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LINK INEQUALITIES FOR LINEAR AND MAXIMAL OPERATORS ON WEIGHTED SEQUENCE SPACES

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Abstract. The article analyzes linear and maximal operators on weighted sequence spaces associated with real valued kernels. The main result characterizes the weights and kernels that yield inequalities which link such operators.

1. Introduction

All analysts spend half their life hunting through the literature for inequalities which they want to use and cannot prove.

G. H. Hardy, A Mathematician's Apology, 1940

Let $\mathrm{L}(\mathbb{Z}_+)$ be the real linear space of sequences $u:\mathbb{Z}_+\to(-\infty,\infty)$, where $\mathbb{Z}_+=\{1,2,3,\dots\}$. Consider a weight function $w:\mathbb{Z}_+\to(0,\infty)$ and define the correspondimg measure μ on $\mathbb{Z}_+$ and associated integrals by

$$\mu(\Omega)=\sum_{n\in\Omega}w(n),\quad \int_\Omega u(n)\,d\mu(n)=\sum_{n\in\Omega}u(n)\,w(n), \tag{1.1}$$

for any set $\Omega\subseteq\mathbb{Z}_+$ and appropriate $u\in\mathrm{L}(\mathbb{Z}_+)$. Suppose that $\kappa:\mathbb{Z}_+\times\mathbb{Z}_+\to(-\infty,\infty)$ is a real valued kernel on the measure space $(\mathbb{Z}_+,\mu)$, and let $\mathfrak{L}(\kappa)$ be the linear operator given by

$$\mathfrak{L}(\kappa)u(m)=\int_{\mathbb{Z}_+}\kappa(m,n)u(n)\,d\mu(n),\ \ u\in\mathrm{Dom}\,\mathfrak{L}(\kappa)\subseteq\mathrm{L}(\mathbb{Z}_+),\ \ m\in\mathbb{Z}_+, \tag{1.2}$$

with domain $\mathrm{Dom}\,\mathfrak{L}(\kappa)$ consisting of $u\in\mathrm{L}(\mathbb{Z}_+)$ such that $\mathfrak{L}(\kappa)u\in\mathrm{L}(\mathbb{Z}_+)$.

Related to kernel κ we next introduce the collection of sets

$$\Omega(\kappa)[m,t]=\{n\in\mathbb{Z}_+:|\kappa(m,n)|\geqslant t\},\quad m\in\mathbb{Z}_+,\quad t\in(0,\infty), \tag{1.3}$$

and assume that

$$\mu\left(\Omega(\kappa)[m,t]\right)<\infty,\quad m\in\mathbb{Z}_+,\quad t\in(0,\infty). \tag{1.4}$$

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Under this assumption, we define the sublinear maximal operator $\mathfrak{M}(\kappa)$ with domain $\mathrm{Dom}\,\mathfrak{M}(\kappa) \subseteq \mathrm{L}(\mathbb{Z}_+)$ by

$$\mathfrak{M}(\kappa)u(m) = \sup_{t\in(0,\infty)} \frac{1}{\mu(\Omega(\kappa)[m,t])} \int_{\Omega(\kappa)[m,t]} |u(n)|\, d\mu(n), \quad m \in \mathbb{Z}_+, \tag{1.5}$$

where $u \in \mathrm{Dom}\,\mathfrak{M}(\kappa)$ provided that $\mathfrak{M}(\kappa)u \in \mathrm{L}(\mathbb{Z}_+)$.

Based on standard definitions in measure theory, corresponding to the measure space $(\mathbb{Z}_+,\mu)$ we set up the Banach spaces $\mathrm{L}^p(\mathbb{Z}_+,\mu)$ of p-integrable sequences, $1 \leqslant p \leqslant \infty$, with norms denoted by $||\cdot||_p$. The specific goal of the article is to identify properties of kernel κ and measure μ that lead to pointwise estimates of $\mathfrak{L}(\kappa)u$, when $u \in \mathrm{Dom}\,\mathfrak{M}(\kappa) \cap \mathrm{L}^p(\mathbb{Z}_+,\mu)$, in terms of $\mathfrak{M}(\kappa)u$ and $||u||_p$. Let $|\kappa|$ and $|u|$ be the absolute values of kernel κ and sequence u. Observe that operator $\mathfrak{M}(\kappa)$ is defined using $|\kappa|$, $\mathfrak{M}(\kappa)u = \mathfrak{M}(\kappa)|u|$, $||u||_p = |||u|||_p$, and $|\mathfrak{L}(\kappa)u(m)| \leqslant \mathfrak{L}(|\kappa|)|u|(m)$, $m \in \mathbb{Z}_+$. Therefore, with regard to our purposes it would be enough to estimate $\mathfrak{L}(|\kappa|)|u|$. This will be the issue investigated in Section 2. Section 3 is concerned with applications of the main result for the class of convolution kernels and the counting measure.

