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N–SYMMETRIC HOLOMORPHIC FUNCTIONS AND A REVISIT OF THE SERIES $\sum_{k=1}^{\infty} k^m z^k$

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Abstract. In this note we study which series of the form $\sum_{j=0}^{\infty} a_j(\sqrt[n]{z})^j$ have an analytic extension from the slitted unit disk S to the whole disk. We also give a short proof that the function

$$R_n(z) := \sum_{k\in\mathbb{Z}} \frac{1}{(\log z + 2k\pi i)^n},$$

for $n \in \mathbb{N}_2$, is the trace on the slitted plane of a rational function with a single pole of order n at 1, and that $(n-1)!R_n(z)$ coincides on S with $(-1)^n \sum_{k=1}^{\infty} k^{n-1} z^k$.

1. Introduction

Throughout the note, by $\mathbb{N}$ we denote the set of positive natural numbers $\{1,2,3,\ldots\}$, by $\mathbb{Z}$ the set of integers, by $\mathbb{N}_k$ the set $\{n \in \mathbb{Z} : n \geqslant k\}$ where $k \in \mathbb{Z}$, by $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, that is, the set $\{z \in \mathbb{C} : |z| < 1\}$, and by S the slitted disk $\mathbb{D}\setminus(-1,0]$. The main object of the note, denoted by $q_n(z)$, will be the principal branch of the n-th root function $\sqrt[n]{z}$ on S, $n \in \mathbb{N}_2$, that is given by

$$q_n(re^{it}) = r^{1/n}e^{it/n} = e^{\frac{\log|z|+i\arg z}{n}},$$

where $z = re^{it}$ with $0 < r < 1$ and $-\pi < t = \arg z < \pi$.

In each introductory course it is proved that q_n is holomorphic in S. However q_n has no holomorphic extension to $\mathbb{D}$. This can be proved in a simple way. Namely, suppose that there is a holomorphic function $q : \mathbb{D} \to \mathbb{C}$ with

$$q^n(z) = z \quad \text{for } z \in \mathbb{D}.$$

The differentiability of q would imply that

$$nq^{n-1}(z)q'(z) = 1 \quad \text{for } z \in \mathbb{D},$$

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from which together with the assumption $n \in \mathbb{N}_2$, an obvious contradiction is obtained, as $q(0)$ equals necessarily 0.

Unfortunately, many publications are written in that way that the n-th root of the identity seems to have been considered as an analytic function on the whole unit disk (or deleted disk $\mathbb{D}\setminus\{0\}$); see, e.g., [2–4, 9]. In these papers formal series of the form

$$F(z) := \sum_{j=0}^{\infty} a_j (\sqrt[n]{z})^j$$

are considered, where $a_j \in \mathbb{C}$ for $j \in \mathbb{N}_0$.

The aim of Section 2 of the note is to give necessary and sufficient conditions on the coefficients a_j so that F admits a holomorphic extension from S to $\mathbb{D}$. Our result could be known, however, we were not able to find a reference for it, and in view of the content of the papers referenced above, it seems not to be common knowledge anymore.

Our result involves n-symmetric functions. Recall that for $\lambda = e^{2\pi i/n}$, where $n \in \mathbb{N}_2$, these are those functions $f : \mathbb{D} \to \mathbb{C}$ which satisfy the relations

$$f(\lambda^j z) = f(z) \quad \text{for } z \in \mathbb{D},$$

and $j = 0, 1, \ldots, n-1$. In particular, the class of 2-symmetric functions coincides with the class of even functions.

