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BOUNDARY VALUES OF SEQUENCES AT INFINITY IN DIFFERENCE SPACES OF THE SOBOLEV TYPE

AIGERIM KALYBAY, DINARA SARSENALY AND AINUR TEMIRKHANOVA

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Abstract. Weighted difference Sobolev space $W^n_{p,v}$ is determined. The mutual relation of this space with the space $\overset{\circ}{W}{}^n{}_{p,v}$ is investigated depending on the behavior of the weight sequences at infinity.

1. Introduction

When considering boundary value problems for differential equations with degeneracy at the boundary, the question arises of describing in terms of boundary values the completion of a set of infinitely differentiable finite functions with respect to the norm of the corresponding weight space. In general, the set of finitely supported functions is not dense in the unweighted space of smooth functions. However,for example, in the case of strong degeneration of the weight in a weighted Sobolev space, the set of finitely supported smooth functions may turn out to be dense. Therefore, an important question is the problem of characterizing the closure of finitely supported smooth functions in the space under consideration (see, for example, [2], [4], [6], [7], [8], [9]). Studies of the relationship of a weighted space with a set of finitely supported functions play an important role in determining the degree of degeneracy of coefficients of differential equations at the boundary of the domain, the degree of singularity of weight functions in weight inequalities, the influence of the degree of singularity of coefficients in the spectral theory of differential operators. In this direction, several works have been published in recent years [1], [3], [5], [10]. As for the discrete case of this problem, it has been little studied. This paper is devoted to the study of the relationship between the weighted difference Sobolev space and the closure of a set of finite sequences depending on the behavior of the weight sequences at infinity.

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$. Let $\{f_i\}_{i=0}^{\infty}$ be a sequence of real numbers, let us define the difference operation:

$$\Delta f_i = f_{i+1} - f_i, \quad i \geqslant 0, \qquad \Delta^n f_i = \Delta\left(\Delta^{n-1} f_i\right), \quad n = 1,2,\ldots.$$

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Let $E_\circ$ be the set of sequences of real numbers $\{f_i\}_{i=0}^{\infty}$ for which $f_k = 0,\ k = 0,1,\ldots,n-1$, which is equivalent to the condition $\Delta^i f_0 = 0,\ i = 0,1,\ldots,n-1$. Let us define a weighted difference space on this set $E_\circ$.

The sequence of real numbers $\{\upsilon_i\}_{i=0}^{\infty}$ is called weighted sequence, if $\upsilon_i > 0$, $i \geqslant 0$. Let $n \geqslant 1$, $W_{p,\upsilon}^n$ be the set of all sequences from $E_\circ$ satisfying the condition

$$\|\Delta^n f\|_{p,\upsilon} = \left(\sum_{i=0}^{\infty} \upsilon_i |\Delta^n f_i|^p\right)^{\frac{1}{p}} < \infty,$$

and we define the norm of this set as follows:

$$\|f\|_{W_{p,\upsilon}^n} = \|\Delta^n f\|_{p,\upsilon}, \forall f \in W_{p,\upsilon}^n. \tag{1}$$

The fulfillment of conditions $\|\lambda f\|_{W_{p,\upsilon}^n} = |\lambda| \|f\|_{W_{p,\upsilon}^n}$ and $\|f_1 + f_2\|_{W_{p,\upsilon}^n} \leqslant \|f_1\|_{W_{p,\upsilon}^n} + \|f_2\|_{W_{p,\upsilon}^n}$ of the norm is obvious. Let $f \in W_{p,\upsilon}^n$ and $\|f\|_{W_{p,\upsilon}^n} = 0$. Then from $\upsilon_i > 0$, $i \geqslant 0$ it follows that $\Delta^n f_k = 0, \forall k \geqslant 0$, which with $\Delta^i f_0 = 0,\ i = 0,1,\ldots,n-1$ gives $f_k = 0,\ \forall k \geqslant 0$. Then the pair $\left(W_{p,\upsilon}^n, \|f\|_{W_{p,\upsilon}^n}\right)$ forms a normed space and this space $W_{p,\upsilon}^n$ is called weighted difference Sobolev space. By standard method it can be proved that the space $W_{p,\upsilon}^n$ is complete.

