Mathematical
Inequalities
& Applications
Volume 29, Number 4 (2026), 415–425
doi:10.7153/mia-2026-29-25

SOME OPERATOR INEQUALITIES INVOLVING SPECIAL CLASSES OF FUNCTIONS

HUIYING HAN, JUNJIAN YANG*, QI SONG AND JIQIN CHEN

(*Communicated by J.-C. Bourin*)

Abstract. In this note, we obtain several operator inequalities related to operator convex-log functions and geodesically convex functions, which complement some recent results due to Kaleibary et al. [Linear Multilinear Algebra **71** (2023) 1280–1294].

1. Introduction

Let $\mathbb{M}_n$ be the space of all $n\times n$ complex matrices. For $A\in\mathbb{M}_n$, the singular values and eigenvalues of A are denoted by $s_j(A)$ and $\lambda_j(A)$, respectively, $j=1,\dots,n$. The singular values $s_1(A),s_2(A),\dots,s_n(A)$ of a matrix A are the eigenvalues of $|A|=(A^*A)^{\frac{1}{2}}$ arranged in non-increasing order and repeated according to multiplicity. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$. Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\|=\|A\|$ for any $A\in\mathbb{M}_n$ and unitary matrices $U,V\in\mathbb{M}_n$. Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on an n-dimensional complex Hilbert space $\mathscr{H}$. Throughout the paper, a capital letter means an operator in $\mathscr{B}(\mathscr{H})$ or an $n\times n$ matrix. Let p,q be scalars and I be the identity operator (matrix). An operator A is called positive operator (in symbol: $A\geqslant 0$) if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$. We write $A>0$ if it is a strictly positive operator. For self-adjoint operators $A,B\in\mathscr{B}(\mathscr{H})$, we say $B\geqslant A$ or $A\leqslant B$ if $B-A\geqslant 0$ or $A-B\leqslant 0$, i.e., $\langle Bx,x\rangle\geqslant\langle Ax,x\rangle$ or $\langle Ax,x\rangle\leqslant\langle Bx,x\rangle$ for all $x\in\mathscr{H}$. For an arbitrary operator $A\in\mathscr{B}(\mathscr{H})$, we denote by $s_j(A)$ and $\lambda_j(A)$ its singular values and eigenvalues, respectively, $j=1,\dots,n$. The singular values $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)$ of A are the eigenvalues of $|A|=(A^*A)^{\frac{1}{2}}$ arranged in non-increasing order. For positive operators $A,B\in\mathscr{B}(\mathscr{H})$, $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$ are the eigenvalues of A arranged in non-increasing order. If

$$\sum_{j=1}^{k}\lambda_j(A)\leqslant\sum_{j=1}^{k}\lambda_j(B),\quad k=1,\dots,n,$$

Mathematics subject classification (2020): 47A63, 15A42, 15A60.
Keywords and phrases: Operator convex-log function, operator log-convex functions, geodesically convex functions.

* Corresponding author.

© ELEMENT, Zagreb
Paper MIA-29-25

we say that A is weakly majorized by B, denoted by $A \prec_w B$. If, in addition, the last inequality is an equality, i.e., $\sum_{j=1}^{n} \lambda_j(A) = \sum_{j=1}^{n} \lambda_j(B)$, we say that A is majorized by B, written as $A \prec B$. Moreover, if

$$\prod_{j=1}^{k} \lambda_j(A) \leqslant \prod_{j=1}^{k} \lambda_j(B), \quad k = 1, \ldots, n,$$

we say that A is weakly log majorized by B, denoted by $A \prec_{w\log} B$. If, in addition, the last inequality is an equality, i.e., $\prod_{j=1}^{n} \lambda_j(A) = \prod_{j=1}^{n} \lambda_j(B)$, we say that A is log majorized by B, written as $A \prec_{\log} B$. It is well known that for positive operators $A, B \in \mathscr{B}(\mathscr{H})$, $A \prec_{w\log} B$ implies $A \prec_w B$ [16, p. 39]. On the other hand, for positive operators $A, B \in \mathscr{B}(\mathscr{H})$, $A \preceq_{ols} B$ if and only if $A^r \leqslant B^r$ for every $r \geqslant 1$ [14]. For a real-valued function f and a self-adjoint operator $A \in \mathscr{B}(\mathscr{H})$, the value $f(A)$ is defined via the identity $f(A) = U\mathrm{diag}(f(\lambda_i(A)))U^*$, where $U\mathrm{diag}(\lambda_i(A))U^*$ is a spectral decomposition of A and the diagonal matrix $\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))$ is written as $\mathrm{diag}(\lambda_i(A))$, in which U is unitary and $\{\lambda_i(A) : i = 1, \ldots, n\} = \lambda(A)$. For two self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$, a function f is said to be operator monotone (see, e.g. [4, p. 112]) if it is monotone with respect to A and B, i.e., if $A \leqslant B$ implies $f(A) \leqslant f(B)$. Conversely, it is operator monotone decreasing. If strictly positive operators $A, B \in \mathscr{B}(\mathscr{H})$ and $\lambda \in (0,1)$ is a real number, then the following inequalities

