Mathematical
Inequalities
& Applications
Volume 29, Number 4 (2026), 471–484
doi:10.7153/mia-2026-29-30

NOTE ON THE NORM ESTIMATES FOR THE WEIGHTED DIFFERENTIATION COMPOSITION OPERATOR BETWEEN SOME SPACES OF ANALYTIC FUNCTIONS ON THE UNIT DISK

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(*Communicated by I. Perić*)

Abstract. We establish an estimation of an upper bound for the norm of a specific linear operator of considerable recent interest that combines three important fundamental linear operators, namely the differentiation operator, the multiplication operator, and the composition operator (collectively, the so-called *weighted differentiation composition operator*). This operator acts between two spaces of analytic functions on the unit disk in the complex plane. In a more general setting, we impose a condition on the linear operator ensuring that the aforementioned estimate is bounded by a specific quantity, which, in a way, generalizes the concept of operator norm in a certain sense. As a corollary of these two estimate type results, we calculate the exact norm of the linear operator between these spaces, in the case where they both are normed linear vector spaces of analytic functions. We also present some noteworthy remarks of independent interest, for the case where one of the spaces is not a normed linear vector space of analytic functions, but rather a Fréchet space.

1. Introduction

We begin by defining all the notions that we use throughout this note and recalling some basic facts on them. Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk centered at the origin in the complex plane, for which we use the standard notation $\mathbb{C}$, $\partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, that is, the unit circle in the complex plane, which forms, of course, the boundary of the open unit disk $\mathbb{U}$ in $\mathbb{C}$, and let $\overline{\mathbb{U}}$ be the closed unit disk (i.e., the set $\mathbb{U} \cup \partial\mathbb{U}$). Furthermore, by Ω we denote a domain in $\mathbb{C}$, i. e., an open and connected subset of $\mathbb{C}$.

Let $H(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For some basic properties of analytic functions on $\mathbb{U}$ or on other domains in the complex plane, we refer the reader to classical texts such as [1], [6], and [17].

Mathematics subject classification (2020): 47B38.

Keywords and phrases: Operator norm, differentiation operator, multiplication operator, unit disk, weighted Bergman space.

Bratislav Iričanin was partially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia under contract number: 451-03-34/2026-03/200103, and was also partially supported by the Science Fund of the Republic of Serbia, GRANT No. 7632, Project "Mathematical Methods in Image Processing under Uncertainty" – MaMIPU.

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Paper MIA-29-30

The standard Lebesgue area measure is denoted by $dm(z)$. The space of complex-valued measurable functions f on domain $\Omega \subset \mathbb{C}$ such that

$$\|f\|_p = \left(\int_\Omega |f(z)|^p dm(z) \right)^{1/p} < \infty$$

we denote by $L^p(\Omega)$.

Some basic properties of these functions and the space of functions can be found, for example, in [17]. Recall that for $p \geqslant 1$, the map/functional $\|\cdot\|_p$ is a norm on the space $L^p(\Omega)$, and that with the norm the space is Banach one (i.e., a complete normed space), while for $p \in (0,1)$, the functional $\|\cdot\|_p^p$ is a translation invariant metric on the space, which with the metric becomes an F-space (i.e., a complete metrizable space; see, e. g., [5] or [18]).

The family of all complex Borel measures on $\partial\mathbb{U}$, that is, the functions μ on Borel subsets of $\partial\mathbb{U}$ such that $\mu(\emptyset) = 0$ and $\mu(\cup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \mu(E_j)$, for any sequence $(E_j)_{j\in\mathbb{N}}$ of pairwise disjoint Borel sets, we denote by $\mathscr{M}(\partial\mathbb{D}) = \mathscr{M}$ (a $\mu \in \mathscr{M}$ is a measure on the standard measurable space $(\partial\mathbb{U}, \sigma_A)$, that is, σ_A is the smallest σ-algebra containing all open sets of $\partial\mathbb{U}$). It is easy to see that $\mathscr{M}$ is a vector space. A Borel measure μ is positive if $\mu(E) \geqslant 0$ for any Borel set $E \subset \partial\mathbb{U}$. By $\mathscr{M}_+(\partial\mathbb{D}) = \mathscr{M}_+$ we denote the family of all positive measures on $\partial\mathbb{U}$. Any $\mu \in \mathscr{M}$ can be uniquely written in the form

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4) \tag{1}$$

for some $\mu_j \in \mathscr{M}_+$, $j = 1, \ldots, 4$, such that $\mu_1 \perp \mu_2$ and $\mu_3 \perp \mu_4$.

