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ASYMPTOTIC EXPANSION, CONTINUED FRACTION APPROXIMATION AND INEQUALITIES FOR RAÏSSOULI–REZGUI MEAN

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Abstract. In 2016 Raïssouli and Rezgui introduced a new bivariate mean Z given by

$$\forall s,t>0, \quad Z(s,t)=\frac{2(s-t)}{\exp(1-t/s)-\exp(1-s/t)}, \quad \text{with} \quad Z(s,s)=s,$$

which should be called *the Raïssouli-Rezgui mean*. In this paper, we present asymptotic expansion and continued fraction approximation for the Raïssouli-Rezgui mean. We establish inequalities for the Raïssouli-Rezgui mean

$$C(s,t)<Z(s,t)<N(s,t)$$

for $s,t>0$ with $s\neq t$, where $C(s,t)=\frac{2(s^2+st+t^2)}{3(s+t)}$ is the centroidal mean, and $N(s,t)=\frac{s^2+t^2}{s+t}$ is the contraharmonic mean.

1. Introduction

A mean in two variables is any function $M:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ (where $\mathbb{R}_+$ denotes the set of positive real numbers), satisfying for all $s,t>0$ the following conditions:

(i) $M(s,t)=M(t,s)$ (symmetry property);

(ii) $\min(s,t)\leqslant M(s,t)\leqslant\max(s,t)$ (mean value property).

Sometimes, condition (ii) is replaced by the weaker requirement (e.g. [11]).

(iii) $M(s,s)=s$ (reflexivity property).

Clearly, (ii) implies (iii), but the converse is not always true.

For $s,t>0$ with $s\neq t$, Raïssouli and Rezgui [12, Proposition 3.6] (see also [13]) introduced a new bivariate mean Z given by

$$Z(s,t)=\frac{2(s-t)}{\exp(1-t/s)-\exp(1-s/t)},$$

which should be called *the Raïssouli-Rezgui mean*. The Raïssouli-Rezgui mean $Z(s,t)$ is a symmetric homogeneous mean satisfying

$$A(s,t)<Z(s,t), \tag{1}$$

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where $A(s,t)=(s+t)/2$ is the arithmetic mean.

Asymptotic expansion of a mean M has the following form [14]:

$$M(x+s,x+t)\sim x+\frac{s+t}{2}+\sum_{n=1}^{\infty}\frac{a_n(s,t)}{x^n},\qquad x\to\infty,$$

where $a_n(s,t)$ is a polynomial of order n. In [1–3, 5–10, 14], some asymptotic expansions of bivariate means were presented.

In this paper, we present asymptotic expansion and continued fraction approximation for the Raïssouli-Rezgui mean. Based on the obtained result, we establish inequalities for the Raïssouli-Rezgui mean.

The numerical values given have been calculated using the computer program MAPLE 11.

2. Lemmas

LEMMA 1. ([4]) *Let $a_1\neq 0$ and*

$$A(x)\sim\sum_{j=1}^{\infty}\frac{a_j}{x^j},\qquad x\to\infty \tag{2}$$

be a given asymptotic expansion. Define the function B by

$$A(x)=\frac{a_1}{B(x)}.$$

Then the function $B(x)=a_1/A(x)$ has asymptotic expansion of the following form

$$B(x)\sim x+\sum_{j=0}^{\infty}\frac{b_j}{x^j},\qquad x\to\infty,$$

where

$$b_0=-\frac{a_2}{a_1},\quad b_j=-\frac{1}{a_1}\left(a_{j+2}+\sum_{k=1}^{j}a_{k+1}b_{j-k}\right),\qquad j\geqslant 1. \tag{3}$$

