Mathematical
Inequalities
& Applications
Volume 29, Number 3 (2026), 367–400 doi:10.7153/mia-2026-29-23

APPROXIMATION BY PARTIAL INTEGRALS OF DOUBLE FOURIER INTEGRALS

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(*Communicated by C. P. Niculescu*)

Abstract. In this paper, by giving rate of convergence under certain conditions we approximate the function of two variables by partial integrals of its Fourier integral. Also, in particular as a corollary, we get a two-dimensional analogue of the Dirichlet-Jordan theorem for Fourier integrals.

1. Introduction

The Dirichlet-Jordan theorem (see, e.g., [16, p. 57, Theorem 8.1]) asserts that the Fourier series of 2π-periodic function f of bounded variation on $[-\pi,\pi]$ converges at each point and the convergence is uniform on closed intervals of continuity of f. R. Bojanić [4] quantified this result by estimating the rate of convergence of the Fourier series at that point. Later on, R. Bojanić and S. M. Mazhar [5] proved a similar result for Nörlund means of Fourier series of functions of bounded variation. Furthermore, F. Móricz [9] obtained a two-dimensional analogue of the result established by R. Bojanić [4]. Also for non-periodic functions, the Dirichlet-Jordan theorem for Fourier integrals was given in (see, e.g., [15, p. 13, Theorem 3]). Recently, P. A. Bhuva and B. L. Ghodadra [3] have quantified this result by estimating the rate of convergence of the Fourier integrals of functions of bounded variations over $\mathbb{R} := (-\infty,\infty)$ at that point. Further, they [3] have given the rate of convergence of Nörlund means of Fourier integral of f, where f is of bounded variation over every finite interval $[a,b]$, $-\infty < a < b < \infty$. Here, we are going to establish a two-dimensional analogue of the result of P. A. Bhuva and B. L. Ghodadra by giving the rate of convergence of partial integrals of double Fourier integral of f, where f is of bounded variation over $\mathbb{R}^2$. This result also serves as the Fourier integral analogue of the theorem of F. Móricz [9].

Mathematics subject classification (2020): 42B10, 41A25, 26B30, 40A10.
Keywords and phrases: Rate of convergence, pointwise convergence, function of bounded variation on $\mathbb{R}^2$, double Fourier integral.

2. Notations and definitions

First, we recall the definition of Fourier series of f. Let f be a Lebesgue integrable function over $L^1(\mathbb{T})$; $\mathbb{T} := [-\pi,\pi)$. Then its Fourier series is defined as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where a_n and b_n are given by the usual Euler-Fourier formulae. If S_k is the kth partial sum of Fourier series of f then,

$$S_k(f;x) := S_k(x) = \frac{1}{2}a_0 + \sum_{n=1}^{k}(a_n \cos nt + b_n \sin nt), \quad k \in \mathbb{N} \cup \{0\}.$$

The double Fourier series of a function $f(x_1,x_2) \in L^1(\mathbb{T}^2)$, $\mathbb{T} := [-\pi,\pi)$, is given by (see, e.g., [12])

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{m,n}(x_1,x_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{m,n}[\alpha_{m,n} \cos mx_1 \cos nx_2 + \beta_{m,n} \sin mx_1 \cos nx_2 \\ + \gamma_{m,n} \cos mx_1 \sin nx_2 + \delta_{m,n} \sin mx_1 \sin nx_2],$$

where

$$\alpha_{m,n} = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(t_1,t_2) \cos mt_1 \cos nt_2 dt_1 dt_2,$$
$$\beta_{m,n} = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(t_1,t_2) \sin mt_1 \cos nt_2 dt_1 dt_2,$$
$$\gamma_{m,n} = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(t_1,t_2) \cos mt_1 \sin t_2 dt_1 dt_2,$$
$$\delta_{m,n} = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(t_1,t_2) \sin mt_1 \sin nt_2 dt_1 dt_2,$$

and

$$\lambda_{m,n} = 1, \quad \lambda_{m,0} = \lambda_{0,n} = \frac{1}{2}, \quad m,n \in \mathbb{N}, \quad \lambda_{0,0} = \frac{1}{4}.$$

If $S_{m,n}$ is the mnth partial sum of double Fourier series of f then,

$$S_{mn}(f;x_1,x_2) := S_{mn}(x_1,x_2) = \sum_{i=0}^{m}\sum_{j=0}^{n} A_{i,j}(x_1,x_2), \quad m,n \in \mathbb{N} \cup \{0\}.$$

The Fourier integral of a function $f(x) \in L^1(\mathbb{R})$, is given by (see [15, p. 1])

$$\int_0^{\infty}(a(u)\cos ux + b(u)\sin ux)du = \int_0^{\infty} A(u,x)du, \quad \text{say},$$

where

$$a(u) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\cos ut dt \qquad \text{and} \qquad b(u) = \frac{1}{\pi}\int_{\mathbb{R}} f(t)\sin ut dt.$$

If $S(f;y,x)$ is the yth partial integral of the Fourier integral of f then,

$$S(f;y,x) := S(y,x) = \int_0^y A(u,x)du, \quad y \in \mathbb{R}^+.$$

The double Fourier integral of a function $f(x_1,x_2) \in L^1(\mathbb{R}^2)$, is given by

$$\int_0^\infty \int_0^\infty (\alpha(u,v)\cos ux_1 \cos vx_2 + \beta(u,v)\cos ux_1 \sin vx_2 + \gamma(u,v)\sin ux_1 \cos vx_2 \\ + \delta(u,v)\sin ux_1 \sin vx_2)dudv = \int_0^\infty \int_0^\infty A(u,v,x_1,x_2)dudv, \quad \text{say},$$

where

$$\alpha(u,v) = \frac{1}{\pi^2}\int\int_{\mathbb{R}^2} f(t_1,t_2)\cos ut_1 \cos vt_2 dt_1 dt_2,$$
$$\beta(u,v) = \frac{1}{\pi^2}\int\int_{\mathbb{R}^2} f(t_1,t_2)\cos ut_1 \sin vt_2 dt_1 dt_2,$$
$$\gamma(u,v) = \frac{1}{\pi^2}\int\int_{\mathbb{R}^2} f(t_1,t_2)\sin ut_1 \cos vt_2 dt_1 dt_2,$$
$$\delta(u,v) = \frac{1}{\pi^2}\int\int_{\mathbb{R}^2} f(t_1,t_2)\sin ut_1 \sin vt_2 dt_1 dt_2.$$

If $S(f;y_1,y_2,x_1,x_2)$ is the y_1y_2-th partial integral of the Fourier integral of f then,

$$S(f;y_1,y_2,x_1,x_2) := S(y_1,y_2,x_1,x_2) = \int_0^{y_1}\int_0^{y_2} A(u,v,x_1,x_2)dudv, \quad y_1,y_2 \in \mathbb{R}^+.$$

It is important to note that the double Fourier integral may not exists in Lebesgue's or other sense, (e.g. as an improper integral).

Now, we recall the definiton of functions of bounded variation over $\mathbb{R}$. For that, first we give the definition of functions of bounded variation over $[a,b]$, $-\infty < a < b < \infty$.

A function $f : [a,b] \to \mathbb{R}$ is said to be of bounded variation over the interval $[a,b]$, in symbols: $f \in BV[a,b]$, if

$$V_1(f,[a,b]) := \sup\left\{\sum_{k=1}^n |f(x_k) - f(x_{k-1})|\right\} < \infty,$$

where supremum is taken over all partitions $a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$. The total variation function $V_f(x)$ of $f(x)$ on $[a,b]$ is defined by

$$V_f(x) = V_1(f,[a,x]), \quad x \in [a,b].$$

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation over $\mathbb{R}$, in symbols: $f \in BV(\mathbb{R})$, if

$$V_1(f,\mathbb{R}) := \sup_T \left\{\sum_n |f(y_n) - f(x_n)|\right\} < \infty,$$

where T is a finite collection of non-overlapping subintervals $[x_n,y_n]$ in $\mathbb{R}$.

It is clear that above definition of bounded variation over $\mathbb{R}$ can be reformulated equivalently as follows. A function $f:\mathbb{R}\to\mathbb{R}$ is of bounded variation if $f\in BV[a,b]$ for all $-\infty<a<b<\infty$ and $\sup\{V_1(f,[a,b]):-\infty<a<b<\infty\}$ is finite. Furthermore, if this is the case, then the total variation $V_1(f):=V_1(f,\mathbb{R})$ is given by

$$V_1(f)=\sup\{V_1(f,[a,b]):-\infty<a<b<\infty\}.$$

Similarly, we can define functions of bounded variation over $[a,\infty)$, for any $a\in\mathbb{R}$.

Now, we recall the concept of bounded variation in the sense of Hardy over $\mathbb{R}^2$ (see, e.g., [7]). For that, first we give the definition of functions of bounded variation in the sense of Hardy over $R:=[a_1,b_1]\times[a_2,b_2]$.

A function $f(x,y)$ defined on R is said to be of bounded variation over R in the sense of Vitali, in symbol: $f(x,y)\in BV_V(R)$, if

$$V_2(f,R):=\sup_{Q_1\times Q_2}\sum_{i=1}^{m}\sum_{j=1}^{n}|f(x_i,y_j)-f(x_{i-1},y_j)-f(x_i,y_{j-1})+f(x_{i-1},y_{j-1})|<\infty,$$

where the supremum is extended for all finite partitions $Q_1:a_1=x_0<x_1<x_2<\ldots<x_m=b_1$ and $Q_2:a_2=y_0<y_1<y_2<\ldots<y_n=b_2$ of the intervals $[a_1,b_1]$ and $[a_2,b_2]$, respectively. A function f in the class $BV_V(R)$ is said to be of bounded variation over R in the sense of Hardy and Krause, in symbol: $f(x,y)\in BV_H(R)$, if the marginal functions $f(\cdot,a_2)$ and $f(a_1,\cdot)$ are of bounded variation over the intervals $[a_1,b_1]$ and $[a_2,b_2]$, respectively. The total variation function $v_f(x,y)$ of $f(x,y)$ on R is defined by

$$v_f(x,y)=V_2(f,[a_1,x]\times[a_2,y]),\quad x\in[a_1,b_1],\quad y\in[a_2,b_2].$$

A function $f:\mathbb{R}^2\to\mathbb{R}$ is said to be of bounded variation over the real plane $\mathbb{R}^2$ in the sense of Vitali, in symbols: $f\in BV_V(\mathbb{R}^2)$, if

$$V_2(f,\mathbb{R}^2):=\sup\sum_{j=1}^{m}\sum_{k=1}^{n}|f(x_j,y_k)-f(x_{j-1},y_k)-f(x_j,y_{k-1})+f(x_{j-1},y_{k-1})|<\infty$$

where the supremum is extended over all finite sequences $-\infty<x_0<x_1<\ldots<x_m<\infty$ and $-\infty<y_0<y_1<\ldots<y_n<\infty$, where $m,n=1,2,\ldots$; while $V_2(f;\mathbb{R}^2)$ is called the total variation of f over $\mathbb{R}^2$. Similarly to the case of functions $f\in BV(\mathbb{R})$, the above definition can be also equivalently reformulated as follows. A function $f:\mathbb{R}^2\to\mathbb{R}$ is of bounded variation over $\mathbb{R}^2$ if and only if f is of bounded variation over all finite rectangles $[a_1,b_1]\times[a_2,b_2]$, $-\infty<a_1<b_1<\infty$ and $-\infty<a_2<b_2<\infty$; and in addition, the set of the total variations of f over all bounded rectangles $[a_1,b_1]\times[a_2,b_2]$ is bounded. Furthermore, if this is the case, then the supremum of the set of these total variations over all bounded rectangles $[a_1,b_1]\times[a_2,b_2]$ is equal to $V_2(f;\mathbb{R}^2)$ defined above. A function $f\in BV_V(\mathbb{R}^2)$ is said to be of bounded variation over $\mathbb{R}^2$ in the sense of Hardy, in symbols: $f\in BV_H(\mathbb{R}^2)$, if the marginal functions $f(\cdot,y)$ and $f(x,\cdot)$ are of bounded variation on $\mathbb{R}$ for any fixed value of $y\in\mathbb{R}$ or $x\in\mathbb{R}$, respectively.