We denote by $\overset{\circ}{E}$ the set of sequences from the set $E_\circ$, which values after the finite number are zero. From these definitions it follows that $\overset{\circ}{E} \subset W_{p,\upsilon}^n$.

By $\overset{\circ}{W}{}^n{}_{p,\upsilon}$ we denote the closure of the set $\overset{\circ}{E}$ by the norm (1) of the space $W_{p,\upsilon}^n$.

The main purpose of this paper is to study the relationship between spaces $W_{p,\upsilon}^n$ and $\overset{\circ}{W}{}^n{}_{p,\upsilon}$, that is, to determine under what conditions they coincide.

2. Auxiliary results

Let us consider matrix operator

$$Af_k = \sum_{i=0}^{\infty} a_{k,i} f_i,$$

where $a_{k,i},\ k,i = 0,1,\ldots$ are elements of the real matrix $(a_{k,i})$, $f = \{f_i\}_{i=0}^{\infty}$ is the sequence of real numbers.

Let $1 \leqslant p \leqslant \infty$ and l_p be the space of the real sequences $f = \{f_i\}_{i=0}^{\infty}$ with finite norm

$$\|f\|_p = \left(\sum_{i=0}^{\infty} |f_i|^p\right)^{\frac{1}{p}}$$

for $1 \leqslant p < \infty$ and

$$\|f\|_p = \sup_{i \geqslant 0} |f_i|$$

for $p = \infty$.

To prove the main results we need the following statement which was presented in [11].

THEOREM A. *Let* $1 < p < \infty$, $p' = \frac{p}{p-1}$. *The matrix operator* A *is bounded from* l_p *to* l_∞, *if and only if*

$$\sup_{k\geqslant 0}\left(\sum_{i=0}^{\infty}|a_{k,i}|^{p'}\right)^{\frac{1}{p'}} < \infty$$

and moreover

$$\|A\|_{p\to\infty} = \sup_{k\geqslant 0}\left(\sum_{i=0}^{\infty}|a_{k,i}|^{p'}\right)^{\frac{1}{p'}},$$

where $\|A\|_{p\to\infty}$ *is the norm of the operator* A *from* l_p *to* l_∞.

Let $k^{(t)} = k(k-1)(k-2)\dots(k-t+1)$. Then

$$|k^{(t_2)}| > |k^{(t_1)}| \quad \text{for } t_2 > t_1 > 0. \tag{2}$$

These relations follow from the equality

$$\begin{aligned} k^{(t_2)} &= k(k-1)(k-2)\dots(k-t_2+1) \\ &= k(k-1)\dots(k-(t_1-1))(k-t_1)\dots(k-t_2+1) \\ &= k^{(t_1)}(k-t_1)\dots(k-t_2+1). \end{aligned}$$

Sequences $f_i = \{f_{k,i}\}_{k=0}^{\infty}$, $f_{k,i} = k^{(i)}$, $i = 0,1,\dots,n-1$, $k^{(0)} = 1$ are fundamental solutions of the difference equation $\Delta^n f_i = 0$, $n \geqslant 1$. Let $f \in E_0$, then for any $k \geqslant 0$ we have

$$f_k = \frac{1}{(n-1)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-1)}\Delta^n f_j, \quad n \geqslant 1. \tag{3}$$

This equality can be easily verified by applying Abel transformations on the right-hand side of (3), i.e. using summation "by parts". For example, applying Abel transformations once, we get

$$\frac{1}{(n-1)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-1)}\Delta^n f_j = \frac{1}{(n-2)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-2)}\Delta^{n-1} f_j.$$

Hence, it can be seen that further applying the Abel transformations another $n-2$ times we get (3).