The motivation for the objective of the article, analyzing sequences and series by relying on links between the operators $\mathfrak{L}(\kappa)$ and $\mathfrak{M}(\kappa)$, originated from results and basic techniques included in Hardy [1] and Hardy, Littlewood, Pólya [3]. The article is a sequel, essentially an addendum, to *Fractional and maximal operators in measure space settings. Link inequalities,* Martin [4]. In contrast to the earlier article, which addresed themes in harmonic and complex analysis, the present one is related to classical real analysis.

2. Link inequalities – statements and proofs

THEOREM 2.1. *Suppose that $\omega \in (1,\infty)$. Let $w : \mathbb{Z}_+ \to (0,\infty)$ be a weight on $\mathbb{Z}_+$ with associated measure μ, and let $\kappa : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (-\infty,\infty)$ be a kernel on the measure space $(\mathbb{Z}_+,\mu)$. The following four statements are equivalent.*

(i) *There exists a function $\gamma_0 : \mathbb{Z}_+ \to (0,\infty)$ such that*

$$\mathfrak{L}(|\kappa|)\,\chi_\Omega(m) \leqslant \gamma_0(m)\cdot[\mu(\Omega)]^{1-1/\omega}, \quad m \in \mathbb{Z}_+, \tag{2.1}$$

for any characteristic functions χ_Ω of sets $\Omega \subseteq \mathbb{Z}_+$ with $0 < \mu(\Omega) < \infty$.

(ii) *There exists a function $\lambda : \mathbb{Z}_+ \to (0,\infty)$ such that*

$$\mu(\Omega(\kappa)[m,t]) \leqslant \lambda(m)\cdot t^{-\omega}, \ \ m \in \mathbb{Z}_+, \ \ t \in (0,\infty). \tag{2.2}$$

(iii) *If $1 \leqslant p < \omega(\omega-1)^{-1}$, there exists a function $\gamma_p : \mathbb{Z}_+ \to (0,\infty)$ such that*

$$\mathfrak{L}(|\kappa|)\,|u|(m) \leqslant \gamma_p(m)\cdot[\mathfrak{M}(\kappa)u(m)]^{p/q}\cdot\|u\|_p^{1-p/q}, \ \ m \in \mathbb{Z}_+, \tag{2.3}$$

for any $u \in \mathrm{Dom}\,\mathfrak{M}(\kappa) \cap \mathrm{L}^p(\mathbb{Z}_+,\mu)$, where $q^{-1} = p^{-1} - (\omega-1)\omega^{-1}$.

(iv) *There exists a function* $\gamma : \mathbb{Z}_+ \to (0,\infty)$ *such that*

$$\mathfrak{L}(|\kappa|)\,|u|(m) \leqslant \gamma(m)\cdot \|u\|_\infty^{1/\omega}\cdot \|u\|_1^{1-1/\omega},\ \ m\in\mathbb{Z}_+, \tag{2.4}$$

for any $u \in \mathrm{L}^1(\mathbb{Z}_+,\mu)\cap \mathrm{L}^\infty(\mathbb{Z}_+,\mu)$.

