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BLOCK DECOMPOSITIONS AND SHARP MATRIX PRODUCT BOUNDS

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Abstract. In this paper, we establish new numerical radius, spectral norm, and singular value inequalities for matrix expressions involving matrix products and sums. In particular, we refine the matrix Cauchy-Schwarz inequality by showing that if A,B,X are complex square matrices, with X positive semidefinite, then

$$\|AXB^*\|^2 \leqslant \frac{1}{4}\big(\omega(XA^*A)+\|XA^*A\|\big)\big(\omega(XB^*B)+\|XB^*B\|\big),$$

where $\|\cdot\|$ is the spectral norm and $\omega(\cdot)$ is the numerical radius.

1. Introduction

Let $M_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices, with zero matrix O, and identity I. For $A\in M_n(\mathbb{C})$, we denote by $\|A\|$ its spectral norm, that is,

$$\|A\| = \max_{\|x\|=1}\|Ax\|.$$

The numerical range of A is defined by

$$W(A) = \{\langle Ax,x\rangle : \|x\| = 1\},$$

and the numerical radius of A is

$$\omega(A) = \max\{|\lambda| : \lambda \in W(A)\}.$$

It is well known that $\omega(\cdot)$ is a norm on $M_n(\mathbb{C})$ satisfying

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|, \tag{1.1}$$

and it admits the equivalent characterizations [21]

$$\omega(A) = \max_{\theta\in\mathbb{R}}\|\Re(e^{i\theta}A)\| = \max_{\theta\in\mathbb{R}}\|\Im(e^{i\theta}A)\|,$$

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where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. It should be noted that the second inequality in (1.1) becomes an equality when A is normal.

For $A \in M_n(\mathbb{C})$, we denote by $s_j(A)$ the j-th singular value of A arranged in decreasing order. A matrix A is called positive semidefinite if $\langle Ax, x\rangle \geqslant 0$ for all vectors $x \in \mathbb{C}^n$.

Inequalities involving the numerical radius and the spectral norm have attracted considerable attention in matrix analysis and matrix theory. They play a central role in perturbation theory, matrix equations, matrix inequalities, and in the study of commutators and anti-commutators. Over the past decades, numerous refinements of (1.1) and its variants have been established (see, for example, [2, 12, 15, 16, 17, 19, 20, 21] and the references therein). More recently, inequalities combining numerical radius techniques with singular value methods and block matrix arguments have led to significant improvements in classical bounds such as the Cauchy-Schwarz matrix inequality and arithmetic-geometric mean type inequalities.

The purpose of this paper is to develop new upper bounds for expressions of the form

$$\omega(AX+YB), \qquad \|AXB^*\|, \qquad \|AXB^* - BYA^*\|,$$

and related quantities, particularly when some matrices are positive semidefinite. Our approach combines block matrix techniques, interpolating inequalities, and refined numerical radius estimates to obtain bounds that improve and extend several known results in the literature.

Among the most interesting contributions of this work is a new parametrized refinement for

$$\omega(AX+YB),$$

which yields, when A, B are positive semidefinite,

$$\omega(AX+YB) \leqslant \sqrt{\|A+B\|\left(\omega(|X^*|^2A) + \omega(|Y|^2B)\right)},$$

providing a substantial counterpart of the classical bound

$$\omega(AX+YB) \leqslant \|AX+YB\|.$$

We further utilize the numerical radius to obtain refined estimates for $\|AXB^*\|$ when X is positive semidefinite, where we show that if X is positive semidefinite, then

$$\|AXB^*\|^2 \leqslant \frac{1}{4}\left(\omega\left(XA^*A\right) + \|XA^*A\|\right)\left(\omega\left(XB^*B\right) + \|XB^*B\|\right).$$

This improves the celebrated matrix Cauchy-Schwarz inequality of Bhatia and Davis [9]. In fact, this follows upon finding a refined version of the interpolation result of [8], where the numerical radius happens to work better than the spectral norm, provided that X is positive semidefinite. In addition, we establish new singular value inequalities for commutator-type expressions such as $AXB^* - CYD^*$, which generalize earlier results on products of matrices and provide quantitative refinements of norm

bounds for anti-commutators. The matrix Cauchy-Schwarz inequality and its refinements have been studied from different perspectives, as can be seen in [4, 8, 9, 18, 23] and the references therein.

The paper is organized as follows. In Section 2, we derive refined numerical radius inequalities for $AX+YB$ and establish new bounds for $\|AXB^*\|$ under positivity assumptions. These results yield improvements of several known inequalities in the literature. In Section 3, we develop singular value inequalities for commutator-type expressions and obtain refined spectral norm bounds for sums and differences of matrix products. The paper concludes with remarks comparing our estimates to existing bounds and highlighting cases where strict improvements occur.

We will also need several auxiliary results from the literature that will serve as key tools in our proofs. In particular, we rely on refined arithmetic-geometric mean type inequalities, comparison principles for the numerical radius under suitable Hermitian-type assumptions, and singular value inequalities for products of matrices and block matrices. These known estimates will be repeatedly invoked to control mixed terms such as $AX+YB$ and AXB^*, and to connect numerical radius bounds with norm and singular value estimates.