In Section 3 we deal with a similar issue involving the logarithm function and that appeared in [12]. There was considered the series

$$R_n(z) := \sum_{k\in\mathbb{Z}} \frac{1}{\big(\log z + 2k\pi i\big)^n},$$

for $n \in \mathbb{N}_2$, which converges only on the slitted plane $\widehat{S} := \mathbb{C}\setminus(-\infty, 0]$, the point 1 deleted (and not on $\mathbb{C}\setminus\{1\}$). We will present a short proof of the amazing known fact that this function is the trace of a rational function with a unique pole at 1 of order n and give its relation with the power series $\sum_{k=1}^{\infty} k^m z^k$. We will also discuss a way to extend this to the case $n = 1$. Finally, the result in Section 2 will be used to show that for $n, N \in \mathbb{N}_2$ the function

$$\sum_{k\in\mathbb{Z}} \frac{1}{\big(\log z + 2kN\pi i\big)^n}$$

has no analytic extension from S to $\mathbb{D}$, and consequently no such one from $\widehat{S}\setminus\{1\}$ to $\mathbb{C}\setminus\{1\}$.

2. N-th roots and N-symmetric functions

Let us go in medias res to our main result.

PROPOSITION 1. *Assume that $n \in \mathbb{N}_2$, and that the function $h(w) := \sum_{k=0}^{\infty} a_k w^k$ converges in $\mathbb{D}$. Then the function $h \circ q_n : S \to \mathbb{C}$ has a holomorphic extension to $\mathbb{D}$ if and only if $a_k = 0$ for $k \not\equiv 0 \pmod n$. That is, $h(w) = \sum_{j=0}^{\infty} b_j w^{nj}$ with $b_j := a_{nj}$, $j \in \mathbb{N}_0$, and h has an n-fold symmetry.*

Proof. "$\Rightarrow$" Let F be a holomorphic extension of $h \circ q_n$ to $\mathbb{D}$. Let $re^{it} \in S$. Then

$$F(re^{it}) = h(r^{1/n}e^{it/n}).$$

Due to the continuity of the function h,

$$\lim_{t\to\pi^-} h(r^{1/n}e^{it/n}) = h(r^{1/n}e^{i\pi/n}) \quad \text{and} \quad \lim_{t\to-\pi^+} h(r^{1/n}e^{it/n}) = h(r^{1/n}e^{-i\pi/n}).$$

As F is holomorphic in $\mathbb{D}$, so continuous, both values coincide with $F(-r)$. In particular, we have $h(\mu x) = h(\overline{\mu}x)$ for $x \in [0,1)$, where $\mu := e^{i\pi/n}$. By the identity theorem, we obtain $h(\mu z) = h(\overline{\mu}z)$ for all $z \in \mathbb{D}$. Taking $z = \mu w$ in the last relation, we get

$$h(\lambda w) = h(w) \text{ for } w \in \mathbb{D}, \tag{1}$$

where $\lambda := e^{i2\pi/n} = \mu^2$ is the first non real n-th root of 1.

From (1) and a simple inductive argument, we have

$$h(\lambda^j w) = h(w) \text{ for } w \in \mathbb{D},$$

and each $j \in \mathbb{N}_0$.

By differentiating relation (1), we have

$$h^{(k)}(w) = \lambda^k h^{(k)}(\lambda w) \text{ for } w \in \mathbb{D},$$

and each $k \in \mathbb{N}_0$, from which for $w = 0$, it follows that

$$h^{(k)}(0) = \lambda^k h^{(k)}(0). \tag{2}$$

Since

$$\lambda^k \neq 1 \quad \text{for } k \not\equiv 0 \pmod{n},$$

from (2), we have

$$a_k = 0 \quad \text{for } k \not\equiv 0 \pmod{n},$$

as claimed. In particular

$$h(w) = \sum_{j=0}^{\infty} a_{nj}(w^n)^j, \quad w \in \mathbb{D}. \tag{3}$$

"$\Leftarrow$" Conversely, if $a_k = 0$ for $k \not\equiv 0 \pmod{n}$, then, with $b_j := a_{nj}$, $h(w) = \sum_{j=0}^{\infty} b_j w^{nj}$ is convergent on $\mathbb{D}$, and so holomorphic on $\mathbb{D}$. Thus, for $z \in S$,