3. Main results

Let $n \geqslant 1$, $0 \leqslant s_0 \leqslant n-1$. At first, let us consider for which weight sequences $\upsilon = \{\upsilon_i\}_{i=0}^{\infty}$ the condition

$$\lim_{k\to\infty}|\Delta^{s_0} f_k| \equiv |\Delta^{s_0} f_\infty| < \infty, \quad \forall f \in W_{p,\upsilon}^n \tag{4}$$

holds.

If $\upsilon_i \equiv 1$, $i \geqslant 0$, i.e. when there is no weight sequence, condition (4) does not hold for any $s_0 \in [0, n-1]$. Indeed, by applying summation by parts s_0 times in (3), for any $f \in E_0$ we have

$$\Delta^{s_0} f_k = \frac{1}{(n-1-s_0)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)} \Delta^n f_j, \tag{5}$$

and in the case when $\{f_k\}$ is a solution of the equation $\Delta^n f_j = 1$ we have $\lim\limits_{k\to\infty} \Delta^{s_0} f_k = \infty$.

Our results read as follows:

THEOREM 1. *Let $n \geqslant 1$, $0 \leqslant s_0 \leqslant n-1$, $1 \leqslant p < \infty$. Then the condition (4) is satisfied if and only if*

$$\lim_{k\to\infty} \sum_{j=0}^{k-1} \left| (k-j-1)^{(n-1-s_0)} \right|^{p'} \upsilon_j^{1-p'} < \infty. \tag{6}$$

holds.

Proof. For $\forall f \in W^n_{p,\upsilon}$ the equation (3) holds.

Let $\upsilon_k^{\frac{1}{p}} \Delta^n f_k = g_k$, $g = \{g_k\} \in l_p$ for $f \in W^n_{p,\upsilon}$, then $\|f\|_{p,\upsilon} = \|g\|_p$. Conversely, if for $\forall g \in l_p$ we define the sequence $\tilde{f}_k = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} \upsilon_j^{-\frac{1}{p}} g_j$, then $\Delta^n \tilde{f}_k = \upsilon_k^{-\frac{1}{p}} g_k$, from this $\upsilon_k^{\frac{1}{p}} \Delta^n \tilde{f}_k = g_k$, $k \geqslant 1$, i.e. $\tilde{f}_k \in W^n_{p,\upsilon}$ and $\left\|\tilde{f}\right\|_{p,\upsilon} = \|g\|_p$. Therefore, for $f \in W^n_{p,\upsilon}$

$$f_k = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} \Delta^n f_j = \frac{1}{(n-1)!} \sum_{j=0}^{k-1} \upsilon_j^{-\frac{1}{p}} (k-j-1)^{(n-1)} g_j.$$

According to (5) we have

$$\Delta^{s_0} f_k = \frac{1}{(n-1-s_0)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)} \upsilon_j^{-\frac{1}{p}} g_j.$$

It follows from this equality that for any $f \in W^n_{p,\upsilon}$ the fulfillment of condition (4) is equivalent to the fulfillment of

$$\lim_{k\to\infty} \left| \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)} \upsilon_j^{-\frac{1}{p}} g_j \right| < \infty, \quad \forall g \in l_p, \tag{7}$$

which in turn is equivalent to the condition

$$\sup_{k\geqslant 1} \left| \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)} \upsilon_j^{-\frac{1}{p}} g_j \right| < \infty, \quad \forall g \in l_p.$$

The last relation means that a matrix operator of the form

$$Ag_k = \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)}\, \upsilon_j^{-\frac{1}{p}} g_j$$

acts from l_p to l_∞. In order for this mapping to be valid, it is necessary and sufficient that the norm of the operator A from l_p to l_∞ is finite. According to the Theorem A the norm of linear matrix operators from l_p to l_∞

$$\|A\|^{p'}_{l_p \to l_\infty} = \sup_{k\geqslant 1} \sum_{j=0}^{k-1} \upsilon_j^{1-p'} \left|(k-j-1)^{(n-1-s_0)}\right|^{p'} < \infty. \tag{8}$$