The first lemma follows from the well-known Cauchy-Schwarz inequality and the second lemma follows from the classical Young's inequality for scalars, noting that $ab=\left(a^q b^{1-q}\right)\left(a^{1-q}b^q\right)$.

LEMMA 1.1. *Let $u,v,w,z\in\mathbb{C}^n$. Then*

$$|\langle u,v\rangle+\langle w,z\rangle|^2\leqslant\left(\langle u,u\rangle+\langle w,w\rangle\right)\left(\langle v,v\rangle+\langle z,z\rangle\right).$$

LEMMA 1.2. *Let a,b be nonnegative real numbers and $q\in[0,1]$. Then*

$$ab\leqslant(qa+(1-q)b)((1-q)a+qb).$$

LEMMA 1.3. [7] *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be such that AB is Hermitian. Then*

$$\omega(AB)\leqslant\omega(BA).$$

LEMMA 1.4. [15] *Let $A,B\in M_n(\mathbb{C})$. Then*

$$\omega(AB)\leqslant\frac{1}{2}\left\|AA^*+B^*B\right\|.$$

LEMMA 1.5. [8] *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ and $q\in[0,1]$. Then*

$$\|AB^*\|^2\leqslant\left\|qA^*A+(1-q)B^*B\right\|\left\|(1-q)A^*A+qB^*B\right\|.$$

LEMMA 1.6. [14] *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be such that AB is Hermitian. Then*

$$\|AB\|\leqslant\|\Re(BA)\|.$$

LEMMA 1.7. [5] *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\|\Re(A)\| \leqslant \frac{1}{2}\left(\omega(A) + \|A\|\right).$$

Replacing A by iA in Lemma 1.7, we have the following lemma.

LEMMA 1.8. *Let $A \in M_n(\mathbb{C})$. Then*

$$\|\Im(A)\| \leqslant \frac{1}{2}\left(\omega(A) + \|A\|\right).$$

LEMMA 1.9. [1] *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite. Then*

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) = \frac{1}{2}\|A+B\|.$$

LEMMA 1.10. [10] *Let $A, B \in M_n(\mathbb{C})$. Then*

$$s_j(AB^*) \leqslant \frac{1}{2} s_j(A^*A + B^*B)$$

for $j = 1, 2, \ldots, n$.

LEMMA 1.11. [22, p. 275] *Let $A, B \in M_n(\mathbb{C})$. Then*

$$s_j(A+B) \leqslant s_j(A) + \|B\|$$

for $j = 1, 2, \ldots, n$.

2. Numerical radius and norm refinements via positivity

We now present our first main result, which establishes a refined interpolation inequality for the numerical radius of mixed matrix expressions of the form $AX + YB$. The parameter $q \in [0,1]$ governs a symmetric interpolation between the contributions of A and B, combining spectral norm and numerical radius terms in a structured product form that separates their individual effects. Unlike the classical estimate $\omega(AX+YB) \leqslant \|AX+YB\|$, the obtained bound incorporates additional structural corrections involving $|X^*|^2 A$ and $|Y|^2 B$, leading to sharper estimates under positivity assumptions. Moreover, this result extends Audenaert's interpolation inequality [8], which bridges the arithmetic-geometric mean and Cauchy-Schwarz norm inequalities, by transferring the interpolative mechanism to the numerical radius setting and providing a more flexible framework that yields several improvements and corollaries as special cases. We point out that this interpolation idea received considerable attention due to its role in bridging the gap between two major matrix inequalities, as one can see in [3, 4, 8, 23]. The significance of the following result lies in how the numerical radius is incorporated into these bounds.

THEOREM 2.1. *Let $A,B,X,Y \in M_n(\mathbb{C})$, with A and B positive semidefinite. If $q \in [0,1]$, then*

$$\omega^2(AX+YB) \leqslant \Big(\|(1-q)A+qB\| + q\,\omega(|X^*|^2A) + (1-q)\,\omega(|Y|^2B) \Big) \times \Big(\|qA+(1-q)B\| + (1-q)\,\omega(|X^*|^2A) + q\,\omega(|Y|^2B) \Big).$$

Proof. Let $x \in \mathbb{C}^n$ be a unit vector. Put $P = B + X^*AX$ and $Q = A + YBY^*$. We notice that P and Q are positive semidefinite. By Lemmas 1.1 and 1.2,

$$\begin{aligned} |\langle (AX+YB)x,x\rangle|^2 &= |\langle A^{1/2}Xx, A^{1/2}x\rangle + \langle B^{1/2}x, B^{1/2}Y^*x\rangle|^2 \\ &\leqslant \langle Px,x\rangle\langle Qx,x\rangle \\ &\leqslant \langle (qP+(1-q)Q)x,x\rangle\langle ((1-q)P+qQ)x,x\rangle. \end{aligned}$$

