_i\lambda_i\right) \leqslant \sum_{i=1}^{\infty} m_i f\left(\lambda_i - \sum_{j=1}^{\infty} m_j f\left(\left|\lambda_j - \sum_{k=1}^{\infty} m_k\lambda_k\right|\right)\right) \quad (3.5)$$

holds for all sequences $\{\lambda_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of non-negative numbers such that $\sum_{i=1}^{\infty} m_i = 1$ (see also formula (6) in [5] for a finite version of (3.5))

Next we formulate the following refined version of Theorem 3.1:

THEOREM 3.9. *Let* (Ω,μ) *be a probability measure space (i.e.* $\mu(\Omega)=1$*).*

(a) If $\alpha_2 \geqslant 2\alpha_1 > 0$*, then*

$$\begin{aligned} P_{\alpha_1} &= \left(\int_\Omega f^{\alpha_1}(x)d\mu(x)\right)^{1/\alpha_1} \quad (3.6)\\ &\leqslant \left(\int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega \left|f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)\right|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2}\\ &\leqslant \left(\int_\Omega f^{\alpha_2}(x)d\mu(x)\right)^{1/\alpha_2} = P_{\alpha_2}. \end{aligned}$$

(b) If $0<\alpha_1<\alpha_2\leqslant 2\alpha_1$*, then*

$$\left(\int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega \left|f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)\right|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2} \leqslant P_{\alpha_1} \leqslant P_{\alpha_2}. \quad (3.7)$$

Proof.

(a) We use Lemma 3.7 for the superquadratic function $\varphi(u)=u^\alpha$, $\alpha=\alpha_2/\alpha_1 \geqslant 2$, see Example 3.6, and replace $f(x)$ by $f^{\alpha_1}(x)$ in (3.4) and obtain that

$$\left(\int_\Omega f^{\alpha}(x)d\mu(x)\right)^{\frac{\alpha_2}{\alpha_1}} \leqslant \int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega \left|f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)\right|^{\frac{\alpha_2}{\alpha_1}}$$

i.e. that

$$P_{\alpha_1} = \left(\int_\Omega f^{\alpha_1}(x)d\mu(x)\right)^{1/\alpha_1} \tag{3.8}$$
$$\leqslant \left(\int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_2}(x)d\mu(x)|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2},$$

and the first inequality in (3.6) is proved. The second inequality in (3.6) is trivial so (3.6) is proved.

(b) In this case we use Lemma 3.7 for the subquadratic function $\varphi(u) = u^\alpha$, $\alpha = \alpha_2/\alpha_1$, $1 < \alpha \leqslant 2$ and find that (3.8) holds in the reversed direction so the first inequality in (3.7) holds and the second inequality follows from Theorem 3.1 so also (3.7) holds and the proof is complete. □

REMARK 3.10. In view of Example 3.6 for the case $\alpha_2 = 2\alpha_1 > 0$ we have equality in the first inequalities in both (3.6) and (3.7).

EXAMPLE 3.11. Inspired by some recent results in [5] we note that the first inequalities in Theorem 3.9 can be rewritten as

$$P_{\alpha_2}^{\alpha_2} - P_{\alpha_1}^{\alpha_2} \geqslant \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)|^{\alpha_2/\alpha_1} d\mu, \tag{3.9}$$

when $\alpha_2 \geqslant 2\alpha_1 > 0$ and as

$$P_{\alpha_1}^{\alpha_2} - P_{\alpha_1}^{\alpha_1} \leqslant \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)|^{\alpha_2/\alpha_1} d\mu(x), \tag{3.10}$$

in the case of $0 < \alpha_1 < \alpha_2 \leqslant 2\alpha_1$.

REMARK 3.12. Another estimate than that in (3.9) in the special case of discrete power means (see (3.1) and Example 3.3) was proved in [5]. However, it should be interesting both to compare the estimates and to generalize the results in [5] to this more general situation. See especially [5, Theorem 5, estimate (12)].

Theorem 4 in [5] also covered the situation $\alpha_2 \leqslant 2\alpha_1 < 0$ where the same estimate was proved to hold. We finish this Section by stating a continuous power mean inequality covering also this situation.