In a similar way, one can define the notion of bounded variation in the sense of Vitali and in the sense of Hardy over $[0,\infty)\times[0,\infty)$.

Next, we recall the definition of the double Riemann-Stieltjes integral (see, e.g., [7]). We say that a function $g(x,y)$ is integrable with respect to the function $f(x,y)$ in the Riemann-Stieltjes sense over R if the limit of the approximating Riemann-Stieltjes sums

$$s_{m,n} := \sum_{i=1}^{m}\sum_{j=1}^{n} g(\xi_i,\eta_j)[f(x_i,y_j)-f(x_{i-1},y_j)-f(x_i,y_{j-1})+f(x_{i-1},y_{j-1})]$$

exists, as the partitions $Q_1 : a_1 = x_0 < x_1 < x_2 < \ldots < x_m = b_1$ and $Q_2 : a_2 = y_0 < y_1 < y_2 < \ldots < y_n = b_2$, get finer and finer in the sense that the mesh size

$$\rho(Q_1\times Q_2) := \max\left\{\max_{1\leqslant i\leqslant m}(x_i-x_{i-1}),\ \max_{1\leqslant j\leqslant n}(y_j-y_{j-1})\right\}\to 0$$

while the intermediate points

$$x_{i-1}\leqslant \xi_i \leqslant x_i \quad\text{and}\quad y_{j-1}\leqslant \eta_j\leqslant y_j,\quad i=1,2,\ldots,m;\quad j=1,2,\ldots,n;$$

are arbitrary. The Riemann-Stieltjes integral of $g(x,y)$ with respect to $f(x,y)$ over R, in symbol:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(x,y)d_xd_yf(x,y)\quad\text{or}\quad \iint_R g(x,y)d_xd_yf(x,y),$$

is defined as the limit of the sums $s_{m,n}$ as $\rho(Q_1\times Q_2)\to 0$ (provided this limit exists).

We note that if f is of bounded variation over $\mathbb{R}$, then $f(x+0)$ and $f(x-0)$ exist at every point $x\in(-\infty,\infty)$. Similarly, if f is of bounded variation in the sense of Hardy over $\mathbb{R}^2$, then $f(x+0,y+0)$, $f(x-0,y+0)$, $f(x+0,y-0)$, and $f(x-0,y-0)$ exist for $(x,y)\in\mathbb{R}^2$.

We use the following notations:

$$f^*(x_1,x_2)=\frac{1}{4}[f(x_1+0,x_2+0)+f(x_1-0,x_2+0)+f(x_1+0,x_2-0)+f(x_1-0,x_2-0)],$$

$$M(g)=\mathrm{lub}_{x\in D}(|g(x)|),\quad D \text{ is the domain of g},$$

$$V_1'(f,[a,b])=M(f)+V_1(f,[a,b]),$$

$$V_2'(f,[a,b],[c,d])=3M(f)+3V_2(f,[a,b]\times[c,d])+V_1(f(\cdot,c),[a,b])+V_1(f(a,\cdot),[c,d]).$$

For $x\in\mathbb{R}$, we define

$$\phi_x(t)=\begin{cases} f(x+t)+f(x-t)-f(x+0)-f(x-0),\ t\in(0,\infty)\\ 0,\ t=0,\end{cases}$$

and for $(x_1,x_2)\in\mathbb{R}^2$, we define

$$\phi_{x_1x_2}(t_1,t_2)=\begin{cases} f(x_1+t_1,x_2+t_2)+f(x_1+t_1,x_2-t_2)+f(x_1-t_1,x_2+t_2)\\ +f(x_1-t_1,x_2-t_2)-4f^*(x_1,x_2),\ \ t_1,t_2\in(0,\infty);\\ f(x_1+0,x_2+t_2)+f(x_1+0,x_2-t_2)+f(x_1-0,x_2+t_2)\\ +f(x_1-0,x_2-t_2)-4f^*(x_1,x_2),\ \ t_1=0 \text{ and } t_2\in(0,\infty);\\ f(x_1+t_1,x_2+0)+f(x_1+t_1,x_2-0)+f(x_1-t_1,x_2+0)\\ +f(x_1-t_1,x_2-0)-4f^*(x_1,x_2),\ \ t_1\in(0,\infty) \text{ and } t_2=0;\\ 0,\ \ t_1=t_2=0.\end{cases}$$

and

$$g_{x_1x_2}(t_1,t_2)=\phi_{x_1x_2}(t_1,t_2)-\phi_{x_1x_2}(0,t_2)-\phi_{x_1x_2}(t_1,0),\ \ t_1,t_2\in\mathbb{R}^+.$$

Jordan proved that if f is a 2π-periodic function of bounded variation on $[-\pi,\pi]$, then its Fourier series converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at each point of x (see, e.g., [16, p. 57, Theorem 8.1]). R. Bojanić [4] give the quantified version of Jordan's theorem for Fourier series as the following theorem.

THEOREM 1. *If f is a 2π-periodic function and is of bounded variation on $[-\pi,\pi]$, then for $x\in[-\pi,\pi]$ and $n\in\mathbb{N}$ we have*

$$\left|S_n(f;x)-\frac{1}{2}[f(x+0)+f(x-0)]\right|\leqslant\frac{3}{n}\sum_{k=1}^{n}V_1\left(\phi_x,\left[0,\frac{\pi}{k}\right]\right).$$

Also, F. Móricz [9] established a two dimension analogue of above theorem, which is stated as follows.

THEOREM 2. *If $f(x,y)$ is 2π-periodic in each variable and is of bounded variation over $[-\pi,\pi]\times[-\pi,\pi]$ in the sense of Hardy and Krause, then for all $m,n\geqslant 0$, we have*

$$\begin{aligned}|S_{mn}(f;x_1,x_2)-f^*(x_1,x_2)|\leqslant&\frac{4(1+\frac{2}{\pi}+\frac{1}{\pi^2})}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n}V_2\left(\phi_{x_1x_2},\left[0,\frac{\pi}{i}\right]\times\left[0,\frac{\pi}{j}\right]\right)\\ &+\frac{2(1+\frac{1}{\pi})}{m+1}\sum_{i=1}^{m}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{\pi}{i}\right]\right)\\ &+\frac{2(1+\frac{1}{\pi})}{n+1}\sum_{j=1}^{n}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{\pi}{j}\right]\right).\end{aligned}$$

For non-periodic functions, Jordan's theorem for functions of bounded variation is as follows (see, e.g., [15, p. 13, Theorem 3]).

THEOREM 3. *Let $f\in L^1(\mathbb{R})$. If f is of bounded variation in an interval including the point x, then*

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos u(x-t)\,dt\,du.$$

If f is continuous and of bounded variation in an interval (a,b) then

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos u(x-t)\,dt\,du$$

the integral converging uniformly in any interval interior to (a,b).

Recently, P. A. Bhuva and B. L. Ghodadra [3] have quantified above result by giving rate of convergence of Fourier integral of functions of bounded variation over $\mathbb{R}$ as the following theorem.

THEOREM 4. *Let $f\in L^1(\mathbb{R})\cap BV(\mathbb{R})$. Then for any $a,y\in\mathbb{R}^+$ with $n\leqslant y<n+1$; $n\in\mathbb{N}\cup\{0\}$ and for any $x\in\mathbb{R}$, we have*

$$\left|S(y,x)-\frac{1}{2}[f(x+0)+f(x-0)]\right|\leqslant\frac{A}{n+1}\sum_{k=0}^{n}V_1\left(\phi_x,\left[0,\frac{a}{k+1}\right]\right)+\frac{A}{ay},$$

where A denotes a positive constant not necessarily the same at each ocurrence.

In this paper, we present a two-dimensional analogue of the above theorem by establishing the rate of convergence of the double Fourier integral of functions of bounded variation over $\mathbb{R}^2$ in the sense of Hardy. This result can also be regarded as the Fourier integral analogue of Theorem 2. Consequently, our theorem shows that any non-periodic function of two variables with bounded variation can be approximated by the partial integrals of its Fourier integral. As a corollary, we obtain Fourier integral analogue of Jordan's theorem for functions of two variables.

3. Results

Our main results is the following theorem.

THEOREM 5. *Let $f\in L^1(\mathbb{R}^2)\cap BV_H(\mathbb{R}^2)$. Then for any $a,b,y_1,y_2\in\mathbb{R}^+$ satisfying $m\leqslant y_1<m+1$ and $n\leqslant y_2<n+1$, where $m,n\in\mathbb{N}\cup\{0\}$, and for any $x_1,x_2\in\mathbb{R}$, we have*

$$\begin{aligned}
&\left|S(y_1,y_2,x_1,x_2)-f^*(x_1,x_2)\right|\\
&=\frac{O(1)}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right)\\
&\quad+\frac{O(1)}{m+1}\sum_{i=0}^{m-1}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{i+1}\right]\right)\\
&\qquad+\frac{O(1)}{n+1}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right)+\frac{O(1)}{y_1y_2}+\frac{O(1)}{y_1}+\frac{O(1)}{y_2}.
\end{aligned}\tag{1}$$

Since continuity of $\phi_{x_1x_2}(t_1,t_2)$ at $t_1=0=t_2$ implies that (see e.g., [9, p. 349])

$$\lim_{\delta_1,\delta_2\to 0} V_2(\phi_{x_1x_2},[0,\delta_1]\times[0,\delta_2]) = \lim_{\delta_1\to 0} V_1(\phi_{x_1x_2}(\cdot,0),[0,\delta_1]) = \lim_{\delta_2\to 0} V_1(\phi_{x_1x_2}(0,\cdot),[0,\delta_2]) = 0,$$

the right hand side of (1) tends to 0 as $y_1,y_2\to\infty$ and hence, we get the following corollary from above result.

COROLLARY 1. *Let $f\in L^1(\mathbb{R}^2)\cap BV_H(\mathbb{R}^2)$. Then the double Fourier integral of f is converges to $f^*(x_1,x_2)$, for every $(x_1,x_2)\in\mathbb{R}^2$.*

4. Proof of the results

We need the following lemmas to prove our theorems.

LEMMA 1. *Consider $n\in\mathbb{N}$. For $j=0,1,\ldots,n$, let a_j and b_j be real numbers. Let $B_j=\sum_{k=j}^n b_k$ and $B'_j=\sum_{k=0}^j b_k$, for $j=0,1,\ldots,n$. Then*

$$\sum_{j=1}^n a_jb_j = \sum_{j=1}^n [a_j-a_{j-1}]B_j + a_0B_1, \tag{2}$$

and

$$\sum_{j=0}^n a_jb_j = \sum_{j=0}^{n-1} (a_j-a_{j+1})B'_j + a_nB'_n. \tag{3}$$

LEMMA 2. *Consider $(m,n)\in\mathbb{N}^2$. For $j=0,1,\ldots,m$ and $k=0,1,\ldots,n$, let $a_{j,k}$ and $b_{j,k}$ be real numbers, and let $B_{m,n}=\sum_{j=0}^m\sum_{k=0}^n b_{j,k}$. Then*

$$\sum_{j=0}^m\sum_{k=0}^n a_{j,k}b_{j,k} = a_{m,n}B_{m,n} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}(a_{j,k}-a_{j+1,k}-a_{j,k+1}+a_{j+1,k+1})B_{j,k} + \sum_{j=0}^{m-1}(a_{j,n}-a_{j+1,n})B_{j,n} + \sum_{k=0}^{n-1}(a_{m,k}-a_{m,k+1})B_{m,k}. \tag{4}$$

Also, if we assume that $B_{j,k}=\sum_{j'=j}^m\sum_{k'=k}^n b_{j',k'}$ and $B_{m+1,n+1}=B_{j,n+1}=B_{m+1,k}=0$, for $j=0,1,\ldots,m$, $k=0,1,\ldots,n$, then

$$\sum_{j=1}^m\sum_{k=1}^n a_{j,k}b_{j,k} = \sum_{j=1}^m\sum_{k=1}^n(a_{j,k}-a_{j-1,k}-a_{j,k-1}+a_{j-1,k-1})B_{j,k} + \sum_{j=1}^m(a_{j,0}-a_{j-1,0})B_{j,1} + \sum_{k=1}^n(a_{0,k}-a_{0,k-1})B_{1,k} + a_{0,0}B_{1,1}. \tag{5}$$