Assuming that $\Gamma_0=\Gamma_0(\kappa,\omega)$, $\Lambda=\Lambda(\kappa,\omega)$, $\Gamma_p=\Gamma_p(\kappa,\omega)$, *and* $\Gamma=\Gamma(\kappa,\omega)$ *are the best functions* γ_0, λ, γ_p, *and* γ *in statements* (i), (ii), (iii), (iv)*, then*

$$\Lambda(m)^{1/\omega} \leqslant \Gamma_0(m) \leqslant \Gamma(m) \leqslant \Gamma_p(m) \leqslant \mathfrak{c}_{p,\omega}\cdot \Lambda(m)^{1/\omega},\quad m\in\mathbb{Z}_+, \tag{2.5}$$

$$\mathfrak{c}_{p,\omega} = \frac{\omega}{p(\omega-1)}\cdot\left[\frac{p}{p-\omega(p-1)}\right]^{1/\omega}.$$

Proof. For convenience, denote $\mathfrak{L}(|\kappa|)=\mathfrak{L}$, $\mathfrak{M}(\kappa)=\mathfrak{M}$, $\Omega(\kappa)[m,t]=\Omega[m,t]$. We divide the proof into four parts.

(i) *implies* (ii). Inequality (2.1) with $\gamma_0=\Gamma_0$ and $\Omega=\Omega[m,t]$ shows that

$$\int_{\Omega[m,t]} |\kappa(m,n)|\,d\mu(n) \leqslant \Gamma_0(m)\cdot[\mu(\Omega[m,t])]^{1-1/\omega},\ \ m\in\mathbb{Z}_+,\ \ t\in(0,\infty).$$

Definition (1.3) of $\Omega[m,t]$ implies

$$\int_{\Omega[m,t]} |\kappa(m,n)|\,d\mu(n) \geqslant t\cdot\mu(\Omega[m,t]),\ \ m\in\mathbb{Z}_+,\ \ t\in(0,\infty),$$

hence $t\cdot\mu(\Omega[m,t]) \leqslant \Gamma_0(m)\cdot[\mu(\Omega[m,t])]^{1-1/\omega}$, an inequality equivalent to (2.2) in statement (ii) with $\lambda(m)=\Gamma_0(m)^\omega$. Consequently, $\Lambda(m)\leqslant\Gamma_0(m)^\omega$, i.e., $\Lambda^{1/\omega}\leqslant\Gamma_0$, as stated in (2.5).

(ii) *implies* (iii). Start with an arbitrary $m\in\mathbb{Z}_+$ and use it to define the non-increasing function

$$\mu_m : (0.\infty)\to(0,\infty),\quad \mu_m(t)=\mu(\Omega[m,t]),\quad t\in(0.\infty). \tag{2.6}$$

Consistent with statement (ii) we know that

$$\mu_m(t)\leqslant\Lambda(m)\cdot t^{-\omega},\quad t\in(0,\infty). \tag{2.7}$$

Associated with the same $m\in\mathbb{Z}_+$ and any sequence $u\in\mathrm{Dom}\,\mathfrak{M}$ we also define the non-increasing functions

$$\mu_{u,m} : (0.\infty)\to[0,\infty),\quad \mu_{u,m}(t)=\int_{\Omega[m,t]}|u(n)|\,d\mu(n),\quad t\in(0,\infty). \tag{2.8}$$

Note that from the definition (1.5) of $\mathfrak{M}u$ in conjunction with (2.6) and (2.8) we get $\mu_{u,m}(t)\leqslant\mathfrak{M}u(m)\cdot\mu_m(t)$, and by (2.7) we have

$$\mu_{u,m}(t)\leqslant\Lambda(m)\cdot\mathfrak{M}u(m)\cdot t^{-\omega},\quad t\in(0,\infty). \tag{2.9}$$

Next, given an arbitrary $\tau \in (0,\infty)$ we introduce the non-negative kernels $\kappa_{\rm inn}, \kappa_{\rm out} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to [0,\infty)$ which depend on τ, defined by

$$\kappa_{\rm inn}(m,n) = |\kappa(m,n)| - \tau \text{ if } n \in \Omega[m,\tau],\ \kappa_{\rm inn}(m,n) = 0 \text{ if } n \in \mathbb{Z}_+ \setminus \Omega[m,\tau],$$

$$\kappa_{\rm out}(m,n) = \tau \text{ if } n \in \Omega[m,\tau],\ \kappa_{\rm out}(m,n) = |\kappa(m,n)| \text{ if } n \in \mathbb{Z}_+ \setminus \Omega[m,\tau].$$