THEOREM 3.13. *Let (Ω, μ) be a probability measure space (i.e. $\mu(\Omega) = 1$).*

(a) If $\alpha_2 \leqslant 2\alpha_1 < 0$, then

$$P_{\alpha_2} = \left(\int_\Omega f^{\alpha_2}(x)d\mu(x)\right)^{1/\alpha_2}$$
$$\leqslant \left(\int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2}. \tag{3.11}$$
$$\leqslant \left(\int_\Omega f^{\alpha_1}(x)d\mu(x)\right)^{1/\alpha_1} = P_{\alpha_1}.$$

(b) If $2\alpha_1 \leqslant \alpha_2 < a_1 < 0$, *then*

$$\left(\int_\Omega f^{\alpha_2}(x)d\mu(x) - \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2} \tag{3.12}$$
$$\leqslant P_{\alpha_2} \leqslant P_{\alpha_1}.$$

Proof.

(a) Since $\alpha_2 \leqslant 2\alpha_1 < 0$ is equivalent to that $-\alpha_2 \geqslant 2(-\alpha_1) > 0$ we can use (3.6) to find that

$$\left(\int_\Omega f^{-\alpha_1}(x)d\mu(x)\right)^{-1/\alpha_1}$$
$$\leqslant \left(\int_\Omega f^{-\alpha_2}(x)d\mu(x) - \int_\Omega |f^{-\alpha_1}(x) - \int_\Omega f^{-\alpha_1}(x)d\mu(x)|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{-1/\alpha_2}$$
$$\leqslant \left(\int_\Omega f^{-\alpha_2}(x)d\mu(x)\right)^{-1/\alpha_2}.$$

By now replacing $f(x)$ by $1/f(x)$ we obtain that

$$\begin{aligned} P_{\alpha_2} &= \left(\int_\Omega f^{\alpha_2}(x)d\mu(x)\right)^{1/\alpha_2} \\ &\leqslant \left(\int_\Omega f^{\alpha_1}(x)d\mu(x) - \int_\Omega |f^{\alpha_1}(x) - \int_\Omega f^{\alpha_1}(x)d\mu(x)|^{\frac{\alpha_2}{\alpha_1}} d\mu(x)\right)^{1/\alpha_2} \\ &\leqslant \left(\int_\Omega f^{\alpha_1}(x)d\mu(x)\right)^{1/\alpha_1} \\ &= P_{\alpha_1}, \end{aligned} \tag{3.13}$$

so (3.11) is proved.

(b) In this case the first of the inequalities in (3.11) holds in the reversed direction since the function $\varphi(u) = u^\alpha$, $\alpha = \alpha_2/\alpha_1$, $1 < \alpha \leqslant 2$, is subadditive. See Example 3.6 and Lemma 3.7. The second inequality follows from Theorem 3.1 so the proof is complete. □

REMARK 3.14. Note that Theorem 3.13 (a) implies that estimate (3.9) holds also for this case. This means that we have an estimate of the same type in both cases covered in Theorem 4 in [5] for the special case with discrete power means studied in this paper. However, no estimates of the type covered by our (b) parts of Theorem 3.9 and Theorem 3.13 can be found there.

4. Some illustrations and applications related to material science and homogenization theory

First, we consider the simple situation where we mix two materials in proportions. m_i, $i = 1,2$ i.e. $m_1 + m_2 = 1$ and heat conduction numbers λ_i, $i = 1,2$. The effective heat conduction number λ_{eff} will then be some power mean $p_\alpha = \left(\sum_{i=1}^{2} m_i \lambda_i^{\alpha}\right)^{1/\alpha}$ for some α, $-1 \leqslant \alpha \leqslant 1$, see e.g. [25]. The extreme situations will be the "laminate situation" in parts a) and b) in the following figure:

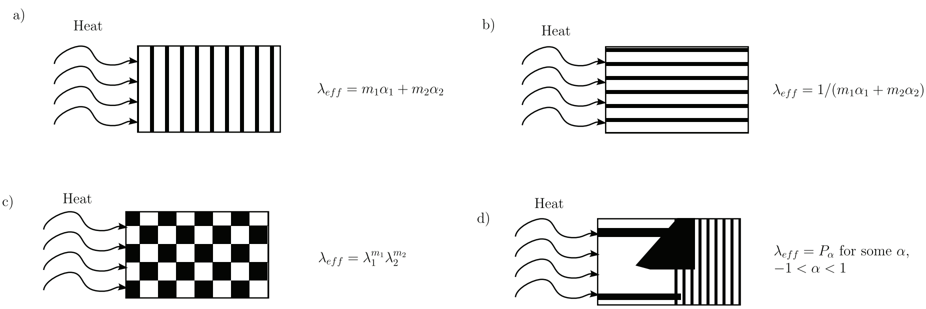


Figure 1: *Heat flow through material with different heat conduction numbers.*

REMARK 4.1. The situation above can easily be generalized to the situation with n, $n = 3,4,\ldots$ materials involved. The investigations in this paper show that we can even have infinitely many materials involved and we get the conclusion that in this case we have that

$$P_{-1} \leqslant \lambda_{eff} \leqslant P_1,$$

with P_α defined in (3.2), see Remark 3.4 and the corresponding more detailed explanations in [11].