For a measure $\mu \in \mathscr{M}$, the total variation measure, denoted by $|\mu|$, is defined for $E \in \sigma_A$ by

$$|\mu|(E) = \sup_{\mathscr{P}} \sum_{j=1}^\infty |\mu(E_j)|,$$

where $\mathscr{P}$ is a Borel measurable partition

$$E_1, E_2, E_3, \ldots$$

of E, that is,

$$E = \bigcup_{j=1}^\infty E_j,$$

and

$$E_j \cap E_l = \emptyset,$$

when $j \neq l$, $j, l \in \mathbb{N}$, and

$$E_j \in \sigma_A$$

for every $j \in \mathbb{N}$.

The total variation measure $|\mu|$ of a complex measure μ is a positive measure on $\partial\mathbb{D}$, and if μ is real, that is, $\mu = \mu_1 - \mu_2$ for some mutually singular $\mu_1, \mu_2 \in \mathscr{M}_+$, we have $|\mu| = \mu_1 + \mu_2$, whereas if (1) holds, then

$$\frac{1}{\sqrt{2}} \sum_{j=1}^{4} \mu_j(E) \leqslant |\mu|(E) \leqslant \sum_{j=1}^{4} \mu_j(E),$$

for every $E \in \sigma_A$.

The vector space $\mathscr{M}$ becomes a normed space if the norm on it is defined as follows

$$\|\mu\|_{\mathscr{M}} := |\mu|(\partial\mathbb{D})$$

and the term "total variation of a measure μ" is frequently used to denote the number $|\mu|(\partial\mathbb{D})$. Thus, here the total variation of μ on $\partial\mathbb{U}$ we denote by $\|\mu\|_{\mathscr{M}}$. Moreover, the space $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ is complete, so a Banach space.

Moreover,

$$|\mu|(\partial\mathbb{D}) < +\infty,$$

from which it follows that every complex measure μ on any σ-algebra is bounded.

If μ is a positive Borel measure that satisfies

$$\|\mu\|_{\mathscr{M}} = 1,$$

then such a measure is called a probability measure. Some additional information on complex Borel measures and related things can be found, for example, in [17].

Assume that $f \in H(\mathbb{U})$ satisfies the following:

$$f(z) = \int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}, \tag{2}$$

for every $z \in \mathbb{U}$, and some $\mu \in \mathscr{M}$.

Then, it is written that $f \in \mathscr{K}$, the family of the so-called Cauchy transforms [3].

Note that for each f analytic in a neighborhood of $\overline{\mathbb{U}}$ we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{U}} \frac{f(\zeta)\overline{\zeta}}{1 - \overline{\zeta} z} d\zeta,$$

so that for each such an analytic function a corresponding Borel measure μ is equal to

$$d\mu(\zeta) = \frac{1}{2\pi i} f(\zeta)\overline{\zeta} d\zeta.$$

This is a concrete example of a Cauchy transform that is obtained from well-known Cauchy's integral formula.

Let

$$C[\mu](z) = \int_{\partial\mathbb{U}} \frac{d\mu(\zeta)}{1 - \overline{\zeta} z},$$

where $\mu \in \mathscr{M}$.

Since a function $f \in \mathscr{K}$ need not have a unique representing measure $\mu \in \mathscr{M}$ satisfying (2), we define the norm on $\mathscr{K}$ by

$$\|f\|_{\mathscr{K}} := \inf_{\mu \in \mathscr{M}} \left\{ \|\mu\|_{\mathscr{M}} \,:\, f = C[\mu] \right\},$$

that is, the infimum is taken over all complex Borel measures such that the representation

$$f = C[\mu].$$

holds.