Let $\mathfrak{L}_{\rm inn} = \mathfrak{L}(\kappa_{\rm inn})$ and $\mathfrak{L}_{\rm out} = \mathfrak{L}(\kappa_{\rm out})$ be the linear operators with kernels $\kappa_{\rm inn}$ and $\kappa_{\rm out}$. We claim that if $1 \leqslant p < \omega(\omega-1)^{-1}$ and $u \in \mathrm{Dom}\,\mathfrak{M} \cap \mathrm{L}^p(\mathbb{Z}_+,\mu)$, then $\mathfrak{L}_{\rm inn}|u|$, $\mathfrak{L}_{\rm out}|u| \in \mathrm{L}(\mathbb{Z}_+)$ and

$$\mathfrak{L}_{\rm inn}|u|(m) \leqslant \frac{\Lambda(m)\cdot \mathfrak{M}u(m)}{\omega-1}\cdot \tau^{-\omega+1},\ \ m \in \mathbb{Z}_+, \tag{2.10}$$

$$\mathfrak{L}_{\rm out}|u|(m) \leqslant \left[\frac{p\,\Lambda(m)}{p-\omega\,(p-1)}\right]^{(p-1)/p}\cdot \|u\|_p \cdot \tau^{[(p-\omega(p-1)]/p},\ \ m \in \mathbb{Z}_+. \tag{2.11}$$

To prove (2.10), we use the definition of kernel $\kappa_{\rm inn}$, the non-increasing functions $\mu_{u,m}$ defined by (3.8), and express $\mathfrak{L}_{\rm inn}|u|(m)$ as a Stieltjes integral,

$$\mathfrak{L}_{\rm inn}|u|(m) = \int_{\Omega[m,\tau]} (\,|\kappa(m,n)| - \tau\,)\,|u(n)|\,d\mu(n) = -\int_\tau^\infty (t-\tau)\,d\mu_{u,m}(t).$$

By inequality (2.9), $\lim\limits_{t\to\infty}(t-\tau)\,\mu_{u,m}(t) = 0$, and integration by parts yields

$$\mathfrak{L}_{\rm inn}|u|(m) = \int_\tau^\infty \mu_{u,m}(t)\,dt.$$

Using (2.9) one more time, we get

$$\mathfrak{L}_{\rm inn}|u|(m) \leqslant \Lambda(m)\cdot\mathfrak{M}u(m)\int_\tau^\infty t^{-\omega}\,dt = \frac{\Lambda(m)\cdot\mathfrak{M}u(m)}{\omega-1}\cdot\tau^{-\omega+1}.$$

The proof of (2.10) is concluded. To get (2.11), we assign to each $m \in \mathbb{Z}_+$ the sequence $\kappa_{\rm out}(m,\cdot) : \mathbb{Z}_+ \to (0,\infty)$, apply Hölder inequality, and obtain

$$\mathfrak{L}_{\rm out}|u|(m) = \int_{\mathbb{Z}_+} \kappa_{\rm out}(m,n)u(n)\,d\mu\,(n) \leqslant \|\,\kappa_{\rm out}(m,\cdot)\|_r \cdot \|u\|_p, \tag{2.12}$$

where r is the conjugate of p. Our subsequent task is to estimate $\|\,\kappa_{\rm out}(m,\cdot)\|_r$. When $p=1$, $r=\infty$ and $\|\,\kappa_{\rm out}(m,\cdot)\|_\infty = \tau$, hence in this case (2.12) reduces to (2.11). When $p>1$, $r = p/(p-1)$ and $p < \omega(\omega-1)^{-1}$ implies $r > \omega$. Using the definition of kernel $\kappa_{\rm out}$, the non-increasing functions μ_m defined by (2.6), an appropriate Stieltjes integral, and integration by parts, we successively have

$$\begin{aligned}\|\,\kappa_{\rm out}(m,\cdot)\|_r^r &= \tau^r\cdot\mu(\,\Omega[m,\tau]\,) + \int_{\mathbb{Z}_+\setminus\Omega[m,\tau]} |\kappa(m,n)|^r\,d\mu(n)\\ &= \tau^r\cdot\mu_m(\tau) - \int_0^\tau t^r\,d\,\mu_m(t) = r\int_0^\tau t^{r-1}\,\mu_m(t)\,dt.\end{aligned} \tag{2.13}$$

Inequality (2.7) implies

$$\|\kappa_{\mathrm{out}}(m,\cdot)\|_r^r \leqslant r\Lambda(m)\cdot\int_0^\tau t^{r-\omega-1}\,dt = \frac{r\Lambda(m)}{r-\omega}\cdot\tau^{r-\omega}.$$

The just found estimate of $\|\kappa_{\mathrm{out}}(m,\cdot)\|_r$, the substitution $r=p/(p-1)$, and straightforward calculations show that (2.11) follows from (2.12).