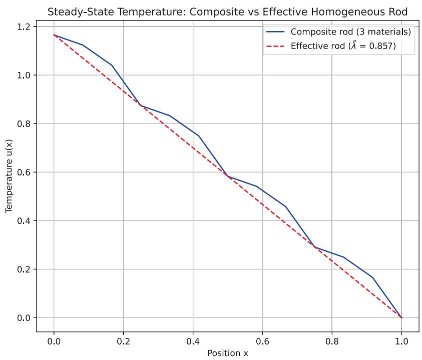
The description above can be applied also in several other physical situations (see Remark 5.1) but as in the introduction of the homogenization books [8] and [25] we continue to concentrate on the heat conduction case. For example the situation in Figure 1 b) can even be illustrated in one dimension via the equation:

$$\frac{d}{dx}\gamma(x)\frac{du}{dx} = f(x) \tag{4.1}$$

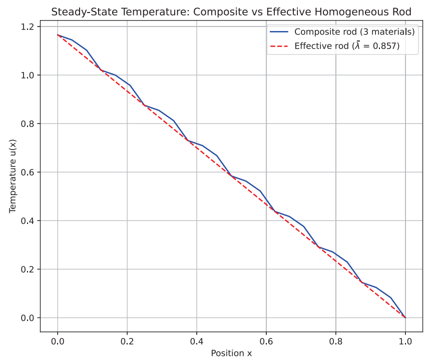
where $\gamma(x)$ is called thermal conductivity (see e.g. introduction of the book [8]). The equation (4.1) can be interpreted as the one-dimensional heat equation when the steady-state situation appeared i.e. when $\frac{d}{dt}u(x,t) = 0$. Moreover, $f(x)$ is a so called internal heat kernel which forces the heat distribution. In the simplest situation in [8] the boundary conditions $u(0) = u(1) = 0$ are used. However, here inspired by the introductory example in the book [25] we do not force the heat distribution by the source term $f(x)$

and instead force it with a left boundary condition. More exactly we study the problem

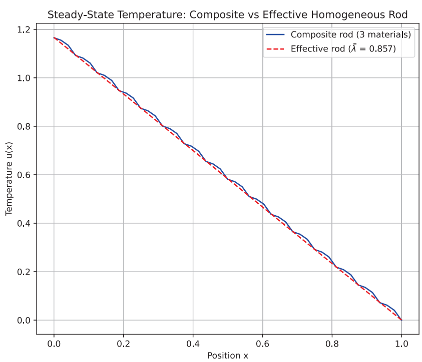
$$\begin{cases} \frac{d}{dx}\left(\gamma(x)\frac{du}{dx}\right) = 0 \\ -\gamma(0)\frac{du}{dx} = 1 \\ \qquad u(1) = 0. \end{cases}$$



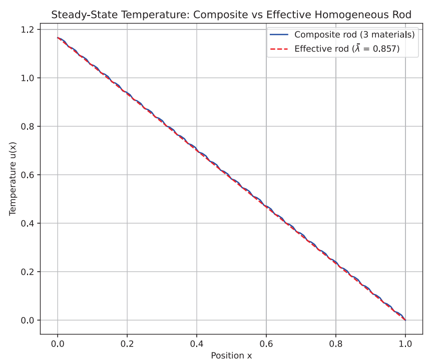
(a) $\varepsilon = 1/4$



(b) $\varepsilon = 1/8$



(c) $\varepsilon = 1/16$



(d) $\varepsilon = 1/32$

Figure 2: *Steady-state thermal converges as $\varepsilon \to 0$ (in blue) vs. homogenized rod (in red dashed).*

In a periodic distribution situation related to homogenization theory this was illustrated in [25] in a concrete situation with two materials involved. However, here we do a concrete illustration with three materials involved: This can be done as a transmission problem.

$$\begin{cases} \frac{d^2u}{dx^2} = 0, \quad \text{on } I_j, \quad j = 1, 2, \ldots, n, \quad n \text{ even}, \quad I_j = (x_{j-1}, x_j), \\ \gamma^+(x)\frac{du^+}{dx}(x) = \gamma^-(x)\frac{du^-}{dx}(x), \\ u^+(x) = u^-(x), \quad x = x_j, \quad j = 1, 2, \ldots n-1. \end{cases}$$

In Figure 2 we illustrate this in a typical way in homogenization theory: We introduce a parameter $\varepsilon = 1/n$, where n is the number of periods of equally distributed materials in each period.