$$\begin{aligned} A\nabla_\lambda B &:= (1-\lambda)A + \lambda B, \\ A!_\lambda B &:= ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1} \\ A\sharp_\lambda B &:= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}}, \end{aligned}$$

are known, in the literature, as the λ-weighted arithmetic, λ-weighted harmonic and λ-weighted geometric matrix means of A and B, respectively. If $\lambda = \frac{1}{2}$, they are simply denoted by $A\nabla B$, $A!B$ and $A\sharp B$, respectively. The following inequalities are well known in the literature:

$$A!_\lambda B \leqslant A\sharp_\lambda B \leqslant A\nabla_\lambda B.$$

Next we introduce the definitions of several special classes of functions [12].

DEFINITION 1. [12] Let f be a nonnegative continuous function on $(0,\infty)$.

(i) The function f is said to be operator convex if and only if

$$f((1-\alpha)A + \alpha B) \leqslant (1-\alpha)f(A) + \alpha f(B) \quad \text{for any } \alpha \in [0,1] \text{ and } A, B > 0\,.$$

(ii) The function f is said to be operator geodesically convex if and only if

$$f(A\sharp_\alpha B) \leqslant f(A)\nabla_\alpha f(B) \qquad \text{for any } \alpha \in [0,1] \text{ and } A, B > 0\,.$$

(*iii*) The function f is said to be operator log-convex when

$$f((1-\alpha)A+\alpha B) \leqslant f(A)\sharp_\alpha f(B) \qquad \text{for any } \alpha \in [0,1] \text{ and } A,B>0\ .$$

If the above inequalities are reversed, then the corresponding types of concave functions are defined.

DEFINITION 2. [12] The nonnegative continuous function $f:[1,\infty)\to[0,\infty)$ is operator convex-log if it can be written in the form $f(t)=h(\log t)$, $t\geqslant 1$, where $h:[0,\infty)\to[0,\infty)$ is an operator convex function. Similarly, a function $g:[1,\infty)\to[0,\infty)$ is called an operator concave-log function if it can be written as $g(t)=\varphi(\log t)$, $t\geqslant 1$, where $\varphi:[0,\infty)\to[0,\infty)$ is an operator concave function.

DEFINITION 3. [13] A nonnegative continuous function f on $(0,\infty)$ is said to be geometrically convex if and only if

$$f(x^{1-\alpha}y^{\alpha}) \leqslant f^{1-\alpha}(x)f^{\alpha}(y) \quad \text{for any } \alpha\in[0,1] \text{ and } x,y\in(0,\infty)\ .$$

Let f be a continuous nonnegative function on $(0,\infty)$. It is known that f is operator monotone if and only if it is operator concave. Ando and Hiai state [1, Theorem 3.1] that f is operator monotone decreasing is equivalent to f is operator convex and numerical function $f(x)$ is non-increasing. For simplicity, we use the notation

$$\mu=\{f:[0,\infty)\to[0,\infty); f \text{ is an operator convex function with } f(\infty)<\infty\}\,.$$

Can numerical function $f(x)\in\mu$ be non-increasing? The answer is yes and this is the content of the next proposition.