Since $|\kappa| = \kappa_{\mathrm{inn}} + \kappa_{\mathrm{inn}}$,

$$\mathfrak{L}|u|(m) = \mathfrak{L}_{\mathrm{inn}}|u|(m) + \mathfrak{L}_{\mathrm{out}}|u|(m), \tag{2.14}$$

for any $m\in\mathbb{Z}_+$ and sequence $u\in \mathrm{Dom}\,\mathfrak{M}\cap \mathrm{L}^p(\mathbb{Z}_+,\mu)$. Assembling (2.14), (2.10), and (2.11) we end up with

$$\mathfrak{L}|u|(m) \leqslant \frac{\Lambda(m)\cdot\mathfrak{M}u(m)}{\omega-1}\cdot\tau^{-\omega+1} + \left[\frac{p\Lambda(m)}{p-\omega(p-1)}\right]^{(p-1)/p}\cdot\|u\|_p\cdot\tau^{[(p-\omega(p-1)]/p}.$$

The value of τ that minimizes the right hand side of this inequality is

$$\tau_* = \left[\frac{\Lambda(m)\cdot\mathfrak{M}u(m)}{\|u\|_p}\right]^{p/\omega}\cdot\left[\frac{p-\omega(p-1)}{p}\right]^{1/\omega}.$$

Elementary computations show that τ_* yields (2.3) in statement (iii) with

$$\gamma_p(m) = \frac{\omega}{p(\omega-1)}\left[\frac{p}{p-\omega(p-1)}\cdot\Lambda(m)\right]^{1/\omega},\quad m\in\mathbb{Z}_+.$$

Hence, the best function Γ_p satisfies the last inequality in (2.5).

(iii) *implies* (iv). We use (2.3) with $\gamma_p=\Gamma_p$ for $u\in \mathrm{L}^1(\mathbb{Z}_+,\mu)\cap\mathrm{L}^\infty(\mathbb{Z}_+,\mu)$, and note that $\mathfrak{M}u(m)\leqslant\|u\|_\infty$ and $\|u\|_p\leqslant\|u\|_1^{1/p}\cdot\|u\|_\infty^{1-1/p}$. Therefore,

$$\mathfrak{L}|u|(m)\leqslant\Gamma_p(m)\cdot\|u\|_\infty^{p/q}\cdot[\,\|u\|_1^{1/p}\cdot\|u\|_\infty^{1-1/p}]^{1-p/q}.$$

Since $q^{-1}=p^{-1}-(\omega-1)\omega^{-1}$, this estimate yields (2.4) in statement (iv) with $\gamma=\Gamma_p$. Thus $\Gamma\leqslant\Gamma_p$, the third inequality in (2.5).

(iv) *implies* (i). We substitute $u=\chi_\Omega$ in (2.4) with $\gamma=\Gamma$, and note that $\|u\|_\infty=1$ and $\|u\|_1=\mu(\Omega)$. Thus we get (2.1) with $\gamma_0=\Gamma$. As a consequence, $\Gamma_0\leqslant\Gamma$, which is the second inequality in (2.5). The proof is complete. □

COROLLARY 2.1. Suppose that the statements in Theorem 2.1 are true.

(i) If one of the best functions $\Gamma_0=\Gamma_0(\kappa,\omega)$, $\Lambda=\Lambda(\kappa,\omega)$, $\Gamma_p=\Gamma_p(\kappa,\omega)$, or $\Gamma=\Gamma(\kappa,\omega)$ is bounded, the three others are bounded too.