Lemma 1 provides a method to converted (2) into a continued fraction of the form

$$A(x)\approx\cfrac{a_1}{x+b_0+\cfrac{b_1}{x+c_0+\cfrac{c_1}{x+d_0+\ddots}}},\qquad x\to\infty, \tag{4}$$

where the elements of the continued fraction are given by the following recurrence

relations:

$$\begin{cases} b_0 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum_{k=1}^{j} a_{k+1} b_{j-k}\right) \\ c_0 = -\frac{b_2}{b_1}, \quad c_j = -\frac{1}{b_1}\left(b_{j+2} + \sum_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\frac{c_2}{c_1}, \quad d_j = -\frac{1}{c_1}\left(c_{j+2} + \sum_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \qquad \cdots \quad \cdots \end{cases} \tag{5}$$

Clearly, since $a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \cdots$, the asymptotic expansion (2) is transformed into the continued-fraction approximation (4), and the constants in the right-hand side of (4) are determined by the system (5).

LEMMA 2. *For $0 < y < 1$, we have*

$$p(y) < e^{-\frac{y}{1-y}} < q(y), \tag{6}$$

where

$$p(y) = \frac{120 - 420y + 492y^2 - 193y^3}{120 - 300y + 252y^2 - 71y^3} \tag{7}$$

and

$$q(y) = \frac{7447440 - 33541080y + 56750340y^2 - 42758620y^3 + 12106583y^4}{7447440 - 26093640y + 34380420y^2 - 20183780y^3 + 4453763y^4}. \tag{8}$$

Proof. Define the functions $p_1(y)$ and $p_2(y)$ by

$$p_1(y) = 120 - 300y + 252y^2 - 71y^3$$

and

$$p_2(y) = 120 - 420y + 492y^2 - 193y^3.$$

We have $p_1(y) > 0$ for $0 < y < 1$. The function p_2 has a unique zero $y_0 = 0.822832\ldots$ and p_2 is positive on $(0, y_0)$ and negative on $(y_0, 1)$. Thus, we have $p(y) = \frac{p_2(y)}{p_1(y)} > 0$ for $0 < y < y_0$ and $p(y) < 0$ for $y_0 < y < 1$. Since $e^{-\frac{y}{1-y}}$ is positive, we see that the left-hand side of (6) holds for $y_0 < y < 1$. The claim remains true at $y = y_0$ because $p(y_0) = 0 < e^{-\frac{y_0}{1-y_0}} = 0.00961\ldots$ Hence, the left-hand side of (6) holds for $y_0 \leqslant y < 1$.

We now prove the left-hand side of (6) for $0 < y < y_0$. Define the functions $f(y)$ by

$$f(y) = \ln p(y) + \frac{y}{1-y}, \quad 0 < y < y_0.$$

Differentiation yields

$$f'(y) = -\frac{y^6}{(1-y)^2 p_1(y) p_2(y)} < 0 \quad \text{for} \quad 0 < y < y_0.$$

Hence, the function $f(y)$ is strictly decreasing for $0 < y < y_0$, and we have

$$f(y) = \ln p(y) + \frac{y}{1-y} < f(0) = 0 \quad \text{for} \quad 0 < y < y_0.$$

This means that the left-hand side of (6) is also valid for $0 < y < y_0$.

Hence, the left-hand side of (6) holds for all $0 < y < 1$.

The upper bound in (6) is obtained by considering the function $g(y)$ defined for $0 < y < 1$ by

$$g(y) = \ln q(y) + \frac{y}{1-y}.$$

Differentiation yields

$$g'(y) = \frac{y^7(16313440 + 3285149y)}{(1-y)^2 g_1(y) g_2(y)},$$

where

$$g_1(y) = 7447440 - 33541080y + 56750340y^2 - 42758620y^3 + 12106583y^4$$

and

$$g_2(y) = 7447440 - 26093640y + 34380420y^2 - 20183780y^3 + 4453763y^4.$$

Noting that $g_1(y) > 0$ and $g_2(y) > 0$ hold for $0 < y < 1$, we obtain $g'(y) > 0$ for $0 < y < 1$. Hence, the function $g(y)$ is strictly increasing for $0 < y < 1$, and we have

$$g(y) = \ln q(y) + \frac{y}{1-y} > g(0) = 0 \quad \text{for} \quad 0 < y < 1.$$

This means that the right-hand side of (6) holds for $0 < y < 1$. The proof is complete. □