PROPOSITION 1. *Let f be a nonnegative operator monotone function on $[0,\infty)$ satisfying $\lim_{t\to\infty} f(t)=L$ with $L\in(0,\infty)$. Then f is non-increasing.*

Proof. We prove it by contradiction. Let $x,y\in[0,\infty)$ with $x<y$. Without loss of generality, we assume that $f(y)>f(x)$ and set

$$\lambda=\frac{f(y)-f(x)}{y-x}.$$

Obviously, $\lambda>0$. Thus by the convexity of f, we have

$$\frac{f(y)-f(x)}{y-x} \leqslant \frac{f(z)-f(y)}{z-y}$$

for $z>y$. This implies that

$$f(z)\geqslant f(y)+\lambda(z-y).$$

When $z \to \infty$, $f(z) = L < \infty$ is bounded, while $f(y) + \lambda(z-y)$ is infinity. This is a contradiction. Therefore, the assumption does not hold. This implies that $f(y) \leqslant f(x)$ for $0 \leqslant x < y$. So, f is non-increasing. $\square$

Consequently, $f \in \mu$ means that f is operator monotone decreasing. For more details on operator monotone and operator convex functions, the interested readers can refer to [1, 3, 4, 5, 8, 15].

Kaleibary et al. [12, Theorem 3.5] obtained the following weak majorization inequality: If $f : [1,\infty) \to [0,\infty)$ is an operator convex-log function, then, for every $A, B \geqslant I$ and $\alpha \in [0,1]$,

$$f(A\sharp_\alpha B) \prec_w f(A)\nabla_\alpha f(B). \tag{1}$$

In [12, Theorem 3.11] they also presented the following eigenvalue inequalities: If $g : [1,\infty) \to [0,\infty)$ is an operator concave-log function and A, B are strictly positive operators such that $e^s I \prec_{ols} e^s A \prec_{ols} B \prec_{ols} e^t A$ for some scalars $0 < s \leqslant t$, and $\alpha \in [0,1]$, then, for every $0 < r \leqslant 1$ and $k = 1,2,\ldots,n$,

$$\lambda_k(g(A)\nabla_\alpha g(B)) \leqslant S(e^{rt})^{\frac{1}{r}} S(e^t)\lambda_k(g(A\sharp_\alpha B)), \tag{2}$$

where $S(t) = \dfrac{t^{\frac{1}{t-1}}}{e\log(t^{\frac{1}{t-1}})}$ for $t > 0$, $t \neq 1$ is the so-called Specht's ratio. It is easy to known that $\lim\limits_{t\to 1} S(t) = 1$, $S(t) = S(1/t) > 1$.

At the same time, they gave a variation of the properties of geodesically convex functions in [12, Theorem 4.4]: Let g be a nonnegative operator monotone decreasing function on $(0,\infty)$ and $0 < sA \leqslant B \leqslant tA$ for some constants $0 < s \leqslant t$. Then, for all $\alpha \in [0,1]$ and $k = 1,2,\ldots,n$,

$$\prod_{j=1}^{k} \lambda_j(g(A\sharp_\alpha B)) \leqslant \mu^k(s,t)\left(\left\{\prod_{j=1}^{k}\lambda_j(g(A))\right\}\sharp_\alpha\left\{\prod_{j=1}^{k}\lambda_j(g(B))\right\}\right). \tag{3}$$

Actually, Kaleibary et al. [12, Lemma 4.2] showed that if g is an increasing geometrically convex function on $(0,\infty)$, then

$$F(A) = \sum_{j=1}^{k} g(\lambda_j(A)), \quad k = 1,2,\ldots,n, \tag{4}$$

is a geodesically convex function.

In this paper, we first give several variations of the operator geodesically convexity and concavity properties which can be considered as complements of (1)–(3). And then we prove (4) holds in case that g is an increasing log-convex function.

2. Main results

Before presenting main results, we list some useful lemmas for our proofs.

LEMMA 1. [10, Lemma 1] *Let* $0 < pA \leqslant B \leqslant qA$, $0 < p \leqslant q$ *and* $\alpha \in [0,1]$. *Then*

$$(1-\alpha)A + \alpha B \leqslant \max\{S(p), S(q)\}(A \sharp_\alpha B),$$

where $S(t)$ *is the so-called Specht ratio.*

LEMMA 2. [6, Proposition 3.5] *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *be strictly positive operators. Then, for* $\alpha \in [0,1]$ *and* $k = 1, \ldots, n$,

$$\left\{\prod_{j=1}^{k} s_j(A)\right\} \sharp_\alpha \left\{\prod_{j=n+1-k}^{n} s_j(B)\right\} \leqslant \prod_{j=1}^{k} s_j(A \sharp_\alpha B) \leqslant \left\{\prod_{j=1}^{k} s_j(A)\right\} \sharp_\alpha \left\{\prod_{j=1}^{k} s_j(B)\right\}.$$

LEMMA 3. [4, Theorem III.4.1] *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *be self-adjoint operators. Then, for* $1 \leqslant k \leqslant n$,

$$\sum_{j=1}^{k} \lambda_j(A+B) \leqslant \sum_{j=1}^{k} \lambda_j(A) + \sum_{j=1}^{k} \lambda_j(B),$$

and the two sums are equal at $k = n$.