The functional $\|\cdot\|_{\mathscr{K}}$ is a norm on the linear space $\mathscr{K}$ and with the norm it is a Banach space. For some basic facts on the space and related things see, for example, [3, 8, 21].

Take $p > 0$ and let w be a positive continuous function on $\mathbb{U}$ belonging to the space $L^1(\mathbb{U})$.

If $f \in H(\mathbb{U})$ satisfies the condition

$$\|f\|_{A^p_w} := \left(\int_{\mathbb{U}} |f(z)|^p w(z) dm(z) \right)^{1/p} < +\infty,$$

we say that it belongs to the weighted Bergman space, which we denote by $A^p_w(\mathbb{U}) = A^p_w$. For this, related or more general spaces and some linear operators from or to them, see, e.g., [2, 4, 6, 9, 11–14, 17, 19, 28–30, 38].

When $p \geqslant 1$, the functional

$$\|\cdot\|_{A^p_w} : A^p_w \to \mathbb{R}_+$$

is a norm on A^p_w, and with the norm it is a Banach space, whereas if $p \in (0,1)$, the functional

$$\|\cdot\|^p_{A^p_w} : A^p_w \to \mathbb{R}_+$$

is a translation invariant metric on the space, which with the metric becomes an F-space.

Assume that φ is a self-map of $\mathbb{U}$, $u \in H(\mathbb{U})$ and $m \in \mathbb{N}_0$. The following operator

$$D^m_{\varphi,u} f(z) = u(z) f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{U}), \tag{3}$$

containing the differentiation

$$Df(z) = f'(z),$$

the multiplication

$$M_u f(z) = u(z) f(z),$$

and the composition operator

$$C_\varphi f(z) = f(\varphi(z)),$$

is called the *weighted differentiation composition operator* or the generalized weighted composition operator.

There have been many investigations devoted to the concrete linear operator (see, e.g., [9, 37–40] and the related references cited therein), and to some related operators on the unit disk or the unit ball in $\mathbb{C}^n$ (see, for example, [4, 20, 31–33, 41]). In these papers, various functional-theoretic properties of the operators are characterized in terms of associated symbols and parameters.

Recall that if $A : V \to W$ is a linear operator between normed vector spaces V and W, then the norm of the operator is defined by

$$\|A\|_{V\to W} := \sup_{v\in V\setminus\{0\}} \frac{\|Av\|_W}{\|v\|_V},$$

and if

$$\|A\|_{V\to W} < +\infty,$$

then the linear operator is said to be bounded.

It is easy to see that such operators map bounded sets in V to bounded sets in W, which is sometimes taken as a definition of such operators. In any book on functional analysis it is shown that a linear operator between normed vector spaces is bounded if and only if it is continuous, which is why the bounded linear operators are of great interest in functional analysis (see, for example, [5, 17, 18, 36]). Many recent papers study, among other things, the boundedness of certain linear operators (see [2, 10–12, 15, 16, 20–22, 25–35, 37–41]).

Operator norms have attracted significant attention in functional analysis over the last century. The classical books [5, 17, 18, 36] contain many well-known formulas for norms of some linear operators of various kinds between normed vector spaces, whereas some new ones for spaces of analytic functions can be found, for instance, in the following recent articles: [10, 16, 22, 26, 27, 34, 35].

The following recent studies, which are devoted, among other things, to the calculation of norms of some specific linear operators between some spaces of functions, attracted our attention: [21, 22, 34, 35] (the articles [23] and [24] deal with related problems).

These papers, as well as the aforementioned ones dealing with operator (3) and its extensions, prompted us to provide estimates for the norm of the following operator:

$$D^m_{\varphi,u} : \mathscr{K} \to A^p_w. \tag{4}$$

The problem relates to those discussed in [21–24, 34, 35], as well as to the problems of calculating the norm of the operators investigated in [10, 16, 22, 26, 27, 35], but it seems not to have been considered so far in the literature.