Taking the maximum over all unit vectors $x \in \mathbb{C}^n$ gives

$$\begin{aligned} \omega^2(AX+YB) &\leqslant \big\|q(B+X^*AX)+(1-q)(A+YBY^*)\big\| \\ &\quad \times \big\|(1-q)(B+X^*AX)+q(A+YBY^*)\big\| \qquad (2.1) \\ &\leqslant \Big(\|qB+(1-q)A\| + \|qX^*AX\| + \|(1-q)YBY^*\| \Big) \\ &\quad \times \Big(\|(1-q)B+qA\| + \|(1-q)X^*AX\| + \|qYBY^*\| \Big) \\ &= \Big(\|qB+(1-q)A\| + q\,\|X^*AX\| + (1-q)\,\|YBY^*\| \Big) \\ &\quad \times \Big(\|(1-q)B+qA\| + (1-q)\,\|X^*AX\| + q\,\|YBY^*\| \Big) \\ &= \Big(\|qB+(1-q)A\| + q\,\omega(X^*AX) + (1-q)\,\omega(YBY^*) \Big) \\ &\quad \times \Big(\|(1-q)B+qA\| + (1-q)\,\omega(X^*AX) + q\,\omega(YBY^*) \Big) \\ &\leqslant \Big(\|qB+(1-q)A\| + q\,\omega(|X^*|^2A) + (1-q)\,\omega(|Y|^2B) \Big) \\ &\quad \times \Big(\|(1-q)B+qA\| + (1-q)\,\omega(|X^*|^2A) + q\,\omega(|Y|^2B) \Big), \end{aligned}$$

where the last step uses Lemma 1.3. This proves the desired inequality. □

The following corollary provides a particularly transparent consequence of Theorem 2.1 obtained by choosing $q = \frac{1}{2}$ and optimizing with respect to a scalar parameter. It yields symmetric and practically applicable upper bounds for $\omega(AX+YB)$ and $\omega(AX+X^*B)$ expressed solely in terms of $\|A+B\|$ and refined numerical radius corrections.

COROLLARY 2.2. *Let $A,B,X,Y \in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\omega(AX+YB) \leqslant \sqrt{\|A+B\| \left(\omega\left(|X^*|^2A\right) + \omega\left(|Y|^2B\right) \right)} \qquad (2.2)$$

and

$$\omega(AX+X^*B) \leqslant \sqrt{\|A+B\|\,\omega\left(|X^*|^2(A+B)\right)}. \tag{2.3}$$

Proof. Letting $q=\frac{1}{2}$ in Theorem 2.1, we have

$$\omega(AX+YB) \leqslant \frac{1}{2}\left(\|A+B\|+\omega\left(|X^*|^2A\right)+\omega\left(|Y|^2B\right)\right).$$

Now, by replacing X by tX and Y by tY and taking the minimum over $t>0$, we obtain

$$\omega(AX+YB) \leqslant \sqrt{\|A+B\|\left(\omega\left(|X^*|^2A\right)+\omega\left(|Y|^2B\right)\right)},$$

which proves (2.2). To prove (2.3), let $q=\frac{1}{2}$ and $Y=X^*$ in (2.1) and use the triangle inequality to obtain

$$\begin{aligned}\omega(AX+X^*B) &\leqslant \frac{1}{2}\left(\|A+B\|+\|X^*(A+B)X\|\right)\\ &= \frac{1}{2}\left(\|A+B\|+\omega\left(X^*(A+B)X\right)\right)\\ &\leqslant \frac{1}{2}\left(\|A+B\|+\omega\left(|X^*|^2(A+B)\right)\right),\end{aligned}$$

where we have used Lemma 1.3 to obtain the last inequality. Again, by replacing X by tX and taking the minimum over $t>0$, we have

$$\omega(AX+X^*B) \leqslant \sqrt{\|A+B\|\,\omega\left(|X^*|^2(A+B)\right)}$$

which proves (2.3). This completes the proof. □

REMARK 2.3. Let $A,B,X\in M_n(\mathbb{C})$ be such that A and B are positive semidefinite. It follows by (1.1), the triangle inequality, and the submultiplicativity of the spectral norm that

$$\omega(AX+X^*B) \leqslant \|X\|(\|A\|+\|B\|). \tag{2.4}$$

The inequality (2.2) refines (2.4). In fact, by letting $Y=X^*$ in (2.2), we have

$$\begin{aligned}\omega(AX+X^*B) &\leqslant \sqrt{\|A+B\|\left(\omega(|X^*|^2A)+\omega(|X^*|^2B)\right)}\\ &\leqslant \sqrt{\|A+B\|\left(\||X^*|^2A\|+\||X^*|^2B\|\right)}\\ &\leqslant \sqrt{\|A+B\|\,\|X\|^2(\|A\|+\|B\|)}\\ &\leqslant \sqrt{\|X\|^2(\|A\|+\|B\|)^2}\\ &= \|X\|(\|A\|+\|B\|),\end{aligned}$$

which proves the assertion.

REMARK 2.4. It follows by (1.1) that $\omega(AX+YB) \leqslant \|AX+YB\|$. Here, we give an example to show that (2.2) may give a better estimate than this bound. Indeed, let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Numerical calculations show that $\omega(AX+YB) \approx 3.16228$,

$$\sqrt{\|A+B\|\left(\omega\left(|X^*|^2A\right)+\omega\left(|Y|^2B\right)\right)} \approx 5.27304, \quad \|AX+YB\| \approx 6.16228.$$

Another related result involving an upper bound for $\omega(AX+YB)$ can be seen in the following theorem which is based on Lemma 1.4.