LEMMA 3. ([10, Lemma 2]) *If $g(t_1,t_2)$ is continuous on the rectangle R and $f(t_1,t_2) \in BV_H(R)$, then $f(t_1,t_2)$ is integrable with respect to $g(t_1,t_2)$ over R in the Riemann-Stieltjes sense, and*

$$\begin{aligned}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)d_{t_1}d_{t_2}g(t_1,t_2) &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} g(t_1,t_2)d_{t_1}d_{t_2}f(t_1,t_2)\\ &+\int_{a_1}^{b_1} f(t_1,b_2)d_{t_1}g(t_1,b_2)-\int_{a_1}^{b_1} f(t_1,a_2)d_{t_1}g(t_1,a_2)\\ &+\int_{a_2}^{b_2} f(b_1,t_2)d_{t_2}g(b_1,t_2)-\int_{a_2}^{b_2} f(a_1,t_2)d_{t_2}g(a_1,t_2)\\ &-f(b_1,b_2)g(b_1,b_2)+f(a_1,b_2)g(a_1,b_2)\\ &+f(b_1,a_2)g(b_1,a_2)-f(a_1,a_2)g(a_1,a_2). \end{aligned} \tag{6}$$

LEMMA 4. ([10, Lemma 3]) *If $f(t_1,t_2) \in BV_H(R)$ and*

$$g(t_1,t_2) = \int_{a_1}^{t_1}\int_{a_2}^{t_2} h(x,y)dxdy + C,\ (t_1,t_2) \in R$$

for some function $h(t_1,t_2) \in L^1(R)$ and constant C, then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)d_{t_1}d_{t_2}g(t_1,t_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)h(t_1,t_2)dt_1dt_2.$$

LEMMA 5. (see [6]; see also [10, Lemma 1]; and see proof of [7, Lemma 3.5]) *If $g(t_1,t_2)$ is continuous on the rectangle R and $f(t_1,t_2) \in BV_V(R)$, then $g(t_1,t_2)$ is integrable with respect to $f(t_1,t_2)$ over R in the Riemann-Stieltjes sense and*

$$\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2} g(t_1,t_2)d_{t_1}d_{t_2}f(t_1,t_2)\right| \leqslant \int_{a_1}^{b_1}\int_{a_2}^{b_2} |g(t_1,t_2)|d_{t_1}d_{t_2}v_f(t_1,t_2),$$

where $v_f(t_1,t_2) = V_2(f,[a_1,t_1]\times[a_2,t_2])$.

LEMMA 6. *Suppose $y,a \in \mathbb{R}^+$. Then for any $a \leqslant c \leqslant \infty$, we get*

$$\left|\int_a^c \frac{\sin yt}{t}\right| \leqslant \frac{2}{ay}. \tag{7}$$

Proof. Let $a,y \in \mathbb{R}^+$ be fixed. Since $\frac{1}{t}$ is a non-negative and non-increasing function, using second mean value theorem (see, e.g., [2, p. 3]) there exists $k \in [a,c]$ such that

$$\int_a^c \frac{\sin yt}{t}dt = \frac{1}{a}\int_a^k \sin yt dt = \frac{1}{ay}[\cos ya - \cos yk].$$

Hence, for $a \leqslant c < \infty$,

$$\left|\int_a^c \frac{\sin yt}{t}\right| = \left|\frac{1}{ay}[\cos ya - \cos yk]\right| \leqslant \frac{2}{ay}. \tag{8}$$

If $c = \infty$, then using (8) we get

$$\left|\int_a^\infty \frac{\sin yt}{t}dt\right| = \lim_{c_1\to\infty}\left|\int_a^{c_1} \frac{\sin yt}{t}dt\right| \leqslant \lim_{c_1\to\infty}\frac{2}{ay} = \frac{2}{ay}. \quad \square$$

LEMMA 7. *Let $g(t) \in BV[a,\infty)$ with $a > 0$. Then for any $y \in \mathbb{R}^+$, we have*

$$\left|\int_a^\infty g(t)\frac{\sin yt}{t}dt\right| \leqslant \frac{2}{ay}[M(g) + V_1(g,[a,\infty))] = \frac{2}{ay}V_1'(g,[a,\infty)).$$

Proof. Let $a, y \in \mathbb{R}^+$ be fixed. Observe that

$$\int_a^\infty g(t)\frac{\sin yt}{t}dt = \lim_{c\to\infty}\int_a^c g(t)\frac{\sin yt}{t}dt.$$

Since $g \in BV[a,\infty)$ and by Lemma 6,

$$\left|\int_a^t \frac{\sin y\mu}{\mu}d\mu\right| \leqslant \frac{2}{ay} < \infty,$$

we get (see e.g., [1, p. 158, Theorem 7.26]),

$$\left|\int_a^\infty g(t)\frac{\sin yt}{t}dt\right| = \left|\lim_{c\to\infty}\int_a^c g(t)d_t\left(\int_a^t \frac{\sin y\mu}{\mu}d\mu\right)\right|. \tag{9}$$

Since $g(t)$ and $\int_a^t \frac{\sin y\mu}{\mu}d\mu$ are of bounded variation on $[a,c]$ and $\int_a^t \frac{\sin y\mu}{\mu}d\mu$ is continuous, by partial integration formula (see e.g., [14, p. 122]), we have

$$\begin{aligned}
&\left|\int_a^\infty g(t)\frac{\sin yt}{t}dt\right| \\
&= \left|\lim_{c\to\infty}\left[g(c)\int_a^c \frac{\sin y\mu}{\mu}d\mu - g(a)\int_a^a \frac{\sin y\mu}{\mu}d\mu - \int_a^c\left(\int_a^t \frac{\sin y\mu}{\mu}d\mu\right)dg(t)\right]\right| \\
&\leqslant \lim_{c\to\infty}|g(c)|\left|\int_a^c \frac{\sin y\mu}{\mu}d\mu\right| + \lim_{c\to\infty}\left|\int_a^c\left(\int_a^t \frac{\sin y\mu}{\mu}d\mu\right)dg(t)\right|.
\end{aligned} \tag{10}$$

Using Lemma 6 and the fact that, a function of bounded variation is bounded, we get

$$\lim_{c\to\infty}|g(c)|\left|\int_a^c \frac{\sin y\mu}{\mu}d\mu\right| \leqslant M(g)\frac{2}{ay}. \tag{11}$$

Since $g(t) \in BV[0,\infty)$ and $\int_a^t \frac{\sin y\mu}{\mu}d\mu$ is continuous, using [14, Theorem 6.29] in the second term of the right hand side (10), we get

$$\lim_{c\to\infty}\left|\int_a^c\left(\int_a^t \frac{\sin y\mu}{\mu}d\mu\right)dg(t)\right| \leqslant \lim_{c\to\infty}\int_a^c\left|\int_a^t \frac{\sin y\mu}{\mu}d\mu\right|dV_g(t).$$

Using Lemma 6 and then as $g \in BV[a,\infty)$ and 1 is a continuous function, using (see e.g., [11, p. 232, Theorem 1]), we get

$$\lim_{c\to\infty}\left|\int_a^c\left(\int_a^t \frac{\sin y\mu}{\mu}d\mu\right)dg(t)\right| \leqslant \lim_{c\to\infty}\frac{2}{ay}\int_a^c dv_g(t) \leqslant \lim_{c\to\infty}\frac{2}{ay}V_1(g,[a,c]) = \frac{2}{ay}V_1(g,[a,\infty)). \tag{12}$$

Using (11) and (12) in (10), we get

$$\left|\int_a^\infty g(t)\frac{\sin yt}{t}dt\right| \leqslant \frac{2}{ay}[M(g)+V_1(g,[a,\infty))]. \quad \square$$

LEMMA 8. *Let $f \in BV_H(R)$, with $a_1, a_2 > 0$. Then for any $y_1, y_2 \in \mathbb{R}^+$, we have*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2$$
$$\leqslant \frac{4}{a_1a_2y_1y_2}[3M(f)+V_1(f(\cdot,b_2),[a_1,b_1])+V_1(f(b_1,\cdot),[a_2,b_2])+V_2(f,[a_1,b_1]\times[a_2,b_2])].$$

Proof. Let $a_1,a_2,y_1,y_2 \in \mathbb{R}^+$ be fixed. Using Lemma 6,

$$\left|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\sin y_1\mu}{\mu}\frac{\sin y_2\nu}{\nu}d\mu d\nu\right| \leqslant \frac{2}{a_1y_1}\frac{2}{a_2y_2} < \infty.$$

Since $f \in BV_H(R)$ and $\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2} \in L^1(R)$, using Lemma 4, we get

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)dt_1dt_2\left(\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{t_2}\frac{\sin y_2\nu}{\nu}d\nu\right).$$

Since $f \in BV_H(R)$ and $\int_{a_1}^{t_1}\int_{a_2}^{t_2}\frac{\sin y_1\mu}{\mu}\frac{\sin y_2\nu}{\nu}d\mu d\nu$ is continuous over rectangle R, using Lemma 3 in above equality, we get

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \Bigg[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{t_2}\frac{\sin y_2\nu}{\nu}d\nu\right)d_{t_1}d_{t_2}f(t_1,t_2)\\
&\quad + \int_{a_1}^{b_1} f(t_1,b_2)d_{t_1}\left(\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu\right)\\
&\quad + \int_{a_2}^{b_2} f(b_1,t_2)d_{t_2}\left(\int_{a_1}^{b_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{t_2}\frac{\sin y_2\nu}{\nu}d\nu\right)\\
&\quad - f(b_1,b_2)\int_{a_1}^{b_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu\Bigg]\\
&= I_1(y_1,y_2)+I_2(y_1,y_2)+I_3(y_1,y_2)+I_4(y_1,y_2), \quad \text{say.}
\end{aligned} \tag{13}$$

Since $\int_{a_1}^{t_1}\int_{a_2}^{t_2}\frac{\sin y_1\mu}{\mu}\frac{\sin y_2\nu}{\nu}d\mu d\nu$ is continuous over rectangle R and $f(x_1,x_2)\in BV_H(R)$, using Lemma 5 we get

$$|I_1(y_1,y_2)|\leqslant\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left|\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\right|\left|\int_{a_2}^{t_2}\frac{\sin y_2\nu}{\nu}d\nu\right|d_{t_1}d_{t_2}V_f(t_1,t_2).$$

Using Lemma 6 and then as $f\in BV_H(R)$ and the constant function 1 is continuous and bounded, using [7, Eq. (19)], we get

$$\begin{aligned}|I_1(y_1,y_2)|&\leqslant\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{2}{a_1y_1}\frac{2}{a_2y_2}d_{t_1}d_{t_2}V_f(t_1,t_2)\\&\leqslant\frac{4}{a_1a_2y_1y_2}V_2(f,[a_1,b_1]\times[a_2,b_2]).\end{aligned}\tag{14}$$

Since $f(t_1,b_2)$ and $\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu$ are of bounded variation on $[a_1,b_1]$ and $\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\ \times\ \int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu$ is continuous on $[a_1,b_1]$, by partial integration formula (see [14, p. 122]), we have

$$\begin{aligned}|I_2(y_1,y_2)|=\Bigg|&f(b_1,b_2)\left(\int_{a_1}^{b_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu\right)\\&-f(a_1,b_2)\left(\int_{a_1}^{a_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu\right)\\&-\int_{a_1}^{b_1}\left(\int_{a_1}^{t_1}\frac{\sin y_1\mu}{\mu}d\mu\int_{a_2}^{b_2}\frac{\sin y_2\nu}{\nu}d\nu\right)d_{t_1}f(t_1,b_2)\Bigg|.\end{aligned}$$

Using Lemma 6 and the fact that a function of bounded variation is bounded, we get

$$\begin{aligned}|I_2(y_1,y_2)|&\leqslant M(f)\frac{2}{a_1y_1}\frac{2}{a_2y_2}+\int_{a_1}^{b_1}\frac{2}{a_1y_1}\frac{2}{a_2y_2}dv_{f(\cdot,b_2)}(t_1)\\&\leqslant\frac{4M(f)}{a_1a_2y_1y_2}+\frac{4}{a_1a_2y_1y_2}\int_{a_1}^{b_1}dv_{f(\cdot,b_2)}(t).\end{aligned}$$