From this the condition (8) is equivalent to the condition (6). □

COROLLARY 1. *If for* $0 \leqslant s_0 \leqslant n-1$ *the condition* (6) *holds, then the following estimate holds*

$$\sup_{k\geqslant 1} |\Delta^{s_0} f_k| \leqslant \frac{1}{(n-1-s_0)!} \sup_{k\geqslant 1} \left(\sum_{j=0}^{k-1} \left|(k-j-1)^{(n-1-s_0)}\right|^{p'} \upsilon_j^{1-p'} \right)^{\frac{1}{p'}} \|\Delta^n f\|_{p,\upsilon} \tag{9}$$

Proof. Let us consider the following relation from the Theorem 1

$$\Delta^{s_0} f_k = \frac{1}{(n-1-s_0)!} \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)}\, \upsilon_j^{-\frac{1}{p}} g_j, \tag{10}$$

where $g_j = \upsilon_j^{\frac{1}{p}} \Delta^n f_j \in l_p$. The fulfillment of the condition (6) means that the operator A is bounded from l_p to l_∞ and the following estimate holds

$$\sup_{k\geqslant 1} \left| \sum_{j=0}^{k-1} (k-j-1)^{(n-1-s_0)}\, \upsilon_j^{-\frac{1}{p}} g_j \right| \leqslant \sup_{k\geqslant 1} \left(\sum_{j=0}^{k-1} \left|(k-j-1)^{(n-1-s_0)}\right|^{p'} \upsilon_j^{1-p'} \right)^{\frac{1}{p'}} \|g\|_p$$

Hence, taking into account (10), we obtain (9). □

COROLLARY 2. *If for* $0 \leqslant s_0 \leqslant n-1$ *and for* $\forall f \in W^n_{p,\upsilon}$ *there exists finite limit* $\lim\limits_{k\to\infty} |\Delta^{s_0} f_k| < \infty$, *then there exists finite limit*

$$\lim_{k\to\infty} |\Delta^{s} f_k| < \infty$$

for $s \in (s_0, n-1]$ *and for all* $\forall f \in W^n_{p,\upsilon}$.

The statement of the corollary 2 follows from the relation (6) and (2), since $s_0 < s$, then $|(k-j-1)^{(n-1-s)}| < |(k-j-1)^{(n-1-s_0)}|$.

COROLLARY 3. *If there is* $\lim\limits_{k\to\infty}|f_k|<\infty$ *for* $\forall f\in W^n_{p,\upsilon}$, *then the limit*

$$\lim_{k\to\infty}|\Delta^s f_k|<\infty,\ \ \forall f\in W^n_{p,\upsilon},\quad s=1,2,\dots,n-1.$$

exists.

Corollary 3 follows from Corollary 2.

COROLLARY 4. *For* $\forall f\in W^n_{p,\upsilon}$ *the finite limit* $\lim\limits_{k\to\infty}\left|\Delta^{n-1}f_k\right|<\infty$ *exists if and only if*

$$\sum_{j=0}^{\infty}\upsilon_j^{1-p'}<\infty \tag{11}$$

holds.

Corollary 4 follows from Theorem 1, when $s_0=n-1$.

COROLLARY 5. *For each* $s\in[0,n-1]$, *there are elements from* $W^n_{p,\upsilon}$ *for which the condition* $\lim\limits_{k\to\infty}|\Delta^s f_k|<\infty$ *does not hold if and only if*

$$\sum_{j=0}^{\infty}\upsilon_j^{1-p'}=\infty. \tag{12}$$

Indeed, if (12) holds, then

$$\lim_{k\to\infty}\left(\sum_{j=0}^{k-1}\left|(k-j-1)^{(n-1-s)}\right|^{p'}\upsilon_j^{1-p'}\right)^{\frac{1}{p'}}=\infty$$

for all $s\in[0,n-1]$. Then, by Theorem 1, for every $s\in[0,n-1]$ there exists a sequence $f\in W^n_{p,\upsilon}$ such that $\lim\limits_{k\to\infty}|\Delta^s f_k|=\infty$.