THEOREM 2.5. *Let $A,B,X,Y \in M_n(\mathbb{C})$. Then*

$$\omega(AX+YB) \leqslant \sqrt{\|AA^*+B^*B\|\,\|YY^*+X^*X\|}.$$

Proof. We have

$$\begin{aligned} \omega(AX+YB) &= \omega\left(\begin{bmatrix} A & Y \\ O & O \end{bmatrix}\begin{bmatrix} X & O \\ B & O \end{bmatrix}\right) \\ &\leqslant \frac{1}{2}\left\|\begin{bmatrix} A & Y \\ O & O \end{bmatrix}\begin{bmatrix} A^* & O \\ Y^* & O \end{bmatrix}+\begin{bmatrix} X^* & B^* \\ O & O \end{bmatrix}\begin{bmatrix} X & O \\ B & O \end{bmatrix}\right\| \\ &= \frac{1}{2}\|AA^*+YY^*+X^*X+B^*B\| \\ &\leqslant \frac{1}{2}\|AA^*+B^*B\|+\frac{1}{2}\|YY^*+X^*X\|. \end{aligned}$$

Now, by replacing X by tX and Y by tY and taking the minimum over $t>0$, we have

$$\omega(AX+YB) \leqslant \sqrt{\|AA^*+B^*B\|\,\|YY^*+X^*X\|}. \quad \square$$

REMARK 2.6. Let $A,B,X,Y \in M_n(\mathbb{C})$. It is known that [13] (see [18] also)

$$\|AX+YB\| \leqslant \sqrt{\|AA^*+YY^*\|\,\|X^*X+B^*B\|}.$$

Comparing this with Theorem 2.5, it might be thought that one can have the bound

$$\|AX+YB\| \leqslant \sqrt{\|AA^*+B^*B\|\,\|YY^*+X^*X\|}.$$

The following example confutes this claim. Indeed, if we let

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -3 \\ -1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}, \quad Y = \begin{bmatrix} -3 & 3 \\ 3 & 1 \end{bmatrix},$$

we find that

$$\|AX+YB\| \approx 25.0333, \quad \sqrt{\|AA^*+B^*B\|\,\|YY^*+X^*X\|} \approx 22.9734.$$

Another observation is that this example provides a clear significance of the bound we found in Theorem 2.5, as it can provide better estimates than the standard estimate that $\omega(AX+YB) \leqslant \|AX+YB\|$. We emphasize that other examples show opposite conclusions, which reveals that the bounds are incomparable in general with the existing bounds.

We now turn to deriving refined upper bounds for the spectral norm of matrix products of the form $\|AXB^*\|$, with particular emphasis on the case where X is positive semidefinite. Such expressions arise naturally in matrix inequalities, matrix interpolation problems, and in refinements of the classical Cauchy-Schwarz inequality for matrices. Our next result provides a parametrized estimate that interpolates between the contributions of A^*A and B^*B, and combines numerical radius and norm terms in a symmetric product structure. This provides a considerable generalization of [8, Theorem 1], where this result is obtained upon letting $X=I$ in Theorem 2.7.

THEOREM 2.7. *Let $A,B,X \in M_n(\mathbb{C})$ be such that X is positive semidefinite. If $q \in [0,1]$, then*

$$\|AXB^*\|^2 \leqslant \frac{1}{4}\left(\omega\left(X\left(qA^*A+(1-q)B^*B\right)\right)+\|X\left(qA^*A+(1-q)B^*B\right)\|\right)\times \left(\omega\left(X\left((1-q)A^*A+qB^*B\right)\right)+\|X\left((1-q)A^*A+qB^*B\right)\|\right). \tag{2.5}$$

Proof. We have

$$\begin{aligned}
\|AXB^*\|^2 &= \left\|AX^{1/2}(BX^{1/2})^*\right\|^2 \\
&\leqslant \left\|qX^{1/2}A^*AX^{1/2}+(1-q)X^{1/2}B^*BX^{1/2}\right\| \\
&\quad\times \left\|(1-q)X^{1/2}A^*AX^{1/2}+qX^{1/2}B^*BX^{1/2}\right\| \text{ (by Lemma 1.5)} \\
&= \left\|X^{1/2}\left(qA^*A+(1-q)B^*B\right)X^{1/2}\right\|\left\|X^{1/2}\left((1-q)A^*A+qB^*B\right)X^{1/2}\right\| \\
&\leqslant \|\Re\left(X\left(qA^*A+(1-q)B^*B\right)\right)\| \\
&\quad\times \|\Re\left(X\left((1-q)A^*A+qB^*B\right)\right)\| \text{ (by Lemma 1.6)} \\
&\leqslant \frac{1}{4}\left(\omega\left(X\left(qA^*A+(1-q)B^*B\right)\right)+\|X\left(qA^*A+(1-q)B^*B\right)\|\right) \\
&\quad\times\left(\omega\left(X\left((1-q)A^*A+qB^*B\right)\right)+\|X\left((1-q)A^*A+qB^*B\right)\|\right),
\end{aligned}$$

where we have used Lemma 1.7. This completes the proof. □

Letting $q=\frac{1}{2}$ in the inequality (2.5), we have the following corollary.