Since $f(\cdot,b_2)\in BV[a_1,b_1]$ and 1 is continuous, using [14, Theorem 6.29] we get

$$|I_2(y_1,y_2)|\leqslant\frac{4M(f)}{a_1a_2y_1y_2}+\frac{4}{a_1a_2y_1y_2}V_1(f(\cdot,b_2),[a_1,b_1]).\tag{15}$$

Similarly, we can prove that

$$|I_3(y_1,y_2)|\leqslant\frac{4M(f)}{a_1a_2y_1y_2}+\frac{4}{a_1a_2y_1y_2}V_1(f(b_1,\cdot),[a_2,b_2]).\tag{16}$$

Also, using Lemma 6 and the fact that a function of bounded variation is bounded, we get

$$|I_4(y_1,y_2)|\leqslant\frac{4M(f)}{a_1a_2y_1y_2}.\tag{17}$$

Using (14)–(17) in (13), we get

$$\begin{aligned}
&\left|\int_{a_1}^{b_2}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right| \\
&\leqslant \frac{4}{a_1a_2y_1y_2}[3M(f)+V_1(f(\cdot,b_2),[a_1,b_1])+V_1(f(b_1,\cdot),[a_2,b_2]) \\
&\quad +V_2(f,[a_1,b_1]\times[a_2,b_2])]. \quad \square
\end{aligned}$$

LEMMA 9. *Let $f \in BV_H(R)$ with $a_1 = 0$, $a_2 > 0$. Then for any $y_1,y_2 \in \mathbb{R}^+$, we have*

$$\begin{aligned}
&\int_0^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\
&\leqslant \frac{2y_1b_1}{a_2y_2}[3M(f)+V_1(f(\cdot,b_2),[0,b_1])+V_1(f(b_1,\cdot),[a_2,b_2]) \\
&\quad +V_2(f,[0,b_1]\times[a_2,b_2])].
\end{aligned}$$

Proof. Let $y_1,y_2,a_2 \in \mathbb{R}^+$. Then

$$\begin{aligned}
&\left|\int_0^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right| \\
&= \lim_{a_1'\to 0^+}\left|\int_{a_1'}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|.
\end{aligned}$$

Using the same technique as that used to prove Lemma 8, but using the following estimate

$$\left|\int_{a_1'}^{b_1}\frac{\sin y_1t_1}{t_1}dt_1\right| \leqslant \int_{a_1'}^{b_1}\frac{y_1t_1}{t_1}dt_1 \leqslant y_1b_1,$$

instead of the previous one, we obtain

$$\begin{aligned}
&\left|\int_{a_1'}^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right| \\
&\leqslant \frac{2y_1b_1}{a_2y_2}[3M(f)+V_1(f(\cdot,b_2),[a_1',b_1])+V_1(f(b_1,\cdot),[a_2,b_2]) \\
&\quad +V_2(f,[a_1',b_1]\times[a_2,b_2])].
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\int_0^{b_1}\int_{a_2}^{b_2} f(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right| \\
&\leqslant \frac{2y_1b_1}{a_2y_2}[3M(f)+V_1(f(\cdot,b_2),[0,b_1])+V_1(f(b_1,\cdot),[a_2,b_2]) \\
&\quad +V_2(f,[0,b_1]\times[a_2,b_2])]. \quad \square
\end{aligned}$$

LEMMA 10. *Let $f \in BV_H(R)$ with $a_1 > 0$, $a_2 = 0$. Then for any $y_1, y_2 \in \mathbb{R}^+$, we have*

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_0^{b_2} f(t_1,t_2)\frac{\sin y_1 t_1}{t_1}\frac{\sin y_2 t_2}{t_2}dt_1 dt_2 \\
&\leqslant \frac{2y_2 b_2}{a_1 y_1}[3M(f) + V_1(f(\cdot,b_2),[a_1,b_1]) + V_1(f(b_1,\cdot),[0,b_2]) \\
&\quad + V_2(f,[a_1,b_1]\times[0,b_2])].
\end{aligned}$$

Proof. Proof is similar to Lemma 9, we only use the following estimates

$$\left|\int_{a_1}^{b_1}\frac{\sin y_1 t_1}{t_1}dt_1\right| \leqslant \frac{2}{a_1 y_1} \text{ and } \left|\int_0^{b_2}\frac{\sin y_2 v}{v}dv\right| \leqslant \int_{a_2}^{b_2} y_2 dt_2 \leqslant y_2 b_2. \qquad \square$$

LEMMA 11. *Let $f \in BV_H(R)$. Then for any $d_1 \in [a_1,b_1]$ and $d_2 \in [a_2,b_2]$, we have*

$$V_1(f(\cdot,d_2),[a_1,b_1]) \leqslant V_2(f,[a_1,b_1]\times[a_2,d_2]) + V_1(f(\cdot,a_2),[a_1,b_1]) \tag{18}$$

and

$$V_1(f(d_1,\cdot),[a_2,b_2]) \leqslant V_2(f,[a_1,d_1]\times[a_2,b_2]) + V_1(f(a_1,\cdot),[a_2,b_2]). \tag{19}$$

Proof. Let $a_1 = t'_0 < t'_1 < \ldots < t'_m = b_1$ be partition of the interval $[a_1,b_1]$. Then observe that

$$\begin{aligned}
&\sum_{i=1}^m |f(t'_i,d_2) - f(t'_{i-1},d_2)| \\
&= \sum_{i=1}^m |f(t'_i,d_2) - f(t'_{i-1},d_2) - f(t'_i,a_2) + f(t'_{i-1},a_2) + f(t'_i,a_2) - f(t'_{i-1},a_2)| \\
&\leqslant \sum_{i=1}^m |f(t'_i,d_2) - f(t'_{i-1},d_2) - f(t'_i,a_2) + f(t'_{i-1},a_2)| + \sum_{i=1}^m |f(t'_i,a_2) - f(t'_{i-1},a_2)| \\
&\leqslant V_2(f,[a_1,b_1]\times[a_2,d_2]) + V_1(f(\cdot,a_2),[a_1,b_1]).
\end{aligned}$$

Taking the supremum over all partitions of $[a_1,b_1]$ on both sides of the above inequality, we get

$$V_1(f(\cdot,d_2),[a_1,b_1]) \leqslant V_2(f,[a_1,b_1]\times[a_2,d_2]) + V_1(f(\cdot,a_2),[a_1,b_1]).$$

Analogusly, we can prove that

$$V_1(f(d_1,\cdot),[a_2,b_2]) \leqslant V_2(f,[a_1,d_1]\times[a_2,b_2]) + V_1(f(a_1,\cdot),[a_2,b_2]). \qquad \square$$

Proof of Theorem 5. Let $x_1,x_2 \in \mathbb{R}$ and $a,b,y_1,y_2 \in \mathbb{R}^+$ be fixed. Using the formulas of $\alpha(u,v), \beta(u,v), \gamma(u,v)$, and $\delta(u,v)$, we have

$$\begin{aligned}
&S(y_1,y_2,x_1,x_2)\\
&= \int_0^{y_1}\int_0^{y_2} A(u,v,x_1,x_2)dudv\\
&= \int_0^{y_1}\int_0^{y_2} [\alpha(u,v)\cos ux_1 \cos vx_2 + \beta(u,v)\cos ux_1 \sin vx_2 + \gamma(u,v)\sin ux_1 \cos vx_2\\
&\qquad\qquad\qquad\qquad + \delta(u,v)\sin ux_1 \sin vx_2]dudv\\
&= \int_0^{y_1}\int_0^{y_2} \Bigg[\left(\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\cos ut_1 \cos vt_2 dt_1dt_2\right)\cos ux_1 \cos vx_2\\
&\qquad + \left(\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\cos ut_1 \sin vt_2 dt_1dt_2\right)\cos ux_1 \sin vx_2\\
&\qquad + \left(\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\sin ut_1 \cos vt_2 dt_1dt_2\right)\sin ux_1 \cos vx_2\\
&\qquad + \left(\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\sin ut_1 \sin vt_2 dt_1dt_2\right)\sin ux_1 \sin vx_2\Bigg]dudv\\
&= \int_0^{y_1}\int_0^{y_2} \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\cos ut_1 \cos ux_1(\cos v(t_2-x_2))dt_1dt_2\right]dudv\\
&\qquad + \int_0^{y_1}\int_0^{y_2} \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\sin ut_1 \sin ux_1(\cos v(t_2-x_2))dt_1dt_2\right]dudv\\
&= \int_0^{y_1}\int_0^{y_2} \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1,t_2)\cos v(t_2-x_2)\cos u(t_1-x_1)dt_1dt_2\right]dudv\\
&= \int_0^{y_1}\int_0^{y_2} \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1+x_1,t_2+x_2)\cos ut_1 \cos vt_2 dt_1dt_2\right]dudv\\
&= \int_0^{y_1}\int_0^{y_2} \Bigg[\frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty} [f(x_1+t_1,x_2+t_2)+f(x_1+t_1,x_2-t_2)\\
&\qquad + f(x_1-t_1,x_2+t_2)+f(x_1-t_1,x_2-t_2)]\cos ut_1 \cos vt_2 dt_1dt_2\Bigg]dudv\\
&= \frac{1}{\pi^2}\int_0^{y_1}\int_0^{y_2}\left[\int_0^{\infty}\int_0^{\infty} \phi^*_{x_1x_2}(t_1,t_2)\cos ut_1 \cos vt_2 dt_1dt_2\right]dudv, \qquad (20)
\end{aligned}$$

where $\phi^*_{x_1x_2}(t_1,t_2) = f(x_1+t_1,x_2+t_2)+f(x_1+t_1,x_2-t_2)+f(x_1-t_1,x_2+t_2)+f(x_1-t_1,x_2-t_2)$. Since $\phi^*_{x_1x_2}(t_1,t_2)$ is integrable over $\mathbb{R}^+\times\mathbb{R}^+$ and 1 is integrable over $[0,y_1]\times[0,y_2]$ as it is continuous, using (see, e.g., [13, p. 423]), we have

$$\begin{aligned}
&\int_0^{y_1}\int_0^{y_2}\int_0^{\infty}\int_0^{\infty} |\phi^*_{x_1x_2}(t_1,t_2)||\cos ut_1||\cos vt_2|dudvdt_1dt_2\\
&\leqslant \int_0^{y_1}\int_0^{y_2}\int_0^{\infty}\int_0^{\infty} |\phi^*_{x_1x_2}(t_1,t_2)|dudvdt_1dt_2\\
&= \left(\int_0^{\infty}\int_0^{\infty} |\phi^*_{x_1x_2}(t_1,t_2)|dt_1dt_2\right)\times\left(\int_0^{y_1}\int_0^{y_2} dudv\right) \leqslant Ay_1y_2.
\end{aligned}$$

Hence, using Fubini's theorem in (20), we get

$$\begin{aligned} S(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi^*_{x_1x_2}(t_1,t_2)\left[\int_0^{y_1}\int_0^{y_2}\cos ut_1\cos vt_2\,du\,dv\right]dt_1dt_2 \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi^*_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2. \end{aligned} \tag{21}$$