LEMMA 3. *For $0 < y < 1$, we have*

$$P(y) < e^y < Q(y), \tag{9}$$

where

$$P(y) = \frac{1680 + 840y + 180y^2 + 20y^3 + y^4}{1680 - 840y + 180y^2 - 20y^3 + y^4}, \tag{10}$$

and

$$Q(y) = \frac{120 + 60y + 12y^2 + y^3}{120 - 60y + 12y^2 - y^3}. \tag{11}$$

Proof. It suffices to show that for $0 < y < 1$,

$$F_1(y) < 0 \quad \text{and} \quad F_2(y) > 0,$$

where

$$F_1(y) = \ln P(y) - y \quad \text{and} \quad F_2(y) = \ln Q(y) - y.$$

Differentiation yields

$$F_1'(y) = -\frac{y^8}{(1680 - 840y + 180y^2 - 20y^3 + y^4)(1680 + 840y + 180y^2 + 20y^3 + y^4)} < 0$$

and

$$F_2'(y) = \frac{y^6}{(120 - 60y + 12y^2 - y^3)(120 + 60y + 12y^2 + y^3)} > 0$$

for $0 < y < 1$. Hence, $F_1(y)$ is strictly decreasing for $0 < y < 1$, and $F_2(y)$ is strictly increasing for $0 < y < 1$, and we have, for $0 < y < 1$,

$$F_1(y) < F_1(0) = 0 \quad \text{and} \quad F_2(y) > F_2(0) = 0.$$

The proof is complete. □

By Lemmas 2 and 3, we obtain the following

COROLLARY 1. *For $0 < y < 1$, we have*

$$U(y) < e^y - e^{-\frac{y}{1-y}} < V(y), \tag{12}$$

where

$$V(y) = Q(y) - p(y) = \frac{24y(1200 - 3000y + 2540y^2 - 810y^3 + 92y^4 - 11y^5)}{(120 - 60y + 12y^2 - y^3)(120 - 300y + 252y^2 - 71y^3)}$$

and

$$U(y) = P(y) - q(y) = \frac{20yU_1(y)}{U_2(y)U_3(y)},$$

with

$$U_1(y) = 1251169920 - 4383731520y + 5805700320y^2 - 3547381680y^3 + 990211224y^4 - 132936276y^5 + 17689088y^6 - 382641y^7,$$

$$U_2(y) = 1680 - 840y + 180y^2 - 20y^3 + y^4$$

and

$$U_3(y) = 7447440 - 26093640y + 34380420y^2 - 20183780y^3 + 4453763y^4.$$

The proofs of Theorems 3 and 4 make use of inequalities (12).

3. Main results

THEOREM 1. *As $x \to \infty$, we have*

$$Z(x+s,x+t) \sim x + \sum_{j=0}^{\infty} \frac{a_j}{x^j}, \tag{13}$$

with the coefficients $a_j \equiv a_j(s,t)$ given by the recurrence relation

$$a_0 = \frac{s+t}{2}, \quad a_j = -\frac{1}{\lambda_1}\left(\lambda_{j+2} + \sum_{k=1}^{j} \lambda_{k+1} a_{j-k}\right), \qquad j \geqslant 1, \tag{14}$$

where

$$\lambda_k = \sum_{j=0}^{k} \frac{(s-t)^j(-s)^{k-j} - (t-s)^j(-t)^{k-j}}{j!}\binom{k-1}{k-j}, \qquad k \geqslant 1.$$