$$h \circ q_n(z) = \sum_{j=0}^{\infty} b_j z^j,$$

which has an obvious analytic extension to $\mathbb{D}$. $\square$

3. An infinite sum involving different branches of the logarithm

A similar issue involving directly the logarithm appeared in [12]. For $k \in \mathbb{Z}$, let $\log_{(k)} z$ be the k-th canonical branch of the logarithm, defined by

$$\log_{(k)} z := \log|z| + i\arg z + 2k\pi i, \quad -\pi < \arg z < \pi.$$

Note that $\log_{(0)} z = \log z$, the principal branch. Let $\widehat{S} := \mathbb{C} \setminus (-\infty, 0]$ be the slitted plane. There was considered the series (erroneously thought to be defined and holomorphic on $\mathbb{C} \setminus \{0,1\}$)

$$R_n(z) := \sum_{k\in\mathbb{Z}} \frac{1}{\left(\log_{(k)} z\right)^n} = \sum_{k\in\mathbb{Z}} \frac{1}{\left(\log z + 2k\pi i\right)^n}, \quad z \in \widehat{S},$$

for $n \in \mathbb{N}_2$, that converges on $\widehat{S} \setminus \{1\}$ (the point $z = 1$ is a zero of the term $\log_{(0)} z$). Below we give a short proof that $R_n(z)$ has a meromorphic extension $R_n(z)^*$ to a rational function of the form $\frac{P_n(z)}{(1-z)^n}$ for some polynomial P_n of degree $n-1$ without a constant term. More precisely, the following result holds.

PROPOSITION 2. *Let $n \in \mathbb{N}_2$. Then,*

$$R_n(z)^* \underset{\text{(i)}}{\overset{z\in\mathbb{C}\setminus\{1\}}{=}} \frac{(-1)^n}{(n-1)!} \frac{\sum_{j=1}^{n-1} \left[{n-1 \atop j}\right] z^j}{(1-z)^n} \underset{\text{(ii)}}{\overset{|z|<1}{=}} \frac{(-1)^n}{(n-1)!} \sum_{k=1}^{\infty} k^{n-1} z^k, \tag{4}$$

where $\left[{n \atop j}\right]$ are the Eulerian numbers given by

$$\left[{n \atop j}\right] = \sum_{i=0}^{j-1} (-1)^i \binom{n+1}{i} (j-i)^n \quad (j \leqslant n).$$

Before we are going to prove this, let us present a table displaying the first few Eulerian numbers.

$\diagdown$	$\left[{m \atop 1}\right]$	$\left[{m \atop 2}\right]$	$\left[{m \atop 3}\right]$	$\left[{m \atop 4}\right]$	$\left[{m \atop 5}\right]$	$\left[{m \atop 6}\right]$	$\left[{m \atop 7}\right]$
$m=1$	1						
$m=2$	1	1					
$m=3$	1	4	1				
$m=4$	1	11	11	1			
$m=5$	1	26	66	26	1		
$m=6$	1	57	302	302	57	1	
$m=7$	1	120	1191	2416	1191	120	1
$\vdots$							$\ddots$

A thorough study of the Eulerian numbers is given in [11].
We also need the following classical formulas:

LEMMA 1.

(a) Euler's product formula for the sine function and its representation in the form of a Weierstrass product:

$$\sigma(z) := \sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = \pi z \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 + \frac{z}{k}\right) e^{-z/k}. \tag{5}$$

(b) For $n \in \mathbb{N}_2$ and $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$Q(z) := \sum_{k \in \mathbb{Z}} \frac{1}{(k+z)^n} = (-1)^{n-1} \frac{\pi}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (\cot \pi z). \tag{6}$$

Proof. (a) For the formulas in (5), see, e.g., [5, Vol. II, p. 292].