LEMMA 4. [12, Proposition 3.10] *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *be strictly positive operators such that* $e^s A \prec_{ols} B \prec_{ols} e^t A$ *for some scalars* $0 < s \leqslant t$, *and* $\alpha \in [0,1]$. *Then*

$$\lambda_k(\log A \nabla_\alpha \log B) \leqslant \lambda_k(\log(MN(A \sharp_\alpha B))),$$

where $S(t)$ *is the so-called Specht ratio and*

$$M = \max\{S^{\frac{1}{r}}(e^{rs}), S^{\frac{1}{r}}(e^{rt})\}, \quad N = \max\{S(e^s), S(e^t)\}$$

and $0 < r \leqslant 1$.

LEMMA 5. [2, Proposition 1.1] *Let* $f(0,\infty) \to [0,\infty)$ *be continuous. Then*

(1) f *is operator monotone decreasing if and only if* f *is operator convex and* $f(\infty) < \infty$.

(2) f *is operator monotone increasing if and only if* f *is operator concave.*

LEMMA 6. [7, Lemma 2.2] *Let* $\nu \geqslant 1$.

(*a*) *If* $f : [0,\infty) \to [0,\infty)$ *is an operator monotone increasing function, then*

$$f(\nu t) \leqslant \nu f(t).$$

(*b*) *If* $f:[0,\infty)\to[0,\infty)$ *is an operator monotone decreasing function, then*

$$f(\nu t)\geqslant\frac{1}{\nu}f(t).$$

Firstly, we give a higher level inequality than (1) under the assumption of $0<pA\leqslant B\leqslant qA$ which is a variation of the operator geodesically convexity property.

THEOREM 1. *Let* $f(t)=h(\log t)$ *be an operator convex-log function on* $[1,\infty)$ *satisfying* $h\in\mu$ *and let* $0<pA\leqslant B\leqslant qA$ *for some scalars* $0<p\leqslant q$. *Then, for every* $A,B>I$ *and* $\alpha\in[0,1]$,

$$f(A\sharp_\alpha B)\leqslant f(s^{-1}A)\nabla_\alpha f(s^{-1}B),$$

where $S(t)$ *is the so-called Specht ratio and* $s=\max\{S(p),S(q)\}$.

Proof. Since $A,B>I$, then $A\sharp_\alpha B>I$. Compute

$$\begin{aligned}
f(A\sharp_\alpha B)&=h(\log(A\sharp_\alpha B))\\
&\leqslant h(\log(s^{-1}(A\nabla_\alpha B)))\qquad\text{(Lemmas 1 and 5)}\\
&\leqslant h((\log(s^{-1}A))\nabla_\alpha(\log(s^{-1}B)))\qquad\text{(operator concavity of log)}\\
&\leqslant h(\log(s^{-1}A))\nabla_\alpha h(\log(s^{-1}B))\qquad\text{(operator convexity of }h\text{)}\\
&=f(s^{-1}A)\nabla_\alpha f(s^{-1}B),
\end{aligned}$$

we get the desired result. □

Secondly, we give a reversed inequality of (2) under the assumption of operator convex-log functions which can also be considered as a variation of the operator geodesically convexity property.

THEOREM 2. *Let* $f(t)=h(\log t)$ *be an operator convex-log function on* $[1,\infty)$ *satisfying* $h\in\mu$ *and let* A *and* B *be strictly positive operators such that* $e^sI\prec_{ols}e^sA\prec_{ols}B\prec_{ols}e^tA$ *for some scalars* $0<s\leqslant t$, *and* $\alpha\in[0,1]$. *Then, for every* $0<r\leqslant1$ *and* $k=1,2,\ldots,n$, *we have*

$$\lambda_k(f(A\sharp_\alpha B))\leqslant S(e^{rt})^{\frac{1}{r}}S(e^t)\lambda_k(f(A)\nabla_\alpha f(B)).$$