(ii) $\mathrm{Dom}\,\mathfrak{M}(\kappa)\cap\mathrm{L}^p(\mathbb{Z}_+,\mu)\subseteq\mathrm{Dom}\,\mathfrak{L}(\kappa)$ for any $1\leqslant p<\omega(\omega-1)^{-1}$. Moreoevr, assuming that function Γ_p is bounded and operator $\mathfrak{M}(\kappa)$ acts continuously from $\mathrm{Dom}\,\mathfrak{M}(\kappa)\cap\mathrm{L}^p(\mathbb{Z}_+,\mu)$ with norm $\|\cdot\|_p$ into $\mathrm{L}^p(\mathbb{Z}_+,\mu)$, then $\mathfrak{L}(\kappa)$ is a continuos operator from $\mathrm{Dom}\,\mathfrak{M}(\kappa)\cap\mathrm{L}^p(\mathbb{Z}_+,\mu)$ into $\mathrm{L}^q(\mathbb{Z}_+,\mu)$, $q^{-1}=p^{-1}-(\omega-1)\omega^{-1}$.

(iii) $\mathrm{L}^\infty(\mathbb{Z}_+,\mu)\cap\mathrm{L}^1(\mathbb{Z}_+,\mu)\subseteq\mathrm{Dom}\,\mathfrak{L}(\kappa)$ and, if function Γ is bounded, then $\mathfrak{L}(\kappa)$ is a continuos operator from $\mathrm{L}^\infty(\mathbb{Z}_+,\mu)\cap\mathrm{L}^1(\mathbb{Z}_+,\mu)$ with norm $\|\cdot\|_\infty+\|\cdot\|_1$ into $\mathrm{L}^\infty(\mathbb{Z}_+,\mu)$.

Proof. Consequences of (2.5) and statements (iii), (iv) in Theorem 2.1. □

3. Link inequalities for convolution kernels

Throughout this section we will be using the unit weight $w \equiv 1$, the counting measure μ on $\mathbb{Z}_+$, and convolution kernels $\kappa : \mathbb{Z}_+ \times \mathbb{Z}_+ \to (0,\infty)$ defined by

$$\kappa(m,n) = \rho(|m-n|), \;\; m,n \in \mathbb{Z}_+, m \neq n, \quad \kappa(m,m) = 0, \;\; m \in \mathbb{Z}_+, \tag{3.1}$$

where $\rho : (0,\infty) \to (0,\infty)$ is a strictly decreasing continuous function with $\lim_{t\to 0} \rho(t) = \infty$ and $\lim_{t\to\infty} \rho(t) = 0$. Since function ρ is bijective, definition (1.3) of the associated sets $\Omega(\kappa)[m,t] \subseteq \mathbb{Z}_+$ implies

$$\Omega(\kappa)[m,t] = \{\, n \in \mathbb{Z}_+ : |m-n| \leqslant \rho^{-1}(t) \,\} \setminus \{m\}, \quad m \in \mathbb{Z}_+, \;\; t \in (0,\infty).$$

Therefore, $\Omega(\kappa)[m,t] \neq \emptyset$ only if $\rho^{-1}(t) \geqslant 1$, i.e., $t \leqslant \rho(1)$. For each such t, let $k \in \mathbb{Z}_+$ be the integer part of $\rho^{-1}(t)$, and from $t \in (\rho(k+1), \rho(k)]$ we get

$$\Omega(\kappa)[m,t] = \Omega(\kappa)[m,\rho(k)] = [m-k, m+k] \cap \mathbb{Z}_+ \setminus \{m\}. \tag{3.2}$$

Using the notation $\mathfrak{S}(m,k) = [m-k, m+k] \cap \mathbb{Z}_+ \setminus \{m\}$, by equation (1.4) it follows that the maximal operator $\mathfrak{M}(\kappa)$ is given by

$$\mathfrak{M}(\kappa)u(m) = \sup_{k\in\mathbb{Z}_+} \frac{1}{\mu(\,\mathfrak{S}(m,k)\,)} \int_{\mathfrak{S}(m,k)} |u(n)|\, d\mu\,(n), \quad m \in \mathbb{Z}_+. \tag{3.3}$$

Observe that equation (3.3) defines a maximal operator $\mathfrak{M}$ independent of the specific convolution kernel κ. As a quite important property of $\mathfrak{M}$, based on Hardy–Littlewood Maximal Theorem, Hardy, Littlewood [2], we get that $\mathrm{L}^p(\mathbb{Z}_+,\mu)$ is a subspace of $\mathrm{Dom}\,\mathfrak{M}$ for any $p > 1$, and operator $\mathfrak{M}$ acts continuously from $\mathrm{L}^p(\mathbb{Z}_+,\mu)$ into $\mathrm{L}^p(\mathbb{Z}_+,\mu)$.