Proof. By direct calculation, we have

$$\begin{aligned}
\exp\left(1 - \frac{x+t}{x+s}\right) = \exp\left(\frac{s-t}{x+s}\right) &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{s-t}{x+s}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(s-t)^k}{k!x^k}\left(1+\frac{s}{x}\right)^{-k} \\
&= \sum_{k=0}^{\infty} \frac{(s-t)^k}{k!x^k} \sum_{j=0}^{\infty} \binom{-k}{j}\left(\frac{s}{x}\right)^j \\
&= \sum_{k=0}^{\infty} \frac{(s-t)^k}{k!x^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{s^j}{x^j} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(s-t)^j(-s)^{k-j}}{j!}\binom{k-1}{k-j}\frac{1}{x^k}
\end{aligned}$$

and

$$\exp\left(1 - \frac{x+s}{x+t}\right) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(t-s)^j(-t)^{k-j}}{j!}\binom{k-1}{k-j}\frac{1}{x^k}.$$

We then obtain

$$\begin{aligned}
&\exp\left(1 - \frac{x+t}{x+s}\right) - \exp\left(1 - \frac{x+s}{x+t}\right) = \sum_{k=1}^{\infty} \frac{\lambda_k}{x^k} \\
&\quad = \frac{2(s-t)}{x} - \frac{(s-t)(s+t)}{x^2} + \frac{(s-t)(t^2+4st+s^2)}{3x^3} - \frac{st(s-t)(s+t)}{x^4} + \cdots,
\end{aligned}$$

where

$$\lambda_k = \sum_{j=0}^{k} \frac{(s-t)^j(-s)^{k-j} - (t-s)^j(-t)^{k-j}}{j!} \binom{k-1}{k-j}, \qquad k \geqslant 1.$$

By Lemma 1, we have

$$Z(x+s,x+t) = \frac{2(s-t)}{\sum_{k=1}^{\infty} \frac{\lambda_k}{x^k}} \sim x + \sum_{j=0}^{\infty} \frac{a_j}{x^j},$$

where

$$a_0 = \frac{s+t}{2}, \quad a_j = -\frac{1}{\lambda_1}\left(\lambda_{j+2} + \sum_{k=1}^{j} \lambda_{k+1} a_{j-k}\right), \qquad j \geqslant 1.$$

The proof of Theorem 1 is complete. □

The coefficients a_j have a simpler form when they are presented in terms of variables α and β where

$$\alpha = \frac{t+s}{2} \quad \text{and} \quad \beta = \frac{t-s}{2}. \tag{15}$$

The first few coefficients are

$$\begin{aligned} &a_1 = \frac{\beta^2}{3}, \quad a_2 = -\frac{\alpha\beta^2}{3}, \quad a_3 = \frac{\beta^2(15\alpha^2+14\beta^2)}{45}, \quad a_4 = -\frac{\alpha\beta^2(5\alpha^2+14\beta^2)}{15}, \\ &a_5 = \frac{\beta^2(315\alpha^4+1764\alpha^2\beta^2+86\beta^4)}{945}, \quad a_6 = -\frac{\alpha\beta^2(63\alpha^4+588\beta^2\alpha^2+86\beta^4)}{189}. \end{aligned} \tag{16}$$

Based on the asymptotic expansion (13) and using the system (5), we derive continued fraction approximation for the Raïssouli-Rezgui mean given by Theorem 2.

THEOREM 2. *As $x \to \infty$, we have*

$$Z(x+s,x+t) \approx x + \alpha + \cfrac{a_1}{x+\alpha+\cfrac{b_1}{x+\alpha+\cfrac{c_1}{x+\alpha+\ddots}}}, \tag{17}$$

where

$$a_1 = \frac{\beta^2}{3}, \quad b_1 = -\frac{14\beta^2}{15}, \quad c_1 = \frac{157\beta^2}{245}, \ldots \tag{18}$$