Since $y_1,y_2\in\mathbb{R}^+$, there exist $m,n\in\mathbb{N}\cup\{0\}$ such that $m\leqslant y_1<m+1$ and $n\leqslant y_2<n+1$. Now, using (21) and the fact that $\int_0^\infty \frac{\sin yt}{t}dt=\frac{\pi}{2}$ for $y\in\mathbb{R}^+$, we have

$$\begin{aligned} &S(y_1,y_2,x_1,x_2)-f^*(x_1,x_2) \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi^*_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad -\frac{4}{\pi^2}\frac{\pi^2}{4}\frac{1}{4}[f(x_1+0,x_2+0)+f(x_1+0,x_2-0)+f(x_1-0,x_2+0)+f(x_1-0,x_2-0)] \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi^*_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2-\frac{1}{\pi^2}\int_0^\infty\int_0^\infty[f(x_1+0,x_2+0) \\ &\qquad +f(x_1+0,x_2-0)+f(x_1-0,x_2+0)+f(x_1-0,x_2-0)]\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty[\phi_{x_1x_2}(t_1,t_2)-\phi_{x_1x_2}(t_1,0)-\phi_{x_1x_2}(0,t_2)]\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad +\frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad\qquad +\frac{1}{\pi^2}\int_0^\infty\int_0^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad +\frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1+\frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2 \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad +\frac{1}{\pi^2}\int_0^\infty\int_{\frac{b}{n+1}}^{b} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad\qquad +\frac{1}{\pi^2}\int_0^\infty\int_b^\infty g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\qquad\qquad\qquad +\frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1+\frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2 \\ &= \frac{1}{\pi^2}\int_0^{\frac{a}{m+1}}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\ &\quad +\frac{1}{\pi^2}\int_{\frac{a}{m+1}}^{a}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\pi^2}\int_a^{\infty}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_0^{\frac{a}{m+1}}\int_{\frac{b}{n+1}}^{b} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_{\frac{a}{m+1}}^{a}\int_{\frac{b}{n+1}}^{b} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_a^{\infty}\int_{\frac{b}{n+1}}^{b} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_0^{\frac{a}{m+1}}\int_b^{\infty} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_{\frac{a}{m+1}}^{a}\int_b^{\infty} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{\pi^2}\int_a^{\infty}\int_b^{\infty} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&+\frac{1}{2\pi}\int_0^{\infty}\phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1+\frac{1}{2\pi}\int_0^{\infty}\phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2\\
&=S_1(y_1,y_2,x_1,x_2)+S_2(y_1,y_2,x_1,x_2)+S_3(y_1,y_2,x_1,x_2)+S_4(y_1,y_2,x_1,x_2)\\
&\quad+S_5(y_1,y_2,x_1,x_2)+S_6(y_1,y_2,x_1,x_2)+S_7(y_1,y_2,x_1,x_2)+S_8(y_1,y_2,x_1,x_2)\\
&\quad+S_9(y_1,y_2,x_1,x_2)+S_{10}(y_1,x_1,x_2)+S_{11}(y_2,x_1,x_2),\quad \text{say.} \qquad (22)
\end{aligned}
$$

Note that, $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$ as $f\in BV_H(\mathbb{R}^2)$. Since $\phi_{x_1x_2}(0,0)=0$ and $|\sin t|\leqslant t$ for $t\geqslant 0$, we have

$$
\begin{aligned}
|S_1(y_1,y_2,x_1,x_2)| &= \frac{1}{\pi^2}\left|\int_0^{\frac{a}{m+1}}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\leqslant \frac{1}{\pi^2}\int_0^{\frac{a}{m+1}}\int_0^{\frac{b}{n+1}} |\phi_{x_1x_2}(t_1,t_2)-\phi_{x_1x_2}(t_1,0)-\phi_{x_1x_2}(0,t_2)+\phi_{x_1x_2}(0,0)|\\
&\qquad\qquad\times\frac{|\sin y_1t_1|}{t_1}\frac{|\sin y_2t_2|}{t_2}dt_1dt_2\\
&\leqslant \frac{1}{\pi^2}\int_0^{\frac{a}{m+1}}\int_0^{\frac{b}{n+1}} V_2(\phi_{x_1x_2},[0,t_1]\times[0,t_2])y_1y_2dt_1dt_2\\
&\leqslant \frac{1}{\pi^2}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)y_1y_2\frac{a}{m+1}\frac{b}{n+1}\\
&< \frac{ab}{\pi^2}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\\
&= \frac{ab}{\pi^2}\frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\\
&\leqslant \frac{4ab}{(m+1)(n+1)\pi^2}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right). \qquad (23)
\end{aligned}
$$

Further, observe that

$$
\begin{aligned}
S_2(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\int_{\frac{a}{m+1}}^{a}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} \left[g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)+g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\right]\\
&\qquad\qquad \times\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} \left[g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\right]\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&\qquad\qquad +\frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= S_{21}(y_1,y_2,x_1,x_2)+S_{22}(y_1,y_2,x_1,x_2),\quad \text{say.}
\end{aligned}\tag{24}
$$

Since $|\sin t|\leqslant 1$ for $t\in\mathbb{R}$ and $|\sin t|\leqslant t$ for $t\geqslant 0$, we have

$$
\begin{aligned}
&|S_{21}(y_1,y_2,x_1,x_2)|\\
&= \left|\frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} \left[g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\right]\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} \Bigg|\phi_{x_1x_2}(t_1,t_2)-\phi_{x_1x_2}(t_1,0)-\phi_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\\
&\qquad\qquad +\phi_{x_1x_2}\left(\frac{ia}{m+1},0\right)\Bigg|\frac{|\sin y_1t_1|}{t_1}\frac{|\sin y_2t_2|}{t_2}dt_1dt_2\\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},t_1\right]\times[0,t_2]\right)\frac{1}{t_1}y_2dt_1dt_2\\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}}\frac{1}{t_1}y_2dt_1dt_2\\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\\
&\qquad\qquad \times\frac{m+1}{ia}\left[\frac{(i+1)a}{m+1}-\frac{ia}{m+1}\right]\frac{by_2}{n+1}\\
&< \frac{b}{\pi^2}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right).
\end{aligned}\tag{25}
$$

Also, note that

$$
\begin{aligned}
S_{22}(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\sum_{i=1}^{m}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_0^{\frac{b}{n+1}} g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \frac{1}{\pi^2}\int_0^{\frac{b}{n+1}}\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right]\frac{\sin y_2t_2}{t_2}dt_2.
\end{aligned}
\tag{26}
$$

Using (2) of Lemma 1, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right]\right|\\
&=\left|\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},t_2\right)\right]\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right.\\
&\qquad\left.+\,g_{x_1x_2}\left(\frac{0a}{m+1},t_2\right)\sum_{\mu=1}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right|\\
&\leqslant\sum_{i=1}^{m}\left|g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},t_2\right)\right|\left|\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right|,
\end{aligned}
\tag{27}
$$

as $g_{x_1x_2}(0,t_2)=\phi_{x_1x_2}(0,t_2)-\phi_{x_1x_2}(0,t_2)-\phi_{x_1x_2}(0,0)=0$. Since $\frac{ia}{m+1},y_1\in\mathbb{R}^+$ and $\frac{ia}{m+1}<a$ for $i=1,2,3,\ldots,m$; by using Lemma 6 we have

$$
\left|\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right|=\left|\int_{\frac{ia}{m+1}}^{a}\frac{\sin y_1t_1}{t_1}dt_1\right|\leqslant\frac{2}{y_1}\frac{m+1}{ia}\leqslant\frac{4}{ia}.
\tag{28}
$$

Using (28) in (27), we get

$$
\begin{aligned}
&\left|\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right]\right|\\
&\leqslant\frac{4}{a}\sum_{i=1}^{m}\frac{1}{i}\left|g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},t_2\right)\right|\\
&=\frac{4}{a}\sum_{i=1}^{m}\frac{1}{i}\left|\phi_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-\phi_{x_1x_2}\left(\frac{ia}{m+1},0\right)-\phi_{x_1x_2}\left(\frac{(i-1)a}{m+1},t_2\right)\right.\\
&\qquad\left.+\,\phi_{x_1x_2}\left(\frac{(i-1)a}{m+1},0\right)\right|\\
&\leqslant\frac{4}{a}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times[0,t_2]\right).
\end{aligned}
\tag{29}
$$

Using (29) in (26), we get

$$\begin{aligned}
&|S_{22}(y_1,y_2,x_1,x_2)| \\
&\leqslant \frac{4}{a\pi^2}\int_0^{\frac{b}{n+1}}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times[0,t_2]\right)\frac{|\sin y_2t_2|}{t_2}dt_2 \\
&\leqslant \frac{4}{a\pi^2}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\int_0^{\frac{b}{n+1}} y_2dt_2 \\
&\leqslant \frac{4b}{a\pi^2}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right). \qquad (30)
\end{aligned}$$

Using (25) and (30) in (24), we get

$$\begin{aligned}
|S_2(y_1,y_2,x_1,x_2)| &\leqslant \frac{b}{\pi^2}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right) \\
&\quad + \frac{4b}{a\pi^2}\sum_{i=1}^{m}\frac{1}{i}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right) \\
&\leqslant \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\sum_{i=0}^{m}\frac{1}{i+1}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right).
\end{aligned}$$

Using (3) of Lemma 1, we get

$$\begin{aligned}
&|S_2(y_1,y_2,x_1,x_2)| \\
&= \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\sum_{i=0}^{m-1}\left[\frac{1}{i+1}-\frac{1}{i+2}\right]V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\right. \\
&\qquad\left. + \frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\} \\
&= \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\sum_{i=0}^{m-1}\int_{i+1}^{i+2}\frac{1}{t^2}dt\cdot V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\right. \\
&\qquad\left. + \frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\} \\
&\leqslant \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\sum_{i=0}^{m-1}\int_{i+1}^{i+2}\frac{1}{t^2}V_2\left(\phi_{x_1x_2},\left[0,\frac{ta}{m+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)dt\right. \\
&\qquad\left. + \frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\} \\
&= \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\frac{1}{m+1}\int_1^{m+1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t}\right]\times\left[0,\frac{b}{n+1}\right]\right)dt\right. \\
&\qquad\left. + \frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\frac{1}{m+1}\sum_{i=0}^{m-1}\int_{i+1}^{i+2} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t}\right]\times\left[0,\frac{b}{n+1}\right]\right)dt\right.\\
&\qquad\left.+\frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\}\\
&\leqslant \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\frac{1}{m+1}\sum_{i=0}^{m-1} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\right.\\
&\qquad\left.+\frac{1}{m+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\}\\
&= \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\frac{1}{(m+1)n}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{n+1}\right]\right)\right.\\
&\qquad\left.+\frac{1}{(m+1)n}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{n+1}\right]\right)\right\}\\
&\leqslant \frac{2b}{\pi^2}\left[1+\frac{2}{a}\right]\left\{\frac{1}{(m+1)n}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right)\right.\\
&\qquad\left.+\frac{1}{(m+1)n}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{j+1}\right]\right)\right\}\\
&\leqslant \frac{8b}{\pi^2}\left[1+\frac{2}{a}\right]\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right). \qquad (31)
\end{aligned}
$$

Analogously, we can prove that

$$
\begin{aligned}
&|S_4(y_1,y_2,x_1,x_2)|\\
&\leqslant \frac{8a}{\pi^2}\left[1+\frac{2}{b}\right]\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right). \qquad (32)
\end{aligned}
$$

Next, we note that

$$
\begin{aligned}
|S_3(y_1,y_2,x_1,x_2)| &= \left|\frac{1}{\pi^2}\int_a^\infty\int_0^{\frac{b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\leqslant \frac{1}{\pi^2}\left|\int_a^\infty\int_0^{\frac{b}{n+1}} \phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\quad+\frac{1}{\pi^2}\left|\int_a^\infty\int_0^{\frac{b}{n+1}} \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\quad+\frac{1}{\pi^2}\left|\int_a^\infty\int_0^{\frac{b}{n+1}} \phi_{x_1x_2}(0,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&= S_{31}(y_1,y_2,x_1,x_2)+S_{32}(y_1,y_2,x_1,x_2)+S_{33}(y_1,y_2,x_1,x_2), \quad \text{say.} \qquad (33)
\end{aligned}
$$

Using Lemma 6 and $\phi_{x_1x_2}(0,0)=0$, we get

$$\begin{aligned}
S_{33}(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\left|\int_a^\infty \frac{\sin y_1t_1}{t_1}dt_1\right|\left|\int_0^{\frac{b}{n+1}}[\phi_{x_1x_2}(0,t_2)-\phi_{x_1x_2}(0,0)]\frac{\sin y_2t_2}{t_2}dt_2\right| \\
&\leqslant \frac{1}{\pi^2}\frac{2}{ay_1}y_2\int_0^{\frac{b}{n+1}} V_1(\phi_{x_1x_2}(0,\cdot),[0,t_2])dt_2 \\
&\leqslant \frac{2b}{ay_1\pi^2}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{n+1}\right]\right).
\end{aligned} \tag{34}$$