Proof. Since $h\in\mu$, it follows from Lemma 5 that h is an operator monotone decreasing function. Moreover, the condition $e^sI\prec_{ols}e^sA\prec_{ols}B\prec_{ols}e^tA$ with $0<s\leqslant t$

implies $A,B>I$. So,

$$\begin{aligned}
\lambda_k(f(A)\nabla_\alpha f(B)) &= \lambda_k(h(\log(A))\nabla_\alpha h(\log(B))) \\
&\geqslant \lambda_k(h(\log(A)\nabla_\alpha \log(B))) \qquad \text{(op. convexity of } h\text{)} \\
&= h(\lambda_k(\log(A)\nabla_\alpha \log(B))) \\
&\geqslant h(\lambda_k(\log MN(A\sharp_\alpha B))) \quad \text{(Lemmas 4 and 5)} \\
&= \lambda_k(h(\log MN(A\sharp_\alpha B))) \\
&\geqslant \lambda_k(h(MN\log(A\sharp_\alpha B))) \quad \text{(Lemmas 5 and 6)} \\
&\geqslant \frac{1}{MN}\lambda_k(h(\log(A\sharp_\alpha B))) \quad \text{(Lemmas 5 and 6)} \\
&= \frac{1}{MN}\lambda_k(f(A\sharp_\alpha B)),
\end{aligned}$$

where constants $M=\max\{S^{\frac{1}{r}}(e^{rs}),S^{\frac{1}{r}}(e^{rt})\}$, $N=\max\{S(e^s),S(e^t)\}$ are defined in Lemma 4 and $M\geqslant 1$, $N\geqslant 1$ by the definition of so-called Specht ratio and $0<r\leqslant 1$, $0<s\leqslant t$.

On the other hand, since $S(t)$ is an increasing function on $[1,\infty]$ and $1<e^s\leqslant e^t$ for $0<s\leqslant t$, then $MN=S(e^{rt})^{\frac{1}{r}}S(e^t)$. Therefore,

$$\lambda_k(f(A\sharp_\alpha B))\leqslant S(e^{rt})^{\frac{1}{r}}S(e^t)\lambda_k(f(A)\nabla_\alpha f(B)). \quad \square$$

Next, a variation of the operator geodesically concavity property is obtained in the next theorem.

THEOREM 3. *Let f be a nonnegative operator monotone function on $(0,\infty)$ and $0<pA\leqslant B\leqslant qA$ for some constants $0<p\leqslant q$. Then, for all $\alpha\in[0,1]$ and $k=1,2,\ldots,n$,*

$$\left(\sum_{j=n-k+1}^{n}\lambda_j(f(A))\right)\nabla_\alpha\left(\sum_{j=n-k+1}^{n}\lambda_j(f(B))\right)\leqslant s\sum_{j=n-k+1}^{n}\lambda_j(f(A\sharp_\alpha B)),$$

where $s=\max\{S(p),S(q)\}$.

Proof. Since f is a nonnegative operator monotone function on $(0,\infty)$ (i.e., f is operator concave), then combining Lemma 1 and the fact that for every nonnegative concave function f and every $s>1$, $f(sx)\leqslant sf(x)$ yields

$$f(A\nabla_\alpha B)\leqslant f(s(A\sharp_\alpha B))\leqslant sf(A\sharp_\alpha B). \tag{5}$$

By the concavity of f, we get

$$f(A)\nabla_\alpha f(B)\leqslant f(A\nabla_\alpha B). \tag{6}$$

Actually, Lemma 3 implies that

$$\sum_{j=n-k+1}^{n}\lambda_j(A)+\sum_{j=n-k+1}^{n}\lambda_j(B)\leqslant\sum_{j=n-k+1}^{n}\lambda_j(A+B),\quad k=1,\ldots,n.$$

Assume that $X=(1-\alpha)f(A)$, $Y=\alpha f(B)$, where $\alpha\in[0,1]$. By replacing A, B with X, Y in the above inequality, respectively, we have

$$\begin{aligned}\sum_{j=n-k+1}^{n}\lambda_j(X)+\sum_{j=n-k+1}^{n}\lambda_j(Y)&=\sum_{j=n-k+1}^{n}\lambda_j((1-\alpha)f(A))+\sum_{j=n-k+1}^{n}\lambda_j(\alpha f(B))\\&=(1-\alpha)\sum_{j=n-k+1}^{n}\lambda_j(f(A))+\alpha\sum_{j=n-k+1}^{n}\lambda_j(f(B))\\&\leqslant\sum_{j=n-k+1}^{n}\lambda_j((1-\alpha)f(A)+\alpha f(B)),\quad k=1,\ldots,n.\end{aligned}$$