Proof. Denote

$$A(x) = Z(x+s,x+t) - x - \alpha.$$

It follows from (13) that

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \qquad x \to \infty, \tag{19}$$

where the coefficients a_j are given in (14). Then, $A(x)$ has the continued fraction approximation of the form

$$A(x) \approx \cfrac{a_1}{x+b_0+\cfrac{b_1}{x+c_0+\cfrac{c_1}{x+d_0+\ddots}}}, \qquad x \to \infty, \tag{20}$$

where the constants in the right-hand side of (20) can be determined using the system (5). Noting that (16), we obtain from the first recurrence relation in (5) that

$$\begin{aligned}
b_0 &= -\frac{a_2}{a_1} = \alpha,\\
b_1 &= -\frac{a_3+a_2b_0}{a_1} = -\frac{14\beta^2}{15},\\
b_2 &= -\frac{a_4+a_2b_1+a_3b_0}{a_1} = \frac{14\alpha\beta^2}{15},\\
b_3 &= -\frac{a_5+a_2b_2+a_3b_1+a_4b_0}{a_1} = -\frac{2\beta^2(245\alpha^2-157\beta^2)}{525},\\
b_4 &= -\frac{a_6+a_2b_3+a_3b_2+a_4b_1+a_5b_0}{a_1} = \frac{2\alpha\beta^2(245\alpha^2-471\beta^2)}{525}, \quad \ldots
\end{aligned}$$

We obtain from the second recurrence relation in (5) that

$$\begin{aligned}
c_0 &= \alpha,\\
c_1 &= -\frac{b_3+b_2c_0}{b_1} = \frac{157\beta^2}{245},\\
c_2 &= -\frac{b_4+b_2c_1+b_3c_0}{b_1} = -\frac{157\alpha\beta^2}{245}, \quad \ldots
\end{aligned}$$

Continuing the above process, we find

$$d_0 = -\frac{c_2}{c_1} = \alpha, \quad \ldots$$

Formula (20) can be written as (17). The proof is complete. □

The continued fraction approximation in (17) motivated us to establish the following theorem.

THEOREM 3. *Let $s,t>0$ with $s \neq t$. Then for all $x \geqslant 0$,*

$$x+\alpha+\frac{\frac{\beta^2}{3}}{x+\alpha} < Z(x+s,x+t) < x+\alpha+\cfrac{\frac{\beta^2}{3}}{x+\alpha+\cfrac{-\frac{14\beta^2}{15}}{x+\alpha}}, \tag{21}$$

where α and β are given in (15).

REMARK 1. Replacing $x+s$ and $x+t$ by s and t respectively in (21), the double inequality (21) becomes

$$C(s,t) < Z(s,t) < \frac{3(s+t)(s^2+8st+t^2)}{s^2+58st+t^2} \tag{22}$$

for $s,t>0$ with $s \neq t$, where $C(s,t) = \frac{2(s^2+st+t^2)}{3(s+t)}$ is the centroidal mean. Obviously, the upper bound in (22) is a new bivariate mean. Inequalities (22) can be understood as (21) with $x=0$. The first inequality in (22) is sharper than inequality (1). In (22), two variables s and t are symmetric. Without loss of generality, assume $s<t$ and let $s/t=r$. Then $r \in (0,1)$, and (22) becomes

$$\frac{2(r^2+r+1)}{3(r+1)} < \frac{2(1-r)}{\exp(1-r)-\exp(1-\frac{1}{r})} < \frac{3(r+1)(r^2+8r+1)}{r^2+58r+1}, \quad 0<r<1. \tag{23}$$

Proof of Theorem 3. In order to prove Theorem 3, it suffices to show the double inequality (23). By an elementary change of variable $y=1-r$ $(0<r<1)$, (23) becomes

$$\frac{2(y^2-3y+3)}{3(2-y)} < \frac{2y}{e^y-e^{-\frac{y}{1-y}}} < \frac{3(2-y)(10-10y+y^2)}{60-60y+y^2}, \qquad 0<y<1 \tag{24}$$

In order to prove the left-hand side of (24), it suffices to show

$$\omega(y)>0, \qquad 0<y<1. \tag{25}$$

where

$$\omega(y) = \frac{3y(2-y)}{y^2-3y+3} - \left(e^y - e^{-\frac{y}{1-y}}\right).$$