COROLLARY 6. *If the condition* (6) *holds for* $0\leqslant s_0\leqslant n-1$, *then for all* $s=s_0,s_0+1,\dots,n-1$ *the equality* $\lim\limits_{k\to\infty}\Delta^s f_k=0$ *holds for* $\forall f\in\overset{\circ}{W}{}^n_{p,\upsilon}$.

Proof. By definition of the space $\overset{\circ}{W}{}^n_{p,\upsilon}$ it is known that for $\forall f\in\overset{\circ}{W}{}^n_{p,\upsilon}$ there is a sequence $\{h_n=h_n(f)\}$ from $\overset{\circ}{E}$ such that

$$\lim_{n\to\infty}\|h_n-f\|_{W^n_{p,\upsilon}}=0. \tag{13}$$

If the condition (6) holds for $0\leqslant s_0\leqslant n-1$, then according to $|(k-j-1)^{(n-1-s)}|<|(k-j-1)^{(n-1-s_0)}|$, $s>s_0$ the condition (6) also holds for $s\in(s_0,n-1]$. Therefore, the inequality (9) holds for all $s\in[s_0,n-1]$, i.e.

$$\sup_{k\geqslant 1}|\Delta^s f_k|\leqslant C\|\Delta^n f\|_{p,\upsilon}, \tag{14}$$

where

$$C=\frac{1}{(n-1-s)!}\sup_{k\geqslant 1}\left(\sum_{j=0}^{k-1}\left|(k-j-1)^{(n-1-s)}\right|^{p'}\upsilon_j^{1-p'}\right)^{\frac{1}{p'}}.$$

It follows from (13) and (14) that

$$\limsup_{n\to\infty}\sup_{k\geqslant 1}\left|\Delta^s f_k-\Delta^s h_{n,k}\right|\leqslant C\lim_{n\to\infty}\left\|\Delta^n(f-h_n)\right\|_{p,\upsilon}\leqslant C\lim_{n\to\infty}\|f-h_n\|_{W^n_{p,\upsilon}}=0,$$

i.e.

$$\limsup_{n\to\infty}\sup_{k\geqslant 1}\left|\Delta^s f_k-\Delta^s h_{n,k}\right|=0. \tag{15}$$

Now, taking into account that for $\forall n\geqslant 1$ there exists $K=K(n)$ such that $\Delta^s h_{n,k}=0$ for $\forall k\geqslant K$, we get

$$\sup_{k\geqslant 1}\left|\Delta^s f_k-\Delta^s h_{n,k}\right|\geqslant\lim_{k\to\infty}\left|\Delta^s f_k-\Delta^s h_{n,k}\right|=\lim_{k\to\infty}\left\|\Delta^s f_k\right\|.$$

Substituting the last relation into (15) we obtain $\lim_{k\to\infty}\Delta^s f_k=0$. □

Thus, we have obtained that the sequences of the space $W^n_{p,\upsilon}$ take a finite boundary value at infinity, depending on the property of changing the weight sequence $\{\upsilon_i\}_{i=0}^{\infty}$ at neighborhood infinity.

Now consider the conditions for the weight sequence under which $\overset{\circ}{W}{}^n{}_{p,\upsilon}=W^n_{p,\upsilon}$.

THEOREM 2. *Let $1<p<\infty$. Then the equality $\overset{\circ}{W}{}^n{}_{p,\upsilon}=W^n_{p,\upsilon}$ holds if and only if the weight sequence $\{\upsilon_i\}_{i=0}^{\infty}$ satisfies the condition* (12).