In this note we give an upper estimate for the norm of the linear operator defined in (4) for the case $p \geqslant 1$. In a more general case, that is, when the parameter p is a positive number, we impose a condition on the linear operator which guarantees that the aforementioned estimate is bounded by a specific quantity which, in a way, generalizes the notion of operator norm. As a corollary of these two interesting results, we calculate the norm of the linear operator in the case $p \geqslant 1$. We also give some interesting remarks for the case when one of the spaces is not a normed linear vector space but is a Fréchet space.

2. Main results and preliminaries

Prior to stating our main results in this paper, we state a well known result (a general version of Jensen's inequality) which is used in the proof of one of the main results. For a proof of the inequality see, for example, [7] and [17].

LEMMA 1. *Assume that μ is a positive measure on a σ-algebra Σ in a set S, so that*

$$\mu(S) = 1.$$

If f is a real function in $L^1(\mu)$,

$$a < f(x) < b$$

for all $x \in S$, and if ϕ is convex on (a,b), then we have that the following inequality holds

$$\phi\left(\int_S f d\mu\right) \leqslant \int_S (\phi \circ f) d\mu.$$

Now we state the main results in this note, and give detailed proofs of the results. As a consequence of the main results, we obtain a characterization for the boundedness of the operator (4) when $p \geqslant 1$. Moreover, when the operator is bounded, we give an explicit formula for the norm of the linear operator.

The following theorem is our first result in this note. The theorem gives a sufficient condition for the boundedness of the operator (4) in the case $p \geqslant 1$.

THEOREM 1. *Assume that $m \in \mathbb{N}$, $p \geqslant 1$, φ is an analytic self-map of the unit disk $\mathbb{U}$, $u \in H(\mathbb{U})$ and w is a positive continuous function belonging to the space $L^1(\mathbb{U})$.*

If the following condition

$$\sup_{\zeta \in \partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1 - \overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} < +\infty \tag{5}$$

holds, then the operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded and the following inequality holds

$$\|D^m_{\varphi,u}\|_{\mathscr{K} \to A^p_w} \leqslant m! \sup_{\zeta \in \partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1 - \overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p}. \tag{6}$$

Proof. For any $f \in \mathscr{K}$, we have representation (2) and

$$\|f\|_{\mathscr{K}} = \|\mu\|_{\mathscr{M}} \tag{7}$$

for some $\mu \in \mathscr{M}$.

Differentiating (2) it follows that

$$f^{(k)}(z) = k! \int_{\partial\mathbb{U}} \frac{(\overline{\zeta})^k d\mu(\zeta)}{(1 - \overline{\zeta}z)^{k+1}}, \quad z \in \mathbb{U}, \tag{8}$$

for $k \in \mathbb{N}_0$.

By using (8) in the relation

$$\|D^m_{\varphi,u} f\|_{A^p_w} = \left(\int_{\mathbb{U}} |u(z)|^p |f^{(m)}(\varphi(z))|^p w(z) dm(z) \right)^{1/p},$$

we have

$$\|D^m_{\varphi,u} f\|_{A^p_w} = m! \left(\int_{\mathbb{U}} |u(z)|^p \left| \int_{\partial \mathbb{U}} \frac{(\overline{\zeta})^m d\mu(\zeta)}{(1-\overline{\zeta}\varphi(z))^{m+1}} \right|^p w(z) dm(z) \right)^{1/p}, \tag{9}$$

for $f \in \mathscr{K}$.

From (9) and by using the known inequality

$$\left| \int_{\partial \mathbb{U}} \widehat{f} d\mu \right| \leqslant \int_{\partial \mathbb{U}} |\widehat{f}| d|\mu|,$$

it follows that

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \left(\int_{\mathbb{U}} |u(z)|^p \left(\int_{\partial \mathbb{U}} \frac{d|\mu|(\zeta)}{|1-\overline{\zeta}\varphi(z)|^{m+1}} \right)^p w(z) dm(z) \right)^{1/p}, \tag{10}$$

for $f \in \mathscr{K}$.