The following result characterizes the functions ρ for which kernels κ yield operators $\mathfrak{L}(\kappa)$ which satisfy the link inequalities in Theorem 2.1.

THEOREM 3.1. *Statements* (i), (ii), (iii), *and* (iv) *in* Theorem 2.1 *are true for* $\omega \in (1,\infty)$ *and the kernel* κ *defined by* (3.1) *if and only if the sequence*

$$\sigma : \mathbb{Z}_+ \to (0,\infty), \quad \sigma(k) = k \cdot \rho(k)^\omega, \quad k \in \mathbb{Z}_+, \tag{3.4}$$

is bounded. Under this assumption, the best function Λ *corresponding to* κ *and* ω *is given by*

$$\Lambda(m) = \max\{\, \sup_{k\leqslant m-1} 2k \cdot \rho(k)^\omega, \, \sup_{k\geqslant m} (m+k-1) \cdot \rho(k)^\omega \,\}, \;\; m,k \in \mathbb{Z}_+, \tag{3.5}$$

and

$$\Lambda(1) = ||\,\sigma\,||_\infty, \quad \Lambda(m) \leqslant 2\,||\,\sigma\,||_\infty, \quad m \in \mathbb{Z}_+, \;\; m \geqslant 2. \tag{3.6}$$

Proof. It would be enough to determine the property of function ρ that makes statement (ii) true. We note that the best function $\Lambda : \mathbb{Z}_+ \to (0,\infty)$ satisfying inequality (2.2) in Theorem 2.1 yields the equation

$$\Lambda(m) = \sup_{k\in\mathbb{Z}_+} \sup_{t\in(\rho(k+1),\rho(k)]} \mu(\Omega(\kappa)[m,t]) \cdot t^{\omega} = \sup_{k\in\mathbb{Z}_+} \mu(\Omega(\kappa)[m,\rho(k)]) \cdot \rho(k)^{\omega}.$$

By (3.2) we have either $\mu(\Omega(\kappa)[m,\rho(k)]) = 2k$ or $\mu(\Omega(\kappa)[m,\rho(k)]) = m+k-1$, according as $m \geqslant k+1$ or $m \leqslant k$, an observation that leads to formula (3.5). It remains to note that (3.6) follows from (3.5) and the definition (3.4) of σ. The proof is complete. The existence of Λ and the property that σ is bounded are equivalent. □

COROLLARY 3.1. If $1 < p < \omega(\omega-1)^{-1}$, $q^{-1} = p^{-1} - (\omega-1)\omega^{-1}$, and sequence σ is bounded, then $\mathfrak{L}(\kappa)$ is a continuous operator from $\mathrm{L}^p(\mathbb{Z}_+,\mu)$ to $\mathrm{L}^q(\mathbb{Z}_+,\mu)$.

Proof. Consequence of Theorem 3.1 and Corollary 2.1. □

Another consequence of link inequalities for convolution kernels κ defined by bijective strictly decreasing continuous functions ρ is concerned with series

$$\mathfrak{s}(u) = \rho(1)u(1) + \rho(2)u(2) + \ldots + \rho(n)u(n) + \ldots, \quad u \in \mathrm{L}(\mathbb{Z}_+,\mu), \tag{3.7}$$

which a priori might be divergent.