This means that

$$\begin{aligned}&\left(\sum_{j=n-k+1}^{n}\lambda_j(f(A))\right)\nabla_\alpha\left(\sum_{j=n-k+1}^{n}\lambda_j(f(B))\right)\\&\leqslant\sum_{j=n-k+1}^{n}\lambda_j\left(f(A)\nabla_\alpha f(B)\right),\ \ k=1,\ldots,n.\end{aligned}$$

Using inequalities (5) and (6), the above inequality becomes

$$\begin{aligned}&\left(\sum_{j=n-k+1}^{n}\lambda_j(f(A))\right)\nabla_\alpha\left(\sum_{j=n-k+1}^{n}\lambda_j(f(B))\right)\\&\leqslant\sum_{j=n-k+1}^{n}\lambda_j(f(A)\nabla_\alpha f(B))\\&\leqslant s\sum_{j=n-k+1}^{n}\lambda_j(f(A\sharp_\alpha B)),\ \ k=1,\ldots,n.\quad\square\end{aligned}$$

REMARK 1. Suppose $F(A)=\sum_{j=n-k+1}^{n}\lambda_j(f(A))$ in Theorem 3. So,

$$F(A)\nabla_\alpha F(B)\leqslant sF(A\sharp_\alpha B),$$

which provides a variant of the geodesically concavity property for $F(A)=\sum_{j=n-k+1}^{n}\lambda_j(f(A))$.

When $k=n$ in Theorem 3, the next corollary is immediate.

COROLLARY 1. *Let f be a nonnegative operator monotone function on $(0,\infty)$ and $0 < pA \leqslant B \leqslant qA$ for some constants $0 < p \leqslant q$. Then, for all $\alpha \in [0,1]$,*

$$\operatorname{tr}(f(A)\nabla_\alpha f(B)) \leqslant s\operatorname{tr}(f(A\sharp_\alpha B)),$$

where $s = \max\{S(p), S(q)\}$.

Finally, we prove that (4) is also a geodesically convex function under a different condition.

THEOREM 4. *Let f be an increasing log-convex function on $(0,\infty)$. Then*

$$F(A) = \sum_{j=1}^{k} f(\lambda_j(A)), \quad k = 1,2,\ldots,n,$$

is a geodesically convex function.

Proof. For strictly positive operators A, B and $\alpha \in [0,1]$, by Lemma 2, we have

$$\begin{aligned}\prod_{j=1}^{k} \lambda_j(A\sharp_\alpha B) &\leqslant \left\{\prod_{j=1}^{k} \lambda_j(A)\right\} \sharp_\alpha \left\{\prod_{j=1}^{k} \lambda_j(B)\right\} \\ &\leqslant \prod_{j=1}^{k} \lambda_j(A)^{1-\alpha}\lambda_j(B)^{\alpha} \qquad (7) \\ &= \prod_{j=1}^{k} \lambda_j(A^{1-\alpha})\lambda_j(B^{\alpha}).\end{aligned}$$

Now, by applying a classical result on the function f [11, Proposition 4.1.6] and inequality (7), we have

$$\sum_{j=1}^{k} f(\lambda_j(A\sharp_\alpha B)) \leqslant \sum_{j=1}^{k} f(\lambda_j^{1-\alpha}(A)\lambda_j^{\alpha}(B)).$$

Hence, we can write

$$\begin{aligned}\sum_{j=1}^{k} f(\lambda_j(A\sharp_\alpha B)) &\leqslant \sum_{j=1}^{k} f(\lambda_j^{1-\alpha}(A)\lambda_j^{\alpha}(B)) \\ &\leqslant \sum_{j=1}^{k} f((1-\alpha)\lambda_j(A) + \alpha\lambda_j(B)) \qquad \text{(Young inequality)} \\ &\leqslant \sum_{j=1}^{k} f^{1-\alpha}(\lambda_j(A))f^{\alpha}(\lambda_j(B)) \qquad \text{(log-convexity of } f\text{)} \\ &\leqslant \left(\sum_{j=1}^{k} f(\lambda_j(A))\right) \sharp_\alpha \left(\sum_{j=1}^{k} f(\lambda_j(B))\right) \qquad \text{(Hölder inequality)} \\ &\leqslant \left(\sum_{j=1}^{k} f(\lambda_j(A))\right) \nabla_\alpha \left(\sum_{j=1}^{k} f(\lambda_j(B))\right).\end{aligned}$$