Since $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$, $\phi_{x_1x_2}(\cdot,0)\in BV[0,\infty)$. Therefore, using Lemma 7 and also using the fact that, $|\sin t|\leqslant t$ for $t>0$, we get

$$\begin{aligned}
S_{32}(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\left|\int_a^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1\right|\left|\int_0^{\frac{b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\right| \\
&\leqslant \frac{1}{\pi^2}\frac{2}{ay_1}[M(\phi_{x_1x_2}(\cdot,0))+V_1(\phi_{x_1x_2}(\cdot,0),[a,\infty))]\frac{by_2}{n+1} \\
&\leqslant \frac{2b}{ay_1\pi^2}V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)).
\end{aligned} \tag{35}$$

Since $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$, using Lemma 10 we get

$$\begin{aligned}
S_{31}(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\lim_{c_1\to\infty}\left|\int_a^{c_1}\int_0^{\frac{b}{n+1}}\phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right| \\
&\leqslant \frac{1}{\pi^2}\lim_{c_1\to\infty}\left\{\frac{2y_2(\frac{b}{n+1})}{ay_1}\left[3M(\phi_{x_1x_2})+V_2\left(\phi_{x_1x_2},[a,c_1]\times\left[0,\frac{b}{n+1}\right]\right)\right.\right. \\
&\quad\left.\left.+V_1\left(\phi_{x_1x_2}\left(\cdot,\frac{b}{n+1}\right),[a,c_1]\right)+V_1\left(\phi_{x_1x_2}(c_1,\cdot),\left[0,\frac{b}{n+1}\right]\right)\right]\right\}.
\end{aligned}$$

Again, since $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$, using (18) and (19) of Lemma 11, we get

$$\begin{aligned}
S_{31}(y_1,y_2,x_1,x_2) &\leqslant \frac{1}{\pi^2}\lim_{c_1\to\infty}\left\{\frac{2b}{ay_1}\left[3M(\phi_{x_1x_2})+3V_2\left(\phi_{x_1x_2},[a,c_1]\times\left[0,\frac{b}{n+1}\right]\right)\right.\right. \\
&\quad\left.\left.+V_1(\phi_{x_1x_2}(\cdot,0),[a,c_1])+V_1\left(\phi_{x_1x_2}(a,\cdot),\left[0,\frac{b}{n+1}\right]\right)\right]\right\} \\
&= \frac{2b}{ay_1\pi^2}\left[3M(\phi_{x_1x_2})+3V_2\left(\phi_{x_1x_2},[a,\infty)\times\left[0,\frac{b}{n+1}\right]\right)\right. \\
&\quad\left.+V_1(\phi_{x_1x_2}(\cdot,0),[a,\infty))+V_1\left(\phi_{x_1x_2}(a,\cdot),\left[0,\frac{b}{n+1}\right]\right)\right] \\
&= \frac{2b}{ay_1\pi^2}V_2'\left(\phi_{x_1x_2},[a,\infty),\left[0,\frac{b}{n+1}\right]\right).
\end{aligned} \tag{36}$$

Using (33)–(36), we get

$$|S_3(y_1,y_2,x_1,x_2)| \leqslant \frac{2b}{ay_1\pi^2}\left[V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{n+1}\right]\right)+V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty))\right.$$
$$\left.+V_2'\left(\phi_{x_1x_2},[a,\infty),\left[0,\frac{b}{n+1}\right]\right)\right]. \tag{37}$$

Analogusly, using Lemma 9, we get

$$|S_7(y_1,y_2,x_1,x_2)| \leqslant \frac{2a}{by_2\pi^2}\left[V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{m+1}\right]\right)+V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty))]\right.$$
$$\left.+V_2'\left(\phi_{x_1x_2},\left[0,\frac{a}{m+1}\right],[b,\infty)\right)\right]. \tag{38}$$

Now,

$$\begin{aligned}
S_5(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}} g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\left\{\left[g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)\right.\right.\\
&\quad\left.+g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right]+\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right]\\
&\quad\left.+\left[g_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right]+g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right\}\\
&\quad\times\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\\
&= S_{51}(y_1,y_2,x_1,x_2)+S_{52}(y_1,y_2,x_1,x_2)+S_{53}(y_1,y_2,x_1,x_2)\\
&\quad+S_{54}(y_1,y_2,x_1,x_2), \quad \text{say.}
\end{aligned} \tag{39}$$

Observe that

$$\begin{aligned}
&g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)+g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\\
&=\phi_{x_1x_2}(t_1,t_2)-\phi_{x_1x_2}(0,t_2)-\phi_{x_1x_2}(t_1,0)-\phi_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)+\phi_{x_1x_2}(0,t_2)\\
&\quad+\phi_{x_1x_2}\left(\frac{ia}{m+1},0\right)-\phi_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)+\phi_{x_1x_2}(t_1,0)+\phi_{x_1x_2}\left(0,\frac{jb}{n+1}\right)\\
&\quad+\phi_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)-\phi_{x_1x_2}\left(0,\frac{jb}{n+1}\right)-\phi_{x_1x_2}\left(\frac{ia}{m+1},0\right)\\
&=\phi_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)-\phi_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-\phi_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)+\phi_{x_1x_2}(t_1,t_2)
\end{aligned}$$

and hence we have

$$
\begin{aligned}
&|S_{51}(y_1,y_2,x_1,x_2)| \\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\left\{\left|g_{x_1x_2}(t_1,t_2)-g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)\right.\right. \\
&\qquad\left.\left.-g_{x_1x_2}\left(t_1,\frac{jb}{n+1}\right)+g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right|\frac{|\sin y_1t_1|}{t_1}\frac{|\sin y_2t_2|}{t_2}\right\}dt_1dt_2 \\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},t_1\right]\times\left[\frac{jb}{n+1},t_2\right]\right)\frac{1}{t_1t_2}dt_1dt_2 \\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right) \\
&\qquad\times\frac{(m+1)(n+1)}{iajb}\left[\frac{(i+1)a}{m+1}-\frac{ia}{m+1}\right]\left[\frac{(j+1)b}{n+1}-\frac{jb}{n+1}\right] \\
&=\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{ij}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right). \qquad (40)
\end{aligned}
$$

Using (2) of Lemma 1, we get

$$
\begin{aligned}
&S_{52}(y_1,y_2,x_1,x_2) \\
&=\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right] \\
&\qquad\times\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2 \\
&=\frac{1}{\pi^2}\sum_{j=1}^{n}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\left(\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right]\right. \\
&\qquad\left.\times\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right)\frac{\sin y_2t_2}{t_2}dt_2 \\
&=\frac{1}{\pi^2}\sum_{j=1}^{n}\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\left\{\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\right.\right. \\
&\quad\left.-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},t_2\right)+g_{x_1x_2}\left(\frac{(i-1)a}{m+1},\frac{jb}{n+1}\right)\right]\left(\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right) \\
&\quad\left.+\left[g_{x_1x_2}\left(\frac{0a}{m+1},t_2\right)-g_{x_1x_2}\left(\frac{0a}{m+1},\frac{jb}{n+1}\right)\right]\sum_{\mu=1}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\right\} \\
&\qquad\times\frac{\sin y_2t_2}{t_2}dt_2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\pi^2} \sum_{j=1}^{n} \int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}} \left(\sum_{i=1}^{m} \left[\phi_{x_1x_2}\left(\frac{ia}{m+1}, t_2\right) - \phi_{x_1x_2}\left(\frac{ia}{m+1}, \frac{jb}{n+1}\right) \right.\right. \\
&\qquad \left.\left. - \phi_{x_1x_2}\left(\frac{(i-1)a}{m+1}, t_2\right) + \phi_{x_1x_2}\left(\frac{(i-1)a}{m+1}, \frac{jb}{n+1}\right) \right] \sum_{\mu=i}^{m} \int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}} \frac{\sin y_1 t_1}{t_1} dt_1 \right) \\
&\qquad\qquad \times \frac{\sin y_2 t_2}{t_2} dt_2. \qquad (41)
\end{aligned}
$$

Using Lemma 6 in the inner sum of (41), we get

$$
\begin{aligned}
&\left| \sum_{i=1}^{m} \left[\phi_{x_1x_2}\left(\frac{ia}{m+1}, t_2\right) - \phi_{x_1x_2}\left(\frac{ia}{m+1}, \frac{jb}{n+1}\right) - \phi_{x_1x_2}\left(\frac{(i-1)a}{m+1}, t_2\right) \right.\right. \\
&\qquad\qquad \left.\left. + \phi_{x_1x_2}\left(\frac{(i-1)a}{m+1}, \frac{jb}{n+1}\right) \right] \sum_{\mu=i}^{m} \int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}} \frac{\sin y_1 t_1}{t_1} dt_1 \right| \\
&\leqslant \sum_{i=1}^{m} V_2\left(\phi_{x_1x_2}, \left[\frac{(i-1)a}{m+1}, \frac{ia}{m+1}\right] \times \left[\frac{jb}{n+1}, t_2\right]\right) \left| \int_{\frac{ia}{m+1}}^{a} \frac{\sin y_1 t_1}{t_1} dt_1 \right| \\
&\leqslant \frac{4}{a} \sum_{i=1}^{m} \frac{1}{i} V_2\left(\phi_{x_1x_2}, \left[\frac{(i-1)a}{m+1}, \frac{ia}{m+1}\right] \times \left[\frac{jb}{n+1}, t_2\right]\right). \qquad (42)
\end{aligned}
$$

Therefore using (42) in (41), we get

$$
\begin{aligned}
&|S_{52}(y_1, y_2, x_1, x_2)| \\
&\leqslant \frac{4}{a\pi^2} \sum_{j=1}^{n} \int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}} \sum_{i=1}^{m} \frac{1}{i} V_2\left(\phi_{x_1x_2}, \left[\frac{(i-1)a}{m+1}, \frac{ia}{m+1}\right] \times \left[\frac{jb}{n+1}, t_2\right]\right) \frac{|\sin y_2 t_2|}{t_2} dt_2 \\
&\leqslant \frac{4}{a\pi^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{i} V_2\left(\phi_{x_1x_2}, \left[\frac{(i-1)a}{m+1}, \frac{ia}{m+1}\right] \times \left[\frac{jb}{n+1}, \frac{(j+1)b}{n+1}\right]\right) \\
&\qquad\qquad \times \int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}} \frac{|\sin y_2 t_2|}{t_2} dt_2 \\
&\leqslant \frac{4}{a\pi^2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{i} V_2\left(\phi_{x_1x_2}, \left[\frac{(i-1)a}{m+1}, \frac{ia}{m+1}\right] \times \left[\frac{jb}{n+1}, \frac{(j+1)b}{n+1}\right]\right) \\
&\qquad\qquad \times \frac{n+1}{jb} \left[\frac{(j+1)b}{n+1} - \frac{jb}{n+1}\right] \\
&= \frac{4}{a\pi^2} \sum_{i=0}^{m-1} \sum_{j=1}^{n} \frac{1}{(i+1)j} V_2\left(\phi_{x_1x_2}, \left[\frac{ia}{m+1}, \frac{(i+1)a}{m+1}\right] \times \left[\frac{jb}{n+1}, \frac{(j+1)b}{n+1}\right]\right). \qquad (43)
\end{aligned}
$$

Similarly, we can prove that

$$
\begin{aligned}
&|S_{53}(y_1, y_2, x_1, x_2)| \\
&\leqslant \frac{4}{b\pi^2} \sum_{i=1}^{m} \sum_{j=0}^{n-1} \frac{1}{i(j+1)} V_2\left(\phi_{x_1x_2}, \left[\frac{ia}{m+1}, \frac{(i+1)a}{m+1}\right] \times \left[\frac{jb}{n+1}, \frac{(j+1)b}{n+1}\right]\right). \qquad (44)
\end{aligned}
$$