Now note that inequality (10) can be rewritten in the form

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \left(\int_{\mathbb{U}} |u(z)|^p \left(\int_{\partial \mathbb{U}} \frac{\|\mu\|_{\mathscr{M}}}{|1-\overline{\zeta}\varphi(z)|^{m+1}} \frac{d|\mu|(\zeta)}{\|\mu\|_{\mathscr{M}}} \right)^p w(z) dm(z) \right)^{1/p}, \tag{11}$$

for $f \in \mathscr{K}$.

Note that the case when $\|\mu\|_{\mathscr{M}} = 0$ is trivial, so we may assume that

$$\|\mu\|_{\mathscr{M}} > 0.$$

Since

$$d\widetilde{\mu}(\zeta) := \frac{d|\mu|(\zeta)}{\|\mu\|_{\mathscr{M}}}$$

is a probability measure on $\partial \mathbb{U}$, we can apply Jensen's inequality with the function

$$\phi(t) = t^p,$$

and from (11), we obtain

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \left(\int_{\mathbb{U}} |u(z)|^p \int_{\partial \mathbb{U}} \frac{\|\mu\|^p_{\mathscr{M}}}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} \frac{d|\mu|(\zeta)}{\|\mu\|_{\mathscr{M}}} w(z) dm(z) \right)^{1/p},$$

for $f \in \mathscr{K}$, that is,

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \|\mu\|^{1-\frac{1}{p}}_{\mathscr{M}} \left(\int_{\mathbb{U}} |u(z)|^p \int_{\partial \mathbb{U}} \frac{d|\mu|(\zeta)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} w(z) dm(z) \right)^{1/p}, \tag{12}$$

for $f \in \mathscr{K}$.

From (12) by using Fubini's theorem it follows that

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \|\mu\|^{1-\frac{1}{p}}_{\mathscr{M}} \left(\int_{\partial \mathbb{U}} \int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) d|\mu|(\zeta) \right)^{1/p}, \tag{13}$$

for $f \in \mathscr{K}$.

Applying the inequality

$$\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \leqslant \sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z),$$

for $\zeta \in \partial \mathbb{D}$, in (13) we obtain

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \|\mu\|^{1-\frac{1}{p}}_{\mathscr{M}} \left(\int_{\partial \mathbb{U}} \sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) d|\mu|(\zeta) \right)^{1/p},$$

for $f \in \mathscr{K}$, that is,

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \|\mu\|_{\mathscr{M}} \left(\sup_{\zeta \in \partial \mathbb{D}} \int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p}, \tag{14}$$

for $f \in \mathscr{K}$.

Employing (7), (14) and the continuity of the function

$$h(t) = t^{1/p}$$

on the interval $[0, +\infty)$, we obtain

$$\|D^m_{\varphi,u} f\|_{A^p_w} \leqslant m! \|f\|_{\mathscr{K}} \sup_{\zeta \in \partial \mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p}, \tag{15}$$

for every $f \in \mathscr{K}$.

From (15), it follows that

$$\begin{aligned} \|D^m_{\varphi,u}\|_{\mathscr{K} \to A^p_w} &= \sup_{f \in \mathscr{K} \setminus \{0\}} \frac{\|D^m_{\varphi,u} f\|_{A^p_w}}{\|f\|_{\mathscr{K}}} \\ &\leqslant m! \sup_{\zeta \in \partial \mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p}. \end{aligned} \tag{16}$$

Applying assumption (5) in (16), we get

$$\|D^m_{\varphi,u}\|_{\mathscr{K}\to A^p_w} < +\infty,$$

which means that the operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded. Moreover, from (16) we see that inequality (6) holds. □

THEOREM 2. *Let $m \in \mathbb{N}$, $p > 0$, φ be an analytic self-map of $\mathbb{U}$, $u \in H(\mathbb{U})$ and w be a positive continuous function belonging to the space $L^1(\mathbb{U})$. If the following condition*

$$\|D^m_{\varphi,u} g\|_{A^p_w} \leqslant L\|g\|_{\mathscr{K}} \tag{17}$$

holds for some nonnegative constant L and for every function $g \in \mathscr{K}$, then the following relation

$$\sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \leqslant \frac{L}{m!}, \tag{18}$$

holds.