REMARK 4.2. In the example above it is obvious that we get the usual convergence even pointwise in this "homogenization procedure" when we let $\varepsilon \to 0$, i.e. when $n \to \infty$. In more advanced situations in more dimensions, more advanced technical situations, (e.g. fibre composite materials), we need more advanced types of convergence in homogenization theory. For example, in a little more advanced situation of our example above we get only so called weak-type convergence (see e.g. [8, Theorem 5.5]). In other cases we have strong convergence but in homogenization theory even more advanced types of convergence are used e.g. Γ-convergence, G-convergence, two-scale convergence, etc. See [8], [25] and other books in homogenization theory and the references therein.

REMARK 4.3. One novelty of this paper is that the homogenization procedure involved above with three materials involved can be performed not only with finite many materials involved but also with even formally infinitely many materials involved. We believe that this fact has not been clearly mentioned so far in the vast homogenization literature.

5. Final remarks and results

REMARK 5.1. First time we discovered the importance to prove a Hölder inequality on continuous form was when we started to develop our theory for real interpolation between infinite many (indeed continuously many) Banach spaces, see [18] and c.f. also [7]. Our simple first idea was to put our Banach spaces on the unit circle in the complex plane and consider the "intermediate" interpolation spaces inside the unit circle and try to work like in classical harmonic analysis via Poisson kernels etc. In classical interpolation theory there is only two Banach spaces involved which are put e.g. to the points 0 and 1 att the real line and the "intermediate spaces" are those with main index strictly between these points. For a popular description of interpolation theory between infinite many spaces we refer to Appendix of the new book [19].

REMARK 5.2. We pronounce that the results in our Theorem 3.9 and Theorem 3.13 we have equality for the case $\alpha_2 = 2\alpha$. Hence, for this case our result can not be improved and this fact is also of interest for our next remark.

REMARK 5.3. We have pointed out that some estimates in the special case studied in the recent interesting paper [5] (discrete power means involving finite sums) can be compared with our estimates (see Remark 3.12 and Remark 3.14). It is not clear which of the results are more sharp in various situations. However, for the case $a_2 = 2a_1$ the situation is clear (see Remark 5.2). It seems to be an interesting open task to try to generalize also some other refined power mean inequalities in [5] to the more general continuous form studied in this paper.

REMARK 5.4. Concerning the illustrations/applications pointed out in Section 4 we have only considered heat conduction. However, it is obvious that such illustrations/applications can also be given for other important engineering problems, such as effective properties of optimal design, cellular solids, elasticity and strength of composite materials – for previous such results see e.g. [24], [25] and the PhD theses [15], [16], [28] and [4]. In [11], we complement Section 4 by giving similar concrete illustrations of various kinds, e.g. to chosen results in these and related PhD theses. Concerning the importance of inequalities in Mechanics and Physics, we also refer to the book [9] by Duvaut and Lions.

REMARK 5.5. The Hardy operator $H : Hf(x) = \frac{1}{x}\int_0^x f(t)dt$ is an arithmetic mean operator but replacing $f(x)$ by $(f(x))^{\alpha}$ and $H(f(x))$ by $(Hf(x))^{1/\alpha}$ we get a scale of power mean operators which, together with the power mean inequalities given in this paper, can alternatively be used in all results in Hardy-type inequalities, see e.g. the well cited book [14] and also [27] and the references therein. By using also the results in this paper we can obtain new and well known results. We will return to this question in a forthcoming paper. Here we also remark that Hardy type inequalities are also of interest in Homogenization Theory, see e.g. the PhD thesis [13], and in the theory of continuous inequalities, see the book [19].

REMARK 5.6. In Section 3 we used the concept of superquadraticity/subquadraticity to derive our refinements. However, there are many other techniques e.g. to use strong convexity, γ-convexity, γ-superquadraticity, and even superterzatic functions as developed by S. Abramovic and L. E. Persson, see e.g. [3, Theorem 2], where the critical turning point in Jensen's inequality is $p = 3$ instead of $p = 2$ as in the superquadratic case (see Lemma 3.7). In this connection we also refer to the book [19] and papers [2], [20] and [21].

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