Using (5) of Lemma 2, we get

$$
\begin{aligned}
&S_{54}(y_1,y_2,x_1,x_2)\\
&=\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n} g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)\int_{\frac{ia}{m+1}}^{\frac{(i+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\int_{\frac{jb}{n+1}}^{\frac{(j+1)b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\\
&=\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[g_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},\frac{jb}{n+1}\right)\right.\\
&\qquad\left.-g_{x_1x_2}\left(\frac{ia}{m+1},\frac{(j-1)b}{n+1}\right)+g_{x_1x_2}\left(\frac{(i-1)a}{m+1},\frac{(j-1)b}{n+1}\right)\right]\\
&\qquad\times\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\sum_{\nu=j}^{n}\int_{\frac{\nu b}{n+1}}^{\frac{(\nu+1)b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\\
&\qquad+\sum_{i=1}^{m}\left[g_{x_1x_2}\left(\frac{ia}{m+1},0\right)-g_{x_1x_2}\left(\frac{(i-1)a}{m+1},0\right)\right]\sum_{\mu=i}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\\
&\qquad\times\sum_{\nu=1}^{n}\int_{\frac{\nu b}{n+1}}^{\frac{(\nu+1)b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\\
&\qquad+\sum_{j=1}^{n}\left[g_{x_1x_2}\left(0,\frac{jb}{n+1}\right)-g_{x_1x_2}\left(0,\frac{(j-1)b}{n+1}\right)\right]\sum_{\mu=1}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\\
&\qquad\times\sum_{\nu=j}^{n}\int_{\frac{\nu b}{n+1}}^{\frac{(\nu+1)b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\\
&\qquad+g_{x_1x_2}\left(\frac{0a}{m+1},\frac{0b}{n+1}\right)\sum_{\mu=1}^{m}\int_{\frac{\mu a}{m+1}}^{\frac{(\mu+1)a}{m+1}}\frac{\sin y_1t_1}{t_1}dt_1\sum_{\nu=1}^{n}\int_{\frac{\nu b}{n+1}}^{\frac{(\nu+1)b}{n+1}}\frac{\sin y_2t_2}{t_2}dt_2\\
&=\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\phi_{x_1x_2}\left(\frac{ia}{m+1},\frac{jb}{n+1}\right)-\phi_{x_1x_2}\left(\frac{(i-1)a}{m+1},\frac{jb}{n+1}\right)\right.\\
&\qquad\left.-\phi_{x_1x_2}\left(\frac{ia}{m+1},\frac{(j-1)b}{n+1}\right)+\phi_{x_1x_2}\left(\frac{(i-1)a}{m+1},\frac{(j-1)b}{n+1}\right)\right]\\
&\qquad\times\int_{\frac{ia}{m+1}}^{a}\frac{\sin y_1t_1}{t_1}dt_1\int_{\frac{jb}{n+1}}^{b}\frac{\sin y_2t_2}{t_2}dt_2.
\end{aligned}
$$

Using Lemma 6 in above inequality, we get

$$
\begin{aligned}
&|S_{54}(y_1,y_2,x_1,x_2)|\\
&\leqslant\frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times\left[\frac{(j-1)b}{n+1},\frac{jb}{n+1}\right]\right)\frac{m+1}{ia}\frac{2}{y_1}\frac{n+1}{jb}\frac{2}{y_2}
\end{aligned}
$$

$$\leqslant \frac{16}{ab\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{ij}V_2\left(\phi_{x_1x_2},\left[\frac{(i-1)a}{m+1},\frac{ia}{m+1}\right]\times\left[\frac{(j-1)b}{n+1},\frac{jb}{n+1}\right]\right)$$
$$=\frac{16}{ab\pi^2}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{1}{(i+1)(j+1)}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right). \tag{45}$$

Using (39), (40), (43), (44), and (45), we have

$$\begin{aligned}
&|S_5(y_1,y_2,x_1,x_2)|\\
&\leqslant \frac{1}{\pi^2}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{ij}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&\quad+\frac{4}{a\pi^2}\sum_{i=0}^{m-1}\sum_{j=1}^{n}\frac{1}{(i+1)j}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&\quad+\frac{4}{b\pi^2}\sum_{i=1}^{m}\sum_{j=0}^{n-1}\frac{1}{i(j+1)}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&\quad+\frac{16}{ab\pi^2}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{1}{(i+1)(j+1)}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&\leqslant \frac{4}{\pi^2}\left[1+\frac{2}{a}+\frac{2}{b}+\frac{4}{ab}\right]\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{(i+1)(j+1)}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right).
\end{aligned}\tag{46}$$

Using (4) of Lemma 2, we get

$$\begin{aligned}
&\sum_{i=0}^{m}\sum_{j=0}^{n}\frac{1}{(i+1)(j+1)}V_2\left(\phi_{x_1x_2},\left[\frac{ia}{m+1},\frac{(i+1)a}{m+1}\right]\times\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&=\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&\quad+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left[\frac{1}{i+1}-\frac{1}{i+2}\right]\left[\frac{1}{j+1}-\frac{1}{j+2}\right]V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{(j+1)b}{n+1}\right]\right)\\
&\quad+\sum_{i=0}^{m-1}\frac{1}{n+1}\left[\frac{1}{i+1}-\frac{1}{i+2}\right]V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times[0,b]\right)\\
&\quad+\sum_{j=0}^{n-1}\frac{1}{m+1}\left[\frac{1}{j+1}-\frac{1}{j+2}\right]V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{(j+1)b}{n+1}\right]\right)\\
&=\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&\quad+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int_{i+1}^{i+2}\frac{1}{t_1^2}dt_1\int_{j+1}^{j+2}\frac{1}{t_2^2}dt_2\cdot V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times\left[0,\frac{(j+1)b}{n+1}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n+1}\sum_{i=0}^{m-1}\int_{i+1}^{i+2}\frac{1}{t_1^2}dt_1\cdot V_2\left(\phi_{x_1x_2},\left[0,\frac{(i+1)a}{m+1}\right]\times[0,b]\right)\\
&+\frac{1}{m+1}\sum_{j=0}^{n-1}\int_{j+1}^{j+2}\frac{1}{t_2^2}dt_2\cdot V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{(j+1)b}{n+1}\right]\right)\\
=&\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int_{i+1}^{i+2}\int_{j+1}^{j+2}\frac{1}{t_1^2}\frac{1}{t_2^2}V_2\left(\phi_{x_1x_2},\left[0,\frac{t_1a}{m+1}\right]\times\left[0,\frac{t_2b}{n+1}\right]\right)dt_1dt_2\\
&+\frac{1}{n+1}\sum_{i=0}^{m-1}\int_{i+1}^{i+2}\frac{1}{t_1^2}V_2\left(\phi_{x_1x_2},\left[0,\frac{t_1a}{m+1}\right]\times[0,b]\right)dt_1\\
&+\frac{1}{m+1}\sum_{j=0}^{n-1}\int_{j+1}^{j+2}\frac{1}{t_2^2}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{t_2b}{n+1}\right]\right)dt_2\\
=&\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&+\frac{1}{(m+1)(n+1)}\int_1^{m+1}\int_1^{n+1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t_1}\right]\times\left[0,\frac{b}{t_2}\right]\right)dt_1dt_2\\
&+\frac{1}{(m+1)(n+1)}\int_1^{m+1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t_1}\right]\times[0,b]\right)dt_1\\
&+\frac{1}{(m+1)(n+1)}\int_1^{n+1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{t_2}\right]\right)dt_2\\
=&\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&+\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\int_{i+1}^{i+2}\int_{j+1}^{j+2}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t_1}\right]\times\left[0,\frac{b}{t_2}\right]\right)dt_1dt_2\\
&+\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\int_{i+1}^{i+2}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{t_1}\right]\times[0,b]\right)dt_1\\
&+\frac{1}{(m+1)(n+1)}\sum_{j=0}^{n-1}\int_{j+1}^{j+2}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{t_2}\right]\right)dt_2\\
=&\frac{1}{(m+1)(n+1)}V_2(\phi_{x_1x_2},[0,a]\times[0,b])\\
&+\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right)\\
&+\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times[0,b]\right)
\end{aligned}
$$

$$+\frac{1}{(m+1)(n+1)}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},[0,a]\times\left[0,\frac{b}{j+1}\right]\right)$$
$$\leqslant\frac{3}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right). \tag{47}$$

Using (47) in (46), we get

$$|S_5(y_1,y_2,x_1,x_2)|\leqslant\frac{12}{\pi^2}\left[1+\frac{2}{a}+\frac{2}{b}+\frac{4}{ab}\right]$$
$$\times\frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right). \tag{48}$$

Further, we have

$$\begin{aligned}|S_6(y_1,y_2,x_1,x_2)|&=\left|\frac{1}{\pi^2}\int_a^\infty\int_{\frac{b}{n+1}}^b g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\leqslant\left|\frac{1}{\pi^2}\int_a^\infty\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\quad+\left|\frac{1}{\pi^2}\int_a^\infty\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\quad+\left|\frac{1}{\pi^2}\int_a^\infty\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(0,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&=S_{61}(y_1,y_2,x_1,x_2)+S_{62}(y_1,y_2,x_1,x_2)+S_{63}(y_1,y_2,x_1,x_2),\quad\text{say.}\end{aligned} \tag{49}$$

Now,

$$S_{61}(y_1,y_2,x_1,x_2)=\frac{1}{\pi^2}\lim_{c_1\to\infty}\left|\int_a^{c_1}\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|.$$

Since $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$, using Lemma 8, we get

$$S_{61}(y_1,y_2,x_1,x_2)\leqslant\frac{1}{\pi^2}\lim_{c_1\to\infty}\left\{\frac{4(n+1)}{aby_1y_2}\left[3M(\phi_{x_1x_2})+V_1(\phi_{x_1x_2}(\cdot,b),[a,c_1])\right.\right.$$
$$\left.\left.+V_1\left(\phi_{x_1x_2}(c_1,\cdot),\left[\frac{b}{n+1},b\right]\right)+V_2\left(\phi_{x_1x_2},[a,c_1]\times\left[\frac{b}{n+1},b\right]\right)\right]\right\}.$$

Using (18) and (19) of Lemma 11, we get

$$S_{61}(y_1,y_2,x_1,x_2)\leqslant\frac{1}{\pi^2}\lim_{c_1\to\infty}\left\{\frac{8}{aby_1}\left[3M(\phi_{x_1x_2})+V_1(\phi_{x_1x_2}(\cdot,0),[a,c_1])\right.\right.$$
$$\left.\left.+V_1\left(\phi_{x_1x_2}(a,\cdot),\left[\frac{b}{n+1},b\right]\right)+3V_2(\phi_{x_1x_2},[a,c_1]\times[0,b])\right]\right\}$$

$$
\begin{aligned}
&= \frac{8}{aby_1\pi^2}\Big[3M(\phi_{x_1x_2}) + V_1(\phi_{x_1x_2}(\cdot,0),[a,\infty)) \\
&\quad + V_1\left(\phi_{x_1x_2}(a,\cdot),\left[\frac{b}{n+1},b\right]\right) + 3V_2(\phi_{x_1x_2},[a,\infty)\times[0,b])\Big] \\
&= \frac{8}{aby_1\pi^2} V_2'(\phi_{x_1x_2},[a,\infty),[0,b]).
\end{aligned} \tag{50}
$$

Since $\phi_{x_1x_2}(\cdot,0) \in BV[0,\infty)$, using Lemma 7 and also using Lemma 6, we get

$$
\begin{aligned}
|S_{62}(y_1,y_2,x_1,x_2)| &= \left|\frac{1}{\pi^2}\int_a^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1 \int_{\frac{b}{n+1}}^b \frac{\sin y_2t_2}{t_2}dt_2\right| \\
&\leqslant \frac{1}{\pi^2}\frac{2}{ay_1}[M(\phi_{x_1x_2}(\cdot,0)) + V_1(\phi_{x_1x_2}(\cdot,0),[a,\infty))]\frac{2(n+1)}{y_2b} \\
&\leqslant \frac{8}{aby_1\pi^2} V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)).
\end{aligned} \tag{51}
$$

Again using Lemma 6, we get

$$
\begin{aligned}
|S_{63}(y_1,y_2,x_1,x_2)| &= \left|\frac{1}{\pi^2}\int_a^\infty \frac{\sin y_1t_1}{t_1}dt_1 \int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2\right| \\
&\leqslant \frac{2}{ay_1\pi^2}\left|\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2\right|.
\end{aligned}
$$