Proof. Consider the following test functions

$$g_\eta(z) = \frac{1}{1-\overline{\eta}z}, \quad z \in \mathbb{U}. \tag{19}$$

for an $\eta \in \partial\mathbb{U}$.

We have that

$$g_\eta(z) = \int_{\partial\mathbb{U}} \frac{d\mu_\eta(\eta)}{1-\overline{\eta}z}, \quad z \in \mathbb{U},$$

holds for the probability measure μ_η with the unit mass concentrated at the point η.

Hence,

$$\|g_\eta\|_{\mathscr{K}} = \|\mu_\eta\|_{\mathscr{M}} = 1 \tag{20}$$

for $\eta \in \partial\mathbb{U}$.

Note also that

$$g^{(m)}_\eta(z) = \frac{m!\overline{\eta}^m}{(1-\overline{\eta}z)^{m+1}}, \tag{21}$$

for $\eta \in \partial\mathbb{U}$ and $z \in \mathbb{U}$.

Via (21), in the following relation

$$\|D^m_{\varphi,u} g_\eta\|_{A^p_w} = \left(\int_{\mathbb{U}} |u(z) g^{(m)}_\eta(\varphi(z))|^p w(z) dm(z) \right)^{1/p},$$

we obtain

$$\|D^m_{\varphi,u} g_\eta\|_{A^p_w} = m! \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\eta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \tag{22}$$

for $\eta \in \partial\mathbb{D}$.

Using the condition (17), together with the equalities in (20) and (22), it follows that

$$m! \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\eta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \leqslant L, \tag{23}$$

for every $\eta \in \partial\mathbb{U}$.

By taking the supremum in the inequality (23) over the set $\partial\mathbb{U}$, it follows that

$$m! \sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \leqslant L,$$

from which (18) easily follows. □

REMARK 1. Note that Theorem 2 holds also for the case when $p \in (0,1)$, unlike Theorem 1 which is proved only in the case when $p \geqslant 1$. The condition $p \geqslant 1$ was used at the point where Jensen's inequality has been applied during the proof of the theorem, which caused the difference in the formulations of these two theorems.

From Theorem 1 and Theorem 2 a somewhat interesting corollary follows.

COROLLARY 1. *Take $m \in \mathbb{N}$, $p \geqslant 1$, φ be an analytic self-map of $\mathbb{U}$, $u \in H(\mathbb{U})$ and w be a positive continuous function belonging to the space $L^1(\mathbb{U})$. Then, the following statements are true.*

(a) The operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded if and only if

$$\sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} < +\infty.$$

(b) If the operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded, then its norm is

$$\|D^m_{\varphi,u}\|_{\mathscr{K}\to A^p_w} = m! \sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p}. \tag{24}$$

Proof. We establish the equivalence by proving each direction separately.

(a) $\Rightarrow$) This implication has already been proven in Theorem 1.

$\Leftarrow$) Since the linear operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded, then it follows that

$$\|D^m_{\varphi,u} g\|_{A^p_w} \leqslant \|D^m_{\varphi,u}\|_{\mathscr{K}\to A^p_w} \|g\|_{\mathscr{K}}$$

for each $g \in \mathscr{K}$.

The last inequality means that the conditions in Theorem 2 are satisfied with

$$L = \|D^m_{\varphi,u}\|_{\mathscr{K}\to A^p_w}.$$

Hence, from (18) we have

$$\sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \leqslant \frac{\|D^m_{\varphi,u}\|_{\mathscr{K}\to A^p_w}}{m!} < \infty, \tag{25}$$

as claimed.