COROLLARY 2.8. *Let $A,B,X \in M_n(\mathbb{C})$ be such that X is positive semidefinite. Then*

$$\|AXB^*\| \leqslant \frac{1}{4}\left(\omega\left(X\left(A^*A+B^*B\right)\right)+\|X\left(A^*A+B^*B\right)\|\right). \tag{2.6}$$

REMARK 2.9. The positive semidefiniteness of X cannot be dropped in (2.6). Indeed, if we let

$$A=\begin{bmatrix}1&1\\1&0\end{bmatrix},\quad B=\begin{bmatrix}1&1\\0&0\end{bmatrix},\quad X=\begin{bmatrix}0&1\\0&1\end{bmatrix},$$

we can see that X is not positive semidefinite, and that

$$\|AXB^*\|\approx 2.23607,\quad \frac{1}{4}\left(\omega\left(X\left(A^*A+B^*B\right)\right)+\|X\left(A^*A+B^*B\right)\|\right)=2.$$

This means that Theorem 2.7 is not necessarily true if X is not positive semidefinite, because (2.6) is a special case of the theorem.

REMARK 2.10. It was shown in [6] that if $A,B,X\in M_n(\mathbb{C})$ are such that X is positive semidefinite, then

$$s_j\left(AXB^*\right)\leqslant\frac{1}{2}\|A^*A+B^*B\|\,s_j\left(X\right) \tag{2.7}$$

for $j=1,2,\ldots,n$. Letting $j=1$ in (2.7), we have

$$\|AXB^*\|\leqslant\frac{1}{2}\|A^*A+B^*B\|\,\|X\|. \tag{2.8}$$

The inequality (2.6) refines (2.8). In fact, by (2.6), we have

$$\begin{aligned}\|AXB^*\|&\leqslant\frac{1}{4}\left(\omega\left(X\left(A^*A+B^*B\right)\right)+\|X\left(A^*A+B^*B\right)\|\right)\\&\leqslant\frac{1}{4}\left(\|X\left(A^*A+B^*B\right)\|+\|X\left(A^*A+B^*B\right)\|\right)\\&=\frac{1}{2}\|X\left(A^*A+B^*B\right)\|\\&\leqslant\frac{1}{2}\|A^*A+B^*B\|\,\|X\|.\end{aligned}$$

Letting $q=0$ (or $q=1$) in (2.5), we have the following corollary.

Now we are ready to present the following refinement of the celebrated Cauchy-Schwarz inequality for matrices, in the case that X is positive semidefinite.

COROLLARY 2.11. *Let $A,B,X\in M_n(\mathbb{C})$ be such that X is positive semidefinite. Then*

$$\|AXB^*\|^2\leqslant\frac{1}{4}\left(\omega\left(XA^*A\right)+\|XA^*A\|\right)\left(\omega\left(XB^*B\right)+\|XB^*B\|\right). \tag{2.9}$$

Moreover, this bound is sharp.

Proof. The desired inequality follows immediately from Theorem 2.7. For sharpness, one can consider the matrices

$$A=\begin{bmatrix}0&0\\1&1\end{bmatrix},\quad B=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad X=\begin{bmatrix}2&2\\2&2\end{bmatrix}.\quad\square$$

REMARK 2.12. We point out that (2.9) is not necessarily true if X is not positive semidefinite. For example, if we let

$$A=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad B=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad X=\begin{bmatrix}1&1\\0&1\end{bmatrix},$$

we find that

$$\|AXB^*\|^2=2,\quad \frac{1}{4}\left(\omega\left(XA^*A\right)+\|XA^*A\|\right)\left(\omega\left(XB^*B\right)+\|XB^*B\|\right)\approx 1.55902.$$

REMARK 2.13. It was proved in [9] that

$$\|AXB^*\|^2\leqslant\|A^*AX\|\,\|XB^*B\| \tag{2.10}$$

for $A,B,X\in M_n(\mathbb{C})$, which generalizes the well-known Cauchy-Schwarz inequality for matrices. The inequality (2.9) refines (2.10) for the case of positive semidefinite matrix X. In fact, by (2.9), we have

$$\begin{aligned}\|AXB^*\|^2&\leqslant\frac{1}{4}\left(\omega\left(XA^*A\right)+\|XA^*A\|\right)\left(\omega\left(XB^*B\right)+\|XB^*B\|\right)\\&\leqslant\frac{1}{4}\left(\|XA^*A\|+\|XA^*A\|\right)\left(\|XB^*B\|+\|XB^*B\|\right)\\&=\|A^*AX\|\,\|XB^*B\|.\end{aligned}$$

This shows that a particular case of (2.5) presents a refinement of (2.10) for the case of positive semidefinite matrix X.