We first prove (25) for $0<y<0.9$. Using the right-hand side of (12), we have, for $0<y<1$,

$$\omega(y) > \frac{3y(2-y)}{y^2-3y+3} - V(y) = \frac{3y^5 h_1(y)}{h_2(y)},$$

where

$$h_1(y) = 560-840y+246y^2+17y^3,$$
$$h_2(y) = (y^2-3y+3)(120-300y+252y^2-71y^3)(120-60y+12y^2-y^3).$$

Noting that $h_1(y)>0$ for $0<y<0.9$ and $h_2(y)>0$ for $0<y<1$, we have

$$\omega(y)>0 \quad \text{for} \quad 0<y<0.9.$$

Second, we prove (25) for $0.9 \leqslant y<1$. Let

$$\omega_1(y) = \frac{3y(2-y)}{y^2-3y+3} \quad \text{and} \quad \omega_2(y) = e^y - e^{-\frac{y}{1-y}}.$$

Then

$$\omega(y) = \omega_1(y) - \omega_2(y).$$

Differentiation yields

$$\omega_1'(y) = \frac{3(6-6y+y^2)}{(3-3y+y^2)^2} > 0 \quad \text{and} \quad \omega_2'(y) = e^y + \frac{e^{-\frac{y}{1-y}}}{(1-y)^2} > 0 \quad \text{for} \quad 0.9 \leqslant y < 1.$$

Hence, the functions $\omega_1(y)$ and $\omega_2(y)$ are both strictly increasing on $[0.9,1)$.

We divide the interval $[0.9,1)$ into two subintervals:

$$[0.9,1) = [0.9,0.95] \cup (0.95,1).$$

For $y \in [0.9,0.95]$, we get

$$\min_{0.9\leqslant y\leqslant 0.95} \omega_1(y) = \omega_1(0.9) = 2.675\ldots \quad \text{and} \quad \max_{0.9\leqslant y\leqslant 0.95} \omega_2(y) = \omega_2(0.95) = 2.585\ldots$$

Thus, (25) holds for $y \in [0.9,0.95]$. For $y \in [0.95,1)$, we get

$$\min_{0.95\leqslant y<1} \omega_1(y) = \omega_1(0.95) = 2.843\ldots \quad \text{and} \quad \sup_{0.95\leqslant y<1} \omega_2(y) = e = 2.718\ldots$$

Thus, (25) holds for $y \in [0.95,1)$.

Therefore, (25) is valid for all $0 < y < 1$.

We now prove the right-hand side of (24). Using the left-hand side of (12), we have, for $0 < y < 1$,

$$\begin{aligned} e^y - e^{-\frac{y}{1-y}} - \frac{2y(60-60y+y^2)}{3(2-y)(10-10y+y^2)} &> U(y) - \frac{2y(60-60y+y^2)}{3(2-y)(10-10y+y^2)} \\ &= \frac{2y^7 h_3(y)}{3(2-y)(10-10y+y^2)h_4(y)h_5(y)}, \end{aligned}$$

where

$$\begin{aligned} h_3(y) &= 2338496160 - 4685672520y + 2638035320y^2 - 291938580y^3 + 7025467y^4, \\ h_4(y) &= 1680 - 840y + 180y^2 - 20y^3 + y^4, \\ h_5(y) &= 7447440 - 26093640y + 34380420y^2 - 20183780y^3 + 4453763y^4. \end{aligned}$$

Noting that $h_j(y) > 0 \ \ (j = 3,4,5)$ for $0 < y < 1$, we have

$$e^y - e^{-\frac{y}{1-y}} - \frac{2y(60-60y+y^2)}{3(2-y)(10-10y+y^2)} > 0 \quad \text{for} \quad 0 < y < 1.$$

This means that the right-hand side of (24) holds. The proof of Theorem 3 is complete. □

REMARK 2. (21) $\Longleftrightarrow$ (22)$\Longleftrightarrow$(23) $\Longleftrightarrow$ (24). In (22), two variables s and t are symmetric. Without loss of generality, assume $s > t$ and let $s/t = r$. Then the inequality (23) holds for $r > 1$. Hence, the inequality (23) holds for all $0 < r \neq 1$.