(b) If the linear operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ is bounded, then from the inequalities in (16) and (25), we obtain the formula for the operator norm given in (24), finishing the proof of the corollary. □

REMARK 2. Note that from the proof of Theorem 2, we have that condition (17) implies

$$\sup_{\zeta\in\partial\mathbb{U}} \left(\int_{\mathbb{U}} \frac{|u(z)|^p w(z)}{|1-\overline{\zeta}\varphi(z)|^{p(m+1)}} dm(z) \right)^{1/p} \leqslant \frac{L}{m!},$$

for each $p > 0$.

We think this fact is interesting for itself. However, if $p \in (0,1)$, the condition (17) is not the boundedness of the operator $D^m_{\varphi,u} : \mathscr{K} \to A^p_w$ in above-mentioned sense, since for the case, as we have already mentioned, the map $\|\cdot\|_{A^p_w}$ *is not* a norm on the space A^p_w.

On the other hand, if $p \in (0,1)$, the map $\|\cdot\|^p_{A^p_w}$ is a translation invariant metric on the space. However, condition (17) is also not the metrical boundedness, in this case.

Namely, let us remember that if V and W are metric spaces with translation invariant metrics d_V and d_W, respectively, then for a linear operator $A : V \to W$ it is said that it is *metrically bounded* if there is a nonnegative constant C such that

$$d_W(Av, 0) \leqslant C d_V(v, 0), \tag{26}$$

for every $v \in V$ (see, e.g., [33]).

Hence, if $p \in (0,1)$, then (26) should be

$$\|D^m_{\varphi,u} g\|^p_{A^p_w} \leqslant C\|g\|_{\mathscr{K}}, \tag{27}$$

which is different from (17) (the left-hand side of inequality (27) is raised to the pth power).

In fact, if $p \in (0,1)$, then the map $\|\cdot\|_{A^p_w}$ is a quasi-norm, that is, there is a constant $c \geqslant 1$ for which

$$\|f+g\|_{A^p_w} \leqslant c(\|f\|_{A^p_w} + \|g\|_{A^p_w})$$

for every $f, g \in A^p_w$.

Thus, for these values of the parameter p we have that the operator maps a normed space into a quasi-normed one, that is, $\mathscr{K}$ to A^p_w.

All aforementioned facts and observations suggest various possible investigations in several other directions, which can be pursued by an interested reader.

3. Conclusion

In this note, we have established an upper bound for the norm of a specific linear operator that has gained considerable recent attention, involving three fundamental operators: differentiation, multiplication, and composition. This operator acts between two spaces of analytic functions on the unit disk in the complex plane. In a more general setting, we imposed a condition on the linear operator which guarantees that the aforementioned estimate is bounded by a prescribed quantity which, in a sense, generalizes the standard notion of an operator norm. As a corollary of these two results in estimation, we calculate the exact norm of the linear operator between the spaces, for the case when they are both normed vector spaces of analytic functions. To achieve this, we refined and combined various methods, ideas, and details from the literature. We anticipate that these techniques can be further extended so that some more general or related interesting results can be obtained. We propose and encourage this direction to interested experts as a potential avenue of research.

REFERENCES

[1] L. V. AHLFORS, *Complex Analysis*, McGraw-Hill, Inc., 3rd ed., New York et al., 1979.

[2] K. AVETISYAN AND S. STEVIĆ, *Extended Cesàro operators between different Hardy spaces*, Appl. Math. Comput. **207**, 2 (2009), 346–350.

[3] P. BOURDON AND J. CIMA, *On integrals of Cauchy-Stieltjes type*, Houston J. Math. **14**, 4 (1988), 465–474.

[4] J. DU, S. LI AND Z. LIU, *Stević-Sharma-type operators between Bergman spaces induced by doubling weights*, Math. Methods Appl. Sci. **47** (12) (2024), 10006–10023.

[5] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators I*, Interscience Publishers, John Wiley and Sons, New York, 1958.

[6] P. DUREN, *Theory of H^p Spaces*, Dover Publications, Mineola, NY, 2000.

[7] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge Mathematical Library, Cambridge, 1988.