We conclude this section with a refined upper bound for the norm of mixed sums of matrices of the form $AX+BY$. Such expressions arise naturally in perturbation theory, matrix interpolation, and in estimates related to the triangle inequality for matrices. The following result provides another upper bound for $\|AX+BY\|$ by incorporating joint information from AA^*+BB^* and X^*X+Y^*Y, together with a numerical radius correction term. As special cases, we obtain new bounds for $\|A+B\|^2$.

THEOREM 2.14. *Let $A,B,X,Y\in M_n(\mathbb{C})$. Then*

$$\begin{aligned}&\|AX+BY\|^2\\&\leqslant\frac{1}{2}\left(\begin{array}{c}\|AA^*+BB^*\|\,\|X^*X+Y^*Y\|\\+\omega\left(\begin{bmatrix}XX^*A^*A+XY^*B^*A & XX^*A^*B+XY^*B^*B\\YX^*A^*A+YY^*B^*A & YX^*A^*B+YY^*B^*B\end{bmatrix}\right)\end{array}\right),\end{aligned} \tag{2.11}$$

$$\|A+B\|^2\leqslant\frac{1}{2}\left(\begin{array}{c}2\|AA^*+BB^*\|\\+\max\left(\omega\left(A^*A+B^*A\right),\omega\left(A^*B+B^*B\right)\right)\\+\frac{1}{2}\|B^*B+A^*A\|+\|A^*B\|\end{array}\right),$$

and

$$\|A+B\|^2\leqslant\frac{1}{2}\left(\begin{array}{c}2\|AA^*+BB^*\|\\+\max\left(\|A\|^2,\|B\|^2\right)\\+\frac{1}{2}\|B^*B+A^*A\|+2\|A^*B\|\end{array}\right). \tag{2.12}$$

Proof. We have

$$\begin{aligned}\|AB^*\|^2 = \||AB^*|\|^2 &= \left\||AB^*|^2\right\| = \|BA^*AB^*\| \\ &\leqslant \|\Re\left(B^*BA^*A\right)\| \\ &\leqslant \frac{1}{2}\left(\|B^*BA^*A\| + \omega\left(B^*BA^*A\right)\right) \\ &\quad \text{(by Lemma 1.7)} \\ &\leqslant \frac{1}{2}\left(\|B^*B\|\,\|A^*A\| + \omega\left(B^*BA^*A\right)\right).\end{aligned}$$

So, we get

$$\|AB^*\|^2 \leqslant \frac{1}{2}\left(\|B^*B\|\,\|A^*A\| + \omega\left(B^*BA^*A\right)\right). \tag{2.13}$$

Replacing A by $\begin{bmatrix} A & B \\ O & O \end{bmatrix}$ and B by $\begin{bmatrix} X^* & Y^* \\ O & O \end{bmatrix}$ in the inequality (2.13), we have

$$\|AX+BY\|^2 \leqslant \frac{1}{2}\left(\begin{array}{c} \|AA^*+BB^*\|\,\|X^*X+Y^*Y\| \\ +\omega\left(\begin{bmatrix} XX^*A^*A+XY^*B^*A & XX^*A^*B+XY^*B^*B \\ YX^*A^*A+YY^*B^*A & YX^*A^*B+YY^*B^*B \end{bmatrix}\right) \end{array}\right),$$

which proves (2.11). Letting $X = Y = I$ in (2.11), we have

$$\begin{aligned}\|A+B\|^2 &\leqslant \frac{1}{2}\left(\begin{array}{c} 2\|AA^*+BB^*\| \\ +\omega\left(\begin{bmatrix} A^*A+B^*A & A^*B+B^*B \\ A^*A+B^*A & A^*B+B^*B \end{bmatrix}\right) \end{array}\right) \\ &\leqslant \frac{1}{2}\left(\begin{array}{c} 2\|AA^*+BB^*\| \\ +\omega\left(\begin{bmatrix} A^*A+B^*A & O \\ O & A^*B+B^*B \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} O & B^*B \\ A^*A & O \end{bmatrix}\right) \\ +\omega\left(\begin{bmatrix} O & A^*B \\ B^*A & O \end{bmatrix}\right) \end{array}\right) \\ &\leqslant \frac{1}{2}\left(\begin{array}{c} 2\|AA^*+BB^*\| \\ +\max\left(\omega\left(A^*A+B^*A\right), \omega\left(A^*B+B^*B\right)\right) \\ +\frac{1}{2}\|B^*B+A^*A\| + \|A^*B\| \end{array}\right) \\ &\quad \text{(by Lemma 1.9)} \\ &\leqslant \frac{1}{2}\left(\begin{array}{c} 2\|AA^*+BB^*\| \\ +\max\left(\|A\|^2, \|B\|^2\right) + \|A^*B\| \\ +\frac{1}{2}\|B^*B+A^*A\| + \|A^*B\| \end{array}\right) \\ &= \frac{1}{2}\left(\begin{array}{c} 2\|AA^*+BB^*\| \\ +\max\left(\|A\|^2, \|B\|^2\right) \\ +\frac{1}{2}\|B^*B+A^*A\| + 2\|A^*B\| \end{array}\right),\end{aligned}$$

which proves the inequality (2.12). □

3. Singular value inequalities for commutator-type expressions

In this section, we investigate singular value inequalities for commutator-type expressions and mixed matrix differences. Our approach relies on block matrix techniques combined with norm and numerical radius estimates to derive refined upper bounds for quantities of the form $AB^* - BA^*$ and $AXB^* - CYD^*$. These results extend classical inequalities related to the Cauchy-Schwarz inequality and commutator estimates, and provide sharper bounds that incorporate structural information from $A^*A + B^*B$ and related matrix combinations. The obtained inequalities also yield improvements over known norm estimates as particular cases. In what follows, $\oplus$ denotes the standard direct sum of matrices.