THEOREM 3.2. *Let $\mathfrak{C} : \mathrm{L}(\mathbb{Z}_+,\mu) \to \mathrm{L}(\mathbb{Z}_+,\mu)$ be the Cesàro operator,*

$$\mathfrak{C}u(n) = \frac{1}{n}\,[u(1)+u(2)+\ldots+u(n)], \quad u \in \mathrm{L}(\mathbb{Z}_+,\mu),\ n \in Z_+. \tag{3.8}$$

Suppose that sequence σ is bounded and $u \in \mathrm{L}^p(\mathbb{Z}_+,\mu)$, $1 \leqslant p < \omega(\omega-1)^{-1}$, $u \geqslant 0$. Then the series $\mathfrak{s}(u)$ is convergent, $\mathfrak{C}u \in \mathrm{L}^{\infty}(\mathbb{Z}_+)$, and

$$\mathfrak{s}(u) \leqslant \frac{\omega}{p(\omega-1)} \cdot \left[\frac{p}{p-\omega(p-1)}\right]^{1/\omega} \cdot \|\sigma\|_{\infty}^{1/\omega} \cdot \|\mathfrak{C}u\|_{\infty}^{p/q} \cdot \|u\|_p^{1-p/q}, \tag{3.9}$$

where $q^{-1} = p^{-1} - (\omega-1)\omega^{-1}$.

Proof. For each $u \in \mathrm{L}^p(\mathbb{Z}_+,\mu)$, $u \geqslant 0$, denote by $u_+ \in \mathrm{L}^p(\mathbb{Z}_+,\mu)$ the shifted sequence defined by $u_+(1) = 0$, $u_+(n+1) = u(n)$, $n \geqslant 1$. Observe that such sequences u are in $\mathrm{L}^{\infty}(\mathbb{Z}_+,\mu)$ too, and obviously $\|u_+\|_p = \|u\|_p$, $\|u_+\|_{\infty} = \|u\|_{\infty}$. In addition, note that $u_+ \in \mathrm{Dom}\,\mathfrak{M}$, and statement (iii) in Theorem 2.1 implies

$$\mathfrak{L}(\kappa)u_+(1) \leqslant \Gamma_p(1) \cdot [\mathfrak{M}u_+(1)]^{p/q} \cdot \|u_+\|_p^{1-p/q}. \tag{3.10}$$

From the definition of $\mathfrak{L}(\kappa)$ and by using (3.1) we get

$$\mathfrak{L}(\kappa)u_+(1) = \sum_{k\in\mathbb{Z}_+} \kappa(1,k)u_+(k) = \sum_{k\in\mathbb{Z}_+\setminus\{1\}} \rho(k-1)u_+(k) = \sum_{n\in\mathbb{Z}_+} \rho(n)u_+(n+1),$$

an equation that reduces to $\mathfrak{L}(\kappa)u_+(1)=\mathfrak{s}(u)$. Consequently, $\mathfrak{s}(u)$ is a convergent series. Next, by equation (3.3) we have

$$\mathfrak{M}u_+(1)=\sup_{n\in\mathbb{Z}_+}\frac{1}{\mu(\mathfrak{S}(1,n))}\sum_{k\in\mathfrak{S}(1,n)}u_+(k).$$

Recall that $\mathfrak{S}(1,n)=[1-n,1+n]\cap\mathbb{Z}_+\setminus\{1\}=\{2,3,\ldots,n+1\}$, hence

$$\mathfrak{M}u_+(1)=\sup_{n\in\mathbb{Z}_+}\frac{1}{n}\sum_{k=2}^{n+1}u_+(k)=\sup_{n\in\mathbb{Z}_+}\frac{1}{n}\left[u(1)+u(2)+\ldots+u(n)\right]=\|\mathfrak{C}u\|_\infty.$$

The last inequality in (2.5) and the first part of (3.6) imply

$$\Gamma_p(1)\leqslant\mathfrak{c}_{p,\omega}\cdot\Lambda(1)^{1/\omega}=\frac{\omega}{p(\omega-1)}\cdot\left[\frac{p}{p-\omega(p-1)}\right]^{1/\omega}\cdot\|\sigma\|_\infty^{1/\omega}. \tag{3.11}$$

Inequality (3.9) follows from (3.10) and (3.11) in conjunction with the substitutions $\mathfrak{L}(\kappa)u_+(1)=\mathfrak{s}(u)$, $\mathfrak{M}u_+(1)=\|\mathfrak{C}u\|_\infty$, and $\|u_+\|_p=\|u\|_p$. □

Our attempts of "*hunting through the literature for inequalities*" similar to (3.9) in Theorem 3.2, which involve series and sequences, have been rather unsuccessful. Any feedback from the readers will be highly appreciated.

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