[8] R. A. HIBSCHWEILER AND T. H. MACGREGOR, *Fractional Cauchy Transforms*, Monographs and Surveys in Pure and Applied Mathematics 136, Chapman & Hall/CRC, Boca Raton, London, New York, Singapore, 2006.

[9] Q. HU AND X. ZHU, *Compact generalized weighted composition operators on the Bergman space*, Opuscula Math. **37** (2) (2017), 303–312.

[10] H. LI AND S. LI, *Norm of an integral operator on some analytic function spaces on the unit disk*, J. Inequal. Appl. Vol. 2013, Article No. 342, (2013), 7 pp.

[11] S. LI AND S. STEVIĆ, *Integral type operators from mixed-norm spaces to α-Bloch spaces*, Integral Transforms Spec. Funct. **18** (7) (2007), 485–493.

[12] S. LI AND S. STEVIĆ, *Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$*, Bull. Belg. Math. Soc. Simon Stevin **14** (4) (2007), 621–628.

[13] S. LI AND S. STEVIĆ, *Compactness of Riemann-Stieltjes operators between $F(p,q,s)$ and α-Bloch spaces*, Publ. Math. Debrecen **72** (1–2) (2008), 111–128.

[14] S. LI AND S. STEVIĆ, *Riemann-Stieltjes operators between different weighted Bergman spaces*, Bull. Belg. Math. Soc. Simon Stevin **15** (4) (2008), 677–686.

[15] S. LI AND S. STEVIĆ, *Cesàro-type operators on some spaces of analytic functions on the unit ball*, Appl. Math. Comput. **208** (2) (2009), 378–388.

[16] X. LIU AND S. LI, *Norm and essential norm of a weighted composition operator on the Bloch space*, Integral Equations Operator Theory **87** (3) (2017), 309–325.

[17] W. RUDIN, *Real and Complex Analysis*, Higher Mathematics Series, 3rd ed., McGraw-Hill Education, London, New York, Sydney, 1987.

[18] W. RUDIN, *Functional Analysis*, 2nd ed., McGraw-Hill, Inc., New York, 1991.

[19] B. SEHBA AND S. STEVIĆ, *On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces*, Appl. Math. Comput. **233** (2014), 565–581.

[20] A. K. SHARMA, *Products of multiplication, composition and differentiation between weighted Bergman-Nevanlinna and Bloch-type spaces*, Turk. J. Math. **35** (2) (2011), 275–291.

[21] A. K. SHARMA, *Weighted composition operators from Cauchy integral transforms to logarithmic weighted-type spaces*, Ann. Funct. Anal. **4** (1) (2013), 163–174.

[22] A. K. SHARMA, *Norm of a composition operator from the space of Cauchy transforms into Zygmund-type spaces*, Ukrainian Math. J. **71** (12) (2020), 1945–1958.

[23] A. K. SHARMA AND R. KRISHAN, *Difference of composition operators from the space of Cauchy integral transforms to the Dirichlet space*, Complex Anal. Oper. Theory **10** (1) (2016), 141–152.

[24] A. K. SHARMA, R. KRISHAN, AND E. SUBHADARSINI, *Difference of composition operators from the space of Cauchy integral transforms to Bloch-type spaces*, Integral Transforms Spec. Funct. **28** (2) (2017), 145–155.

[25] S. STEVIĆ, *Boundedness and compactness of an integral operator on a weighted space on the poly-disc*, Indian J. Pure Appl. Math. **37** (6) (2006), 343–355.

[26] S. STEVIĆ, *Norms of some operators from Bergman spaces to weighted and Bloch-type space*, Util. Math. **76** (2008), 59–64.

[27] S. STEVIĆ, *Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces*, Appl. Math. Comput. **215** (2) (2009), 818–820.

[28] S. STEVIĆ, *Integral-type operators from a mixed norm space to a Bloch-type space on the unit ball*, Sib. Math. J. **50** (6) (2009), 1098–1105.

[29] S. STEVIĆ, *Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball*, Appl. Math. Comput. **212** (2) (2009), 499–504.

[30] S. STEVIĆ, *On an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball*, Abstr. Appl. Anal. Vol. 2