THEOREM 3.1. *Let $A, B \in M_n(\mathbb{C})$. Then*

$$2s_j(AB^* - BA^*) \leqslant s_j((A^*A + B^*B) \oplus (A^*A + B^*B)) + \|A^*B - B^*A\|$$

for $j = 1, 2, \ldots, n$.

Proof. For $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
2s_j(AB^* - BA^*) &= 2s_j\left(\begin{bmatrix} A & B \\ O & O \end{bmatrix}\begin{bmatrix} B^* & O \\ -A^* & O \end{bmatrix}\right) \\
&\leqslant s_j\left(\begin{bmatrix} A^* & O \\ B^* & O \end{bmatrix}\begin{bmatrix} A & B \\ O & O \end{bmatrix} + \begin{bmatrix} B^* & O \\ -A^* & O \end{bmatrix}\begin{bmatrix} B & -A \\ O & O \end{bmatrix}\right) \\
&\quad \text{(by Lemma 1.10)} \\
&= s_j\left(\begin{bmatrix} A^*A & A^*B \\ B^*A & B^*B \end{bmatrix} + \begin{bmatrix} B^*B & -B^*A \\ -A^*B & A^*A \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} A^*A + B^*B & A^*B - B^*A \\ B^*A - A^*B & A^*A + B^*B \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} A^*A + B^*B & O \\ O & A^*A + B^*B \end{bmatrix} + \begin{bmatrix} O & A^*B - B^*A \\ B^*A - A^*B & O \end{bmatrix}\right) \\
&\leqslant s_j\left(\begin{bmatrix} A^*A + B^*B & O \\ O & A^*A + B^*B \end{bmatrix}\right) + \left\|\begin{bmatrix} O & A^*B - B^*A \\ B^*A - A^*B & O \end{bmatrix}\right\| \\
&\quad \text{(by Lemma 1.11)} \\
&= s_j((A^*A + B^*B) \oplus (A^*A + B^*B)) + \|A^*B - B^*A\|,
\end{aligned}$$

as required. □

We next extend the preceding commutator-type estimates to mixed matrix differences of the form $AXB^* - CYD^*$, where the intermediate matrices X and Y are positive

semidefinite. The following result provides a singular value bound that separates the structural contributions of (A,B) and (C,D) through norm terms involving A^*A+B^*B and C^*C+D^*D, together with a coupling term A^*C-B^*D.

THEOREM 3.2. *Let $A,B,C,D,X,Y \in M_n(\mathbb{C})$ be such that X and Y are positive semidefinite. Then*

$$s_j(AXB^*-CYD^*) \leqslant \frac{1}{2}\begin{pmatrix} \max\left(\|A^*A+B^*B\|, \|C^*C+D^*D\|\right) \\ +\|A^*C-B^*D\| \end{pmatrix} s_j(X\oplus Y) \tag{3.1}$$

for $j=1,2,\ldots,n$. Moreover,

$$\begin{aligned}\|AXB^*-BYA^*\| \leqslant &\left(\|A\|\,\|B\|+\frac{1}{2}\left(\omega(A^*B)+\|A^*B\|\right)\right) \\ &\times \max\left(\|X\|,\|Y\|\right).\end{aligned} \tag{3.2}$$

Proof. For $j=1,2,\ldots,n$, we have

$$\begin{aligned}
s_j(AXB^*-CYD^*) &= s_j\left(\begin{bmatrix}A & C\\ O & O\end{bmatrix}\begin{bmatrix}X & O\\ O & Y\end{bmatrix}\begin{bmatrix}B^* & O\\ -D^* & O\end{bmatrix}\right)\\
&\leqslant \frac{1}{2}\left\|\begin{bmatrix}A^* & O\\ C^* & O\end{bmatrix}\begin{bmatrix}A & C\\ O & O\end{bmatrix}+\begin{bmatrix}B^* & O\\ -D^* & O\end{bmatrix}\begin{bmatrix}B & -D\\ O & O\end{bmatrix}\right\| s_j(X\oplus Y)\\
&\quad \text{(by inequality (2.7))}\\
&= \frac{1}{2}\left\|\begin{bmatrix}A^*A+B^*B & A^*C-B^*D\\ C^*A-D^*B & C^*C+D^*D\end{bmatrix}\right\| s_j(X\oplus Y)\\
&= \frac{1}{2}\left\|\begin{bmatrix}A^*A+B^*B & O\\ O & C^*C+D^*D\end{bmatrix}+\begin{bmatrix}O & A^*C-B^*D\\ C^*A-D^*B & O\end{bmatrix}\right\| s_j(X\oplus Y)\\
&\leqslant \frac{1}{2}\begin{pmatrix}\left\|\begin{bmatrix}A^*A+B^*B & O\\ O & C^*C+D^*D\end{bmatrix}\right\| \\ +\left\|\begin{bmatrix}O & A^*C-B^*D\\ C^*A-D^*B & O\end{bmatrix}\right\|\end{pmatrix} s_j(X\oplus Y)\\
&= \frac{1}{2}\begin{pmatrix}\max\left(\|A^*A+B^*B\|, \|C^*C+D^*D\|\right) \\ +\|A^*C-B^*D\|\end{pmatrix} s_j(X\oplus Y),
\end{aligned}$$