(b) Formula (6) immediately follows from (5) and the fact that $\pi \cot \pi z$ is the logarithmic derivative $\frac{\sigma'(z)}{\sigma(z)}$ of the second infinite product in (5). Formula (6) is also mentioned as formula 5.1.3.4 in [10, p. 527]. □

REMARK 1. A proof of formula (6) with $n = 2$ that uses the calculus of residues, maybe found, e.g., in [7, Problem 5.9.1]. Note that in this case the formula can be written in the form [7, p. 253]:

$$\sum_{k \in \mathbb{Z}} \frac{1}{(k+a)^2} = \frac{\pi^2}{\sin^2 \pi a}. \tag{7}$$

Moreover, for $n = 3$ one has

$$\sum_{k \in \mathbb{Z}} \frac{1}{(k+a)^3} = \pi^3 \frac{\cos \pi a}{\sin^3 \pi a},$$

which may additionally be obtained by directly taking the derivatives with respect to a in (7). For the sake of curiosity, let us mention that via residues one can obtain many of these sum formulae. For instance, if

$$\varphi(z) = \pi \cot \pi z - i\pi = \frac{2\pi i}{\exp(2\pi i z) - 1},$$

and if $f = p/q$ is a rational function with $\deg q \geqslant \deg p + 2$ and such that the zeros a_j, $j = 1, \ldots, \nu$, of q are not integers, then, for each $\theta \in [0, 2\pi]$,

$$\sum_{k \in \mathbb{Z}} f(k) e^{ik\theta} = -\sum_{j=1}^{\nu} \operatorname{Res} \left(f(z) e^{iz\theta} \varphi(z),\ z = a_j\right)$$

(see [1, p. 245–246]). For the proof one considers the boundary of the square symmetric to the origin and with corner $(N + 1/2)(1 + i)$ and let $N \to +\infty$.

Proof of Proposition 2. For an easy proof of the equality (ii) in (4), using the finite difference operator, we refer to [8]. Since

$$\sum_{j=1}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix} = (n-1)!$$

(see [8]), the pole 1 has order n.

Next we verify the identity (i) in (4) for $n=2$, the only case we need; that is

$$R_2(z)^* = \frac{z}{(z-1)^2}. \tag{8}$$

Indeed, since for $z \in \widehat{S} \setminus \{1\}$,

$$R_2(z)^* = \frac{1}{(2\pi i)^2} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{\log z}{2\pi i}+k\right)^2},$$

we may substitute z in (6) by $a := a(z) = \frac{\log z}{2\pi i}$, consequently $\frac{da}{dz} = \frac{1}{2\pi i z}$ and $\frac{dz}{da} = 2\pi i z$, and so, with $n=2$, we obtain

$$R_2(z)^* = -\frac{\pi}{(2\pi i)^2}\frac{d}{da}(\cot \pi a) = -\frac{\pi}{(2\pi i)^2}\frac{d}{dz}\left(i\frac{z+1}{z-1}\right)\frac{dz}{da} = \frac{z}{(z-1)^2}.$$

To prove that $R_n^*(z)$ coincides on $\mathbb{D}$ with the right-hand side, denoted by $E_n(z)$, we differentiate it and obtain

$$E_n'(z) = \frac{(-1)^n}{(n-1)!}\sum_{k=1}^{\infty} k^n z^{k-1} = -\frac{n}{z}\,E_{n+1}(z),$$

and consequently

$$E_{n+1}(z) = -\frac{zE_n'(z)}{n}. \tag{9}$$

On the other hand, $R_n(z)$ satisfies the same recursive relation. Indeed, if $z \in \widehat{S} \setminus \{1\}$, the uniform convergence on small closed disks yields

$$R_n'(z) = \sum_{k\in\mathbb{Z}} (-n)\frac{1}{z}\frac{1}{(\log z + 2k\pi i)^{n+1}} = -\frac{n}{z}\,R_{n+1}(z).$$

Hence,

$$R_{n+1}(z) = -\frac{zR_n'(z)}{n}. \tag{10}$$

From (8) and the well-known formula

$$\sum_{k=1}^{\infty} kz^k = \frac{z}{(z-1)^2}, \quad z \in \mathbb{D},$$

(see, e.g., formula 5.2.2.4 in [10, p. 555]), we have that for $n=2$ both functions $R_2(z)$ and $E_2(z)$ coincide on S. Using this fact, and the relations (9) and (10), the equalities in Proposition 2 follow for all $n\in\mathbb{N}_2$ by the identity theorem. □

For the reader's convenience, let us mention that by using (6), it can be proved in the same way that

$$R_3(z)^* = \frac{1}{2}\,\frac{z+z^2}{(z-1)^3},$$

etc. However, by continuing in this way, one does not get the explicit representation (i).