Estimate for the integral on the right hand side of the above inequality was obtained in [3], where the corresponding term is denoted by $S_2(x,y)$; after writing the exact value of the constant, we obtain

$$
\left|\int_{\frac{b}{n+1}}^b \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2\right| \leqslant \left[2+\frac{4}{b}\right]\sum_{j=0}^n \frac{1}{j+1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)
$$

and hence

$$
|S_{63}(y_1,y_2,x_1,x_2)| \leqslant \frac{2}{ay_1\pi^2}\left[2+\frac{4}{b}\right]\sum_{j=0}^n \frac{1}{j+1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right). \tag{52}
$$

Using (49)–(52), we get

$$
\begin{aligned}
S_6(y_1,y_2,x_1,x_2) &\leqslant \frac{8}{aby_1\pi^2}V_2'(\phi_{x_1x_2},[a,\infty),[0,b]) + \frac{8}{aby_1\pi^2}V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)) \\
&\quad + \frac{1}{\pi^2}\frac{2}{ay_1}\left[2+\frac{4}{b}\right]\sum_{j=0}^n \frac{1}{j+1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right).
\end{aligned} \tag{53}
$$

Using (3) of Lemma 1 in third term of the right hand side of (53), we get

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{1}{j+1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[\frac{jb}{n+1},\frac{(j+1)b}{n+1}\right]\right)\\
&=\sum_{j=0}^{n-1}\left[\frac{1}{j+1}-\frac{1}{j+2}\right]V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{(j+1)b}{n+1}\right]\right)+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&=\sum_{j=0}^{n-1}\int_{j+1}^{j+2}\frac{1}{t^2}dt\cdot V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{(j+1)b}{n+1}\right]\right)+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&\leqslant\sum_{j=0}^{n-1}\int_{j+1}^{j+2}\frac{1}{t^2}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{tb}{n+1}\right]\right)dt+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&=\int_1^{n+1}\frac{1}{n+1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{t}\right]\right)dt+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&=\frac{1}{n+1}\sum_{j=0}^{n-1}\int_{j+1}^{j+2}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{t}\right]\right)dt+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&\leqslant\frac{1}{n+1}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right)+\frac{1}{n+1}V_1(\phi_{x_1x_2}(0,\cdot),[0,b])\\
&\leqslant\frac{2}{n+1}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right).
\end{aligned}$$

Using above inequality in the third term of the right hand side of (53), we get

$$\begin{aligned}
S_6(y_1,y_2,x_1,x_2)\leqslant{}&\frac{8}{aby_1\pi^2}[V_2'(\phi_{x_1x_2},[a,\infty),[0,b])+V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty))]\\
&+\frac{1}{a\pi^2}\left[2+\frac{4}{b}\right]\frac{4}{y_1(n+1)}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right).
\end{aligned}\tag{54}$$

Analogusly, we have

$$\begin{aligned}
S_8(y_1,y_2,x_1,x_2)\leqslant{}&\frac{8}{aby_2\pi^2}\left[V_2'(\phi_{x_1x_2},[0,a],[b,\infty))+V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty))\right]\\
&+\frac{1}{b\pi^2}\left[2+\frac{4}{a}\right]\frac{4}{(m+1)y_2}\sum_{i=0}^{m-1}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{i+1}\right]\right).
\end{aligned}\tag{55}$$

Observe that

$$\begin{aligned}
|S_9(y_1,y_2,x_1,x_2)|&=\left|\frac{1}{\pi^2}\int_a^\infty\int_b^\infty g_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\leqslant\left|\frac{1}{\pi^2}\int_a^\infty\int_b^\infty \phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\
&\quad+\left|\frac{1}{\pi^2}\int_a^\infty\int_b^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|
\end{aligned}$$

$$+\left|\frac{1}{\pi^2}\int_a^\infty\int_b^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|$$
$$= S_{91}(y_1,y_2,x_1,x_2)+S_{92}(y_1,y_2,x_1,x_2)+S_{93}(y_1,y_2,x_1,x_2), \quad \text{say.} \tag{56}$$

Since $\phi_{x_1x_2}(\cdot,0)\in BV[0,\infty)$, using Lemma 7 and also using Lemma 6, we get

$$\begin{aligned}|S_{92}(y_1,y_2,x_1,x_2)| &= \frac{1}{\pi^2}\left|\int_a^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1\int_b^\infty\frac{\sin y_2t_2}{t_2}dt_2\right|\\ &\leqslant \frac{1}{\pi^2}\frac{2}{ay_1}[M(\phi_{x_1x_2}(\cdot,0))+V_1(\phi_{x_1x_2}(\cdot,0),[a,\infty))]\frac{2}{by_2}\\ &\leqslant \frac{4}{aby_1y_2\pi^2}V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty))\end{aligned} \tag{57}$$

and

$$\begin{aligned}S_{93}(y_1,y_2,x_1,x_2) &= \frac{1}{\pi^2}\left|\int_a^\infty\frac{\sin y_1t_1}{t_1}dt_1\int_b^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2\right|\\ &\leqslant \frac{1}{\pi^2}\frac{2}{by_2}[M(\phi_{x_1x_2}(0,\cdot))+V_1(\phi_{x_1x_2}(0,\cdot),[b,\infty))]\frac{2}{ay_1}\\ &= \frac{4}{aby_1y_2\pi^2}V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty)).\end{aligned} \tag{58}$$

Since $\phi_{x_1x_2}\in BV_H([0,\infty)\times[0,\infty))$, using Lemma 8, we get

$$\begin{aligned}S_{91}(y_1,y_2,x_1,x_2) &= \lim_{c_1\to\infty}\lim_{c_2\to\infty}\left|\frac{1}{\pi^2}\int_a^{c_1}\int_b^{c_2}\phi_{x_1x_2}(t_1,t_2)\frac{\sin y_1t_1}{t_1}\frac{\sin y_2t_2}{t_2}dt_1dt_2\right|\\ &\leqslant \frac{1}{\pi^2}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\left\{\frac{4}{aby_1y_2}[3M(\phi_{x_1x_2})+V_1(\phi_{x_1x_2}(\cdot,c_2),[a,c_1])\right.\\ &\quad\left.+V_1(\phi_{x_1x_2}(c_1,\cdot),[b,c_2])+V_2(\phi_{x_1x_2},[a,c_1]\times[b,c_2])]\right\}.\end{aligned}$$

Using (18) and (19) of Lemma 11, we get

$$\begin{aligned}S_{91}(y_1,y_2,x_1,x_2) &\leqslant \frac{1}{\pi^2}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\left\{\frac{4}{aby_1y_2}[3M(\phi_{x_1x_2})+V_1(\phi_{x_1x_2}(\cdot,b),[a,c_1])\right.\\ &\quad\left.+V_1(\phi_{x_1x_2}(a,\cdot),[b,c_2])+3V_2(\phi_{x_1x_2},[a,c_1]\times[b,c_2])]\right\}\\ &= \frac{4}{aby_1y_2\pi^2}[3M(\phi_{x_1x_2})+V_1(\phi_{x_1x_2}(\cdot,b),[a,\infty))\\ &\quad+V_1(\phi_{x_1x_2}(a,\cdot),[b,\infty))+3V_2(\phi_{x_1x_2},[a,\infty)\times[b,\infty))]\\ &= \frac{4}{aby_1y_2\pi^2}V_2'(\phi_{x_1x_2},[a,\infty),[b,\infty)).\end{aligned} \tag{59}$$

Using (56)–(59), we get

$$|S_9(y_1,y_2,x_1,x_2)| \leqslant \frac{4}{aby_1y_2\pi^2}\big[V_2'(\phi_{x_1x_2},[a,\infty),[b,\infty)) + V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)) + V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty))\big]. \tag{60}$$

After carefully reading the proof of Theorem 4, we can write the estimate for $S_{10}(y_1,x_1,x_2)$ as follows by explicitly determining the exact value of the constant and applying Lemma 7 to the last term.

$$\begin{aligned} S_{10}(y_1,x_1,x_2) &= \frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(t_1,0)\frac{\sin y_1t_1}{t_1}dt_1 \\ &\leqslant \frac{1}{\pi}\left[2+a+\frac{4}{a}\right]\frac{1}{m+1}\sum_{i=0}^{m-1}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{i+1}\right]\right) \\ &\quad + \frac{1}{ay_1\pi}V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)) \end{aligned} \tag{61}$$

and similarly, we have

$$\begin{aligned} S_{11}(y_2,x_1,x_2) &= \frac{1}{2\pi}\int_0^\infty \phi_{x_1x_2}(0,t_2)\frac{\sin y_2t_2}{t_2}dt_2 \\ &\leqslant \frac{1}{\pi}\left[2+b+\frac{4}{b}\right]\frac{1}{n+1}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right) \\ &\quad + \frac{1}{by_2\pi}V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty)). \end{aligned} \tag{62}$$

Combining all the inequality, we get

$$\begin{aligned} &|S(y_1,y_2,x_1,x_2) - f^*(x_1,x_2)| \\ &\leqslant \left\{\frac{4ab}{\pi^2} + \frac{8b}{\pi^2}\left[1+\frac{2}{a}\right] + \frac{8a}{\pi^2}\left[1+\frac{2}{b}\right] + \frac{12}{\pi^2}\left[1+\frac{2}{a}+\frac{2}{b}+\frac{4}{ab}\right]\right\} \\ &\quad \times \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V_2\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right) \\ &\quad + \left\{\frac{1}{\pi}\left[2+a+\frac{4}{a}\right] + \frac{4}{b\pi^2y_2}\left[2+\frac{4}{a}\right]\right\}\frac{1}{m+1}\sum_{i=0}^{m-1}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{i+1}\right]\right) \\ &\quad + \left\{\frac{1}{\pi}\left[2+b+\frac{4}{b}\right] + \frac{4}{a\pi^2y_1}\left[2+\frac{4}{b}\right]\right\}\frac{1}{n+1}\sum_{j=0}^{n-1}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right) \\ &\quad + \frac{4}{ab\pi^2}\big[V_2'(\phi_{x_1x_2},[a,\infty),[b,\infty)) + V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)) + V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty))\big]\frac{1}{y_1y_2} \\ &\quad + \left\{\frac{2b}{a\pi^2}V_1\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{n+1}\right]\right) + \left[\frac{2b}{a\pi^2}+\frac{8}{ab\pi^2}+\frac{1}{a\pi}\right]V_1'(\phi_{x_1x_2}(\cdot,0),[a,\infty)) \right. \\ &\quad \left. + \frac{2b}{a\pi^2}V_2'\left(\phi_{x_1x_2},[a,\infty),\left[0,\frac{b}{n+1}\right]\right) + \frac{8}{ab\pi^2}V_2'(\phi_{x_1x_2},[a,\infty),[0,b])\right\}\frac{1}{y_1} \end{aligned}$$

$$
\begin{aligned}
&+\left\{\frac{2a}{b\pi^2}V_1\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{m+1}\right]\right)+\left[\frac{2a}{b\pi^2}+\frac{8}{ab\pi^2}+\frac{1}{b\pi}\right]V_1'(\phi_{x_1x_2}(0,\cdot),[b,\infty))\right.\\
&\left.+\frac{2a}{b\pi^2}V_2'\left(\phi_{x_1x_2},\left[0,\frac{a}{m+1}\right],[b,\infty)\right)+\frac{8}{ab\pi^2}V_2'(\phi_{x_1x_2},[0,a],[b,\infty))\right\}\frac{1}{y_2}\\
=&\frac{O(1)}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}V\left(\phi_{x_1x_2},\left[0,\frac{a}{i+1}\right]\times\left[0,\frac{b}{j+1}\right]\right)\\
&+\frac{O(1)}{n+1}\sum_{j=0}^{n-1}V\left(\phi_{x_1x_2}(0,\cdot),\left[0,\frac{b}{j+1}\right]\right)+\frac{O(1)}{m+1}\sum_{i=0}^{m-1}V\left(\phi_{x_1x_2}(\cdot,0),\left[0,\frac{a}{i+1}\right]\right)\\
&+\frac{O(1)}{y_1y_2}+\frac{O(1)}{y_1}+\frac{O(1)}{y_2}. \quad \square
\end{aligned}
$$

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(Received March 20, 2026)

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