THEOREM 4. *Let $s,t > 0$ with $s \neq t$. We have*

$$Z(s,t) < N(s,t), \tag{26}$$

where $N(s,t) = \frac{s^2+t^2}{s+t}$ is the contraharmonic mean.

Proof. In (26), two variables s and t are symmetric. Without loss of generality, assume $s < t$ and let $s/t = r$. Then $r \in (0,1)$, and (26) becomes

$$\frac{2(1-r)}{\exp(1-r) - \exp(1-\frac{1}{r})} < \frac{r^2+1}{r+1}, \qquad 0 < r < 1. \tag{27}$$

By an elementary change of variable $y = 1-r$ $(0 < r < 1)$, (27) can be written as

$$e^y - e^{-\frac{y}{1-y}} > \frac{2y(2-y)}{2-2y+y^2} \quad \text{for} \quad 0 < y < 1. \tag{28}$$

We first prove (28) for $0 < y < 0.9$. Using the left-hand side of (12), we have, for $0 < y < 1$,

$$\begin{aligned} e^y - e^{-\frac{y}{1-y}} - \frac{2y(2-y)}{2-2y+y^2} &> U(y) - \frac{2y(2-y)}{2-2y+y^2} \\ &= \frac{2y^3 h_6(y)}{(1680-840y+180y^2-20y^3+y^4)(2-2y+y^2)h_7(y)}, \end{aligned}$$

where

$$\begin{aligned} h_6(y) &= 4170566400 - 14612438400y + 18865768320y^2 - 10606387200y^3 \\ &\quad + 2347209440y^4 - 232545900y^5 + 66377134y^6 + 627353y^7, \\ h_7(y) &= 7447440 - 26093640y + 34380420y^2 - 20183780y^3 + 4453763y^4. \end{aligned}$$

Noting that $h_6(y) > 0$ for $0 < y < 0.9$ and $h_7(y) > 0$ for $0 < y < 1$, we have

$$e^y - e^{-\frac{y}{1-y}} - \frac{2y(2-y)}{2-2y+y^2} > 0 \quad \text{for} \quad 0 < y < 0.9.$$

Therefore, (28) holds for $0 < y < 0.9$.

Second, we prove (28) for $0.9 \leqslant y < 1$. Let

$$\omega_2(y) = e^y - e^{-\frac{y}{1-y}} \quad \text{and} \quad \omega_3(y) = \frac{2y(2-y)}{2-2y+y^2}.$$

Differentiation yields

$$\omega_3'(y) = \frac{8(1-y)}{(2-2y+y^2)^2} > 0 \quad \text{for} \quad 0.9 \leqslant y < 1.$$

We see that the functions $\omega_2(y)$ and $\omega_3(y)$ are both strictly increasing on $[0.9,1)$.

For $y \in [0.9,1)$, we get

$$\min_{0.9\leqslant y<1} \omega_2(y) = \omega_2(0.9) = 2.459\ldots \quad \text{and} \quad \sup_{0.9\leqslant y<1} \omega_3(y) = \omega_3(1) = 2.$$

Thus, (28) holds for $y \in [0.9,1)$.

Therefore, (28) is valid for all $0 < y < 1$. The proof is complete. □

REMARK 3. There is no strict comparison between the two upper bounds in (23) and (27).

REMARK 4. It follows from (22) and (26) that

$$C(s,t) < Z(s,t) < N(s,t) \tag{29}$$

for $s,t > 0$ with $s \neq t$, where $C(s,t)$ is the centroidal mean, and $N(s,t)$ is the contraharmonic mean.

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