Next we deal with the $n=1$ case. As the series

$$\sum_{k=1}^{\infty}\frac{1}{\log z+2k\pi i},\quad z\in\widehat{S},$$

diverges, we may consider (as it is custom in Fourier analysis) the symmetric partial sums

$$S_N(z) := \sum_{k=-N}^{N}\frac{1}{\log z+2k\pi i},\quad z\in\widehat{S}\setminus\{1\},\ N\in\mathbb{N},$$

(for a closely related idea, see, e.g., [6, p. 175]).

In this vein, we are going to show the following result. Recall that

$$E_n(z) := \frac{(-1)^n}{(n-1)!}\sum_{k=1}^{\infty}k^{n-1}z^k,\quad n\in\mathbb{N}_1.$$

PROPOSITION 3. *The following assertions hold:*

(a) The limit $R_1(z) := \lim_{N\to+\infty}S_N(z)$ *exists for each* $z\in\widehat{S}\setminus\{1\}$.

(b) $R_1(z)$ *is holomorphic in* $\widehat{S}\setminus\{1\}$.

(c) $R_1(z)$ *has a meromorphic extension to* $\mathbb{C}\setminus\{1\}$ *which is given by*

$$R_1(z)^* = \frac{1}{2}\,\frac{z+1}{z-1} = \frac{z}{z-1}-\frac{1}{2}\overset{|z|<1}{=}E_1(z)-\frac{1}{2}. \tag{11}$$

It is interesting to note that the result involves a shift, not uncommon when considering principal values of integrals, and this reminds us of the Sokhotski-Plemelj formula ([5, Vol. I, p. 316]).

Proof. (a), (b) It is straightforward to verify that for $z\in\widehat{S}\setminus\{1\}$,

$$R_1(z) = \frac{1}{\log z}+\sum_{k=1}^{\infty}\frac{2\log z}{(\log z)^2+4k^2\pi^2},$$

which is convergent, in particular, on S.

(c) Since

$$\sum_{n=1}^{\infty} \frac{1}{a^2+4\pi^2 n^2} = \frac{a\coth(a/2)-2}{4a^2},$$

(see [1, p. 227]), we conclude that for $z \in S$

$$\begin{aligned} R_1(z) &= \frac{1}{\log z} + \frac{2\log z\left(\frac{\sqrt{z}+\frac{1}{\sqrt{z}}}{\sqrt{z}-\frac{1}{\sqrt{z}}}\log z - 2\right)}{4(\log z)^2} \\ &= \frac{1}{\log z} + \frac{1}{2}\,\frac{z+1}{z-1} - \frac{1}{\log z}, \end{aligned}$$

from which the relation in (11) follow. □

We conclude with a nice connection between Propositions 1 and 2.

COROLLARY 1. *Let $n, N \in \mathbb{N}_2$, and let*

$$R_{n,N}(z) := \sum_{k\in\mathbb{Z}} \frac{1}{\left(\log z + 2kN\pi i\right)^n}, \quad z \in S.$$

Then, $R_{n,N}(z)$ has no holomorphic extension to $\mathbb{D}$.

Proof. Just note that for $z \in S$, $\sqrt[N]{z} \in S$ again, and so

$$\begin{aligned} R_{n,N}(z) &= \frac{1}{N^n} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\log \sqrt[N]{z} + 2k\pi i\right)^n} \\ &= \frac{(-1)^n}{(n-1)!\,N^n} \sum_{k=1}^{\infty} k^{n-1} (\sqrt[N]{z})^k, \end{aligned}$$

and apply Proposition 1. □

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