which proves (3.1). To prove (3.2), let $j=1$, $D=A$, and $C=B$ in (3.1). This yields

$$\begin{aligned}\|AXB^*-BYA^*\| &\leqslant \frac{1}{2}\left(\|A^*A+B^*B\|+2\|\Im(A^*B)\|\right)\max\left(\|X\|,\|Y\|\right)\\ &\leqslant \frac{1}{2}\left(\|A\|^2+\|B\|^2+2\|\Im(A^*B)\|\right)\max\left(\|X\|,\|Y\|\right).\end{aligned}$$

Now, by replacing the matrix A by tA and taking the minimum over $t > 0$, we have

$$\begin{aligned}\|AXB^* - BYA^*\| &\leqslant \left(\|A\|\|B\| + \|\Im(A^*B)\|\right)\max\left(\|X\|, \|Y\|\right)\\ &\leqslant \left(\|A\|\|B\| + \frac{1}{2}\left(\omega(A^*B) + \|A^*B\|\right)\right)\max\left(\|X\|, \|Y\|\right)\\ &\quad \text{(by Lemma 1.8).}\end{aligned}$$

This completes the proof. □

REMARK 3.3. Following the proof of Theorem 3.2, we see that

$$\begin{bmatrix} A^*A + B^*B & A^*C - B^*D \\ C^*A - D^*B & C^*C + D^*D \end{bmatrix}$$

is positive semidefinite. Consequently (see [11]),

$$\left\| \begin{bmatrix} A^*A + B^*B & A^*C - B^*D \\ C^*A - D^*B & C^*C + D^*D \end{bmatrix} \right\| \leqslant \|A^*A + B^*B\| + \|C^*C + D^*D\|.$$

This entails the other version

$$s_j(AXB^* - CYD^*) \leqslant \frac{1}{2}\left(\|A^*A + B^*B\| + \|C^*C + D^*D\|\right) s_j(X \oplus Y). \tag{3.3}$$

It is worth mentioning that the two bounds in (3.1) and (3.3) are incomparable, in general. To see this, consider

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It can be seen that, for these matrices,

$$\|A^*A + B^*B\| + \|C^*C + D^*D\| = 13,$$

and

$$\max\left(\|A^*A + B^*B\|, \|C^*C + D^*D\|\right) + \|A^*C - B^*D\| = 14.$$

This example shows that (3.3) is better than (3.1). On the other hand, if we let

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

we find that

$$\|A^*A + B^*B\| + \|C^*C + D^*D\| = 23,$$

and

$$\max\left(\|A^*A + B^*B\|, \|C^*C + D^*D\|\right) + \|A^*C - B^*D\| = 20,$$

showing that (3.1) can provide better estimates than (3.3). In conclusion, we see that (3.1) and (3.3) are incomparable, in general.

REMARK 3.4. The triangle inequality and the submultiplicativity of the spectral norm imply that

$$\|AXB^* - BYA^*\| \leqslant \|A\|\,\|B\|\,(\|X\| + \|Y\|)$$

for $A, B, X, Y \in M_n(\mathbb{C})$. In particular, if $Y = X$, we have

$$\|AXB^* - BXA^*\| \leqslant 2\,\|A\|\,\|B\|\,\|X\|\,. \tag{3.4}$$

For a positive semidefinite X, letting $Y = X$ in (3.2), we have

$$\|AXB^* - BXA^*\| \leqslant \left(\|A\|\,\|B\| + \frac{1}{2}\,(\omega(A^*B) + \|A^*B\|)\right)\|X\|\,, \tag{3.5}$$

which is a refinement of (3.4) when X is positive semidefinite. In fact, by (3.5), we have

$$\begin{aligned}\|AXB^* - BXA^*\| &\leqslant \left(\|A\|\,\|B\| + \frac{1}{2}\,(\omega(A^*B) + \|A^*B\|)\right)\|X\| \\ &\leqslant \left(\|A\|\,\|B\| + \frac{1}{2}\,(\|A^*B\| + \|A^*B\|)\right)\|X\| \\ &= (\|A\|\,\|B\| + \|A^*B\|)\,\|X\| \\ &\leqslant 2\,\|A\|\,\|B\|\,\|X\|\,.\end{aligned}$$

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