So, $F(A)$ is a geodesically convex function. □

REMARK 2. In general, log-convexity and geometric convexity are not related to each other. However, if f is a positive increasing log-convex function on $(0,\infty)$, then it is a geometrically convex function. Indeed, it follows $f(x^{1-\alpha}y^{\alpha}) \leqslant f((1-\alpha)x+\alpha y) \leqslant f^{1-\alpha}(x)f^{\alpha}(y)$ from the monotone increasingness of f and the weighted arithmetic-geometric mean inequality. In addition, a geometrically convex function is not always a log-convex function. For example, $\sinh t$ on $[0,\infty)$ is geometrically convex [13]. However, the calculation $\frac{\mathrm{d}^2(\sinh t)}{\mathrm{d}t^2} = -\frac{1}{\sinh^2 t} < 0$ shows that $\sinh t$ is not log-convex.

Acknowledgement. The work is supported by National Natural Science Foundation of China (grant no. 12661080, 12261030), Hainan Provincial Natural Science Foundation (grant no. 125MS075), Hainan Provincial Natural Science Foundation for High-level Talents (grant no. 124RC503, 123RC474), Hainan Province Flexible Talent Introduction Collaborative Innovation Center (Yu Changbin), the specific research fund of The Innovation Platform for Academicians of Hainan Province (no. YSPTZX202407), the Key Laboratory of Computational Science and Application of Hainan Province and Hainan Provincial Graduate Innovation Research Program (grant no. Hys2025-411).

REFERENCES

[1] T. ANDO, F. HIAI, *Operator log-convex functions and operator means*, Math. Ann. **350** (2011) 611–630.

[2] Y. BEDRANI, F. KITTANEH, M. SABABHEH, *From positive to accretive matrices*, Positivity **25** (2021) 1601–1629.

[3] J. BENDAT, S. SHERMANN, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1965) 58–71.

[4] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[5] R. BHATIA, T. SANO, *Loewner matrices and operator convexity*, Math. Ann. **344** (2009) 703–716.

[6] J. C. BOURIN, F. HIAI, *Jensen and Minkowski inequalities for operator means and anti-norms*, Linear Algebra Appl. **456** (2014) 22–53.

[7] T. H. DINH, H. R. MORADI, M. SABABHEH, *On the Pólya-Szegö operator inequality*, Hacet. J. Math. Stat. **49** (2020) 1744–1752.

[8] W. F. DONOGHUE, *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin, 1974.

[9] T. FURUTA, J. MIĆIĆ, J. E. PEČARIĆ, Y. SEO, *Mond-Pečarić method in operator inequalities*, Zagreb: Element, 2005. (Monographs in Inequalities; 1).

[10] M. B. GHAEMI, V. KALEIBARY, *Some inequalities involving operator monotone functions and operator means*, Math. Inequal. Appl. **19** (2016) 757–764.

[11] F. HIAI, *Matrix analysis: matrix monotone functions, matrix means, and majorization*, Interdiscip. Inf. Sci. **16** (2010) 139–248.

[12] V. KALEIBARY, M. R. JABBARZADEH, S. FURUICHI, *Operator geodesically convex functions and their applications*, Linear Multilinear Algebra. **71** (2023) 1280–1294.

[13] C. P. NICULESCU, *Convexity according to the geometric mean*, Math. Inequal. Appl. **3** (2000) 155–167.

[14] M. P. OLSPN, *The selfadjoint operators of a von Neumann algebra from a conditionally complete lattice*, Proc. Amer. Math. Soc. **28** (1971) 537–544.

[15] M. UCHIYAMA, *Operator monotone functions, positive definite kernels and majorization*, Proc. Amer. Math. Soc. **138** (2010) 3985–3996.
[16] X. ZHAN, *Matrix Theory*, Beijing: Higher Education Press, 2008, (in Chinese).

(Received September 2, 2025)

Huiying Han
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
e-mail: hanhuiying2026@163.com

Junjian Yang
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
and
Key Laboratory of Data Science and Intelligence Education
Hainan Normal University
Ministry of Education
e-mail: junjianyang1981@163.com

Qi Song
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
e-mail: qisong2001@163.com

Jiqin Chen
School of Mathematics and Statistics
Hainan Normal University
Haikou, China
e-mail: jiqinchen2000@163.com