Proof. By definitions of the sets, it is known that $\overset{\circ}{W}{}^n{}_{p,\upsilon}\subset W^n_{p,\upsilon}$. Now, to prove the equality based on condition (12) it is enough to show that $\overset{\circ}{W}{}^n{}_{p,\upsilon}\supset W^n_{p,\upsilon}$.

Since $f_k=0$, $k=0,1,\dots n-1$ for $f\in W^n_{p,\upsilon}$ by definition of the space $W^n_{p,\upsilon}$, then by redefining $f_n=\overline{f}_0$, $f_{n+i}=\overline{f}_i$, $i\geqslant 1$ we can assume that for $f\in W^n_{p,\upsilon}$ there can be $f_0\neq 0$. To each element $f\in W^n_{p,\upsilon}$ we assign a corresponding element $g\in l_p$, where $g_j=\upsilon_j^{\frac{1}{p}}\Delta^n f_j$, $j\geqslant 0$. Conversely, if to each element of $g\in l_p$ we assign the following sequence

$$f_k=\frac{1}{(n-1)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-1)}\upsilon_j^{-\frac{1}{p}}g_j,\quad k\geqslant 1,$$

then $\Delta^n f_k=\upsilon_k^{-\frac{1}{p}}g_k$, $\upsilon_k^{\frac{1}{p}}\Delta^n f_k=g_k$ and $f\in W^n_{p,\upsilon}$. This correspondence $\|f\|_{W^n_{p,\upsilon}}=\|g\|_p$ is isometric. Therefore, we consider the space $W^n_{p,\upsilon}$ as the space l_p. Then $W^n_{p,\upsilon}$ is a reflexive space and for any $F\in\left(W^n_{p,\upsilon}\right)^*$ there exists a unique $z(F)=z\in l_{p'}$ such that the following representation holds

$$F(f)=\sum_{k=0}^{\infty}z_k\upsilon_k^{\frac{1}{p}}\Delta^n f_k,\quad\forall f\in W^n_{p,\upsilon}. \tag{16}$$

Consider the following set

$$\mathscr{B} = \left\{ F \in \left(W_{p,\upsilon}^{n}\right)^{*} : F(\varphi) = 0,\ \forall \varphi \in \overset{\circ}{E} \right\}.$$

This set is the annihilator of the subset $\overset{\circ}{E}$ of the space $W_{p,\upsilon}^{n}$, i.e. $\mathscr{B} = \left[\overset{\circ}{E}\right]^{\perp}$. Since the space $W_{p,\upsilon}^{n}$ is reflexive, and the set $\overset{\circ}{E}$ is dense in the space $\overset{\circ}{W}{}^{n}{}_{p,\upsilon}$, then

$$\mathscr{B}^{\perp} = \left[\left[\overset{\circ}{E}\right]^{\perp}\right]^{\perp} = \overline{\overset{\circ}{E}} = \overset{\circ}{W}{}^{n}{}_{p,\upsilon}$$

or

$$\overset{\circ}{W}{}^{n}{}_{p,\upsilon} = \left\{ f \in W_{p,\upsilon}^{n} : F(f) = 0, \forall F \in \mathscr{B} \right\},$$

i.e. the set $\overset{\circ}{W}{}^{n}{}_{p,\upsilon}$ is the set of $f \in W_{p,\upsilon}^{n}$ such that $F(f) = 0,\ \forall F \in \mathscr{B}$. From which it follows that if for all $f \in W_{p,\upsilon}^{n}$ $F(f) = 0,\ \forall F \in \mathscr{B}$ is fulfilled, then $\overset{\circ}{W}{}^{n}{}_{p,\upsilon} \supset W_{p,\upsilon}^{n}$, i.e. the condition $\overset{\circ}{W}{}^{n}{}_{p,\upsilon} \supset W_{p,\upsilon}^{n}$ is equivalent to the condition

$$F(f) = 0, \quad \forall F \in \mathscr{B}, \quad \forall f \in W_{p,\upsilon}^{n}. \tag{17}$$

Since $\mathscr{B} \subset \left(W_{p,\upsilon}^{n}\right)^{*}$, then for $F \in \mathscr{B}$ there exists a unique $\overline{z}(F) = \overline{z} \in l_{p'}$ such that

$$F(\varphi) = \sum_{k=0}^{\infty} \tilde{z}_k \upsilon_k^{\frac{1}{p}} \Delta^n \varphi_k = 0, \ \ \forall \varphi \in \overset{\circ}{E}. \tag{18}$$

Taking into account that the initial and final terms of the sequence $\overset{\circ}{E}$ are zero, we apply the Abbel transformation to (18), then

$$F(\varphi) = \sum_{k=0}^{\infty} \nabla^n \left(\tilde{z}_k \upsilon_k^{\frac{1}{p}} \right) \varphi_k = 0,\ \forall \varphi \in \overset{\circ}{E},$$

where $\nabla f_k = f_k - f_{k-1}$.

Hence, due to the density of $\overset{\circ}{E}$ in the space l_1, we obtain that the equation $\nabla^n \left(\overline{z}_k \upsilon_k^{\frac{1}{p}} \right) = 0,\ k = 0,1,\ldots$. Since $k^{(0)} = 1,\ k^{(i)},\ i = 1,2,\ldots,n-1$ are linearly independent solutions of this equation, then its general solution will be

$$\tilde{z}_k \upsilon_k^{\frac{1}{p}} = \sum_{i=0}^{n-1} \beta_i k^{(i)}, \ \ k \geqslant 0,$$

where $\beta_i,\ i = 0,1,\ldots,n-1$ are real numbers.

From this

$$\tilde{z}_k = \sum_{i=0}^{n-1} \beta_i \upsilon_k^{-\frac{1}{p}} k^{(i)}, \ \ k \geqslant 0, \tag{19}$$

where in the left part the sequence $\{\overline{z}_k\}$ belongs to the space $l_{p'}$. And since by the condition (12) $\left\{\upsilon_k^{-\frac{1}{p}}\right\} \notin l_{p'}$, $1 = k^{(0)} < |k^{(i)}|$, $i \geqslant 1$, then $\left\{\upsilon_k^{-\frac{1}{p}} k^{(i)}\right\}_{k=0}^{\infty} \notin l_{p'}$, $i = 0, 1, \ldots, n-1$. Thus, the left-hand side of the equality (19) is from the space $l_{p'}$, and none of the terms on the right-hand side belong to the space $l_{p'}$. Therefore, in (19) $\beta_i = 0$, $i = 0, 1, \ldots, n-1$ and $\tilde{z}_k = 0$, $k \geqslant 0$.

Then, from (16) it follows that

$$F(f) = 0, \quad \forall f \in W^n_{p,\upsilon}, \quad F \in \left(W^n_{p,\upsilon}\right)^*. \tag{20}$$

Thus the condition (17) is fulfilled. Then $\overset{\circ}{W}{}^n{}_{p,\upsilon} \supset W^n_{p,\upsilon}$ and $\overset{\circ}{W}{}^n{}_{p,\upsilon} = W^n_{p,\upsilon}$ Theorem 2 has been proven. □

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Aigerim Kalybay
KIMEP University
Abay Avenue, Almaty 050010, Kazakhstan
e-mail: kalybay@kimep.kz

Dinara Sarsenaly
School of Artificial Intelligence and Data Science
Astana IT University
55/11 Mangilik El Avenue, 010000, Astana, Kazakhstan
e-mail: dinara.sarsenali@mail.ru

Ainur Temirkhanova
Department of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St, 010008, Astana, Kazakhstan
e-mail: ainura-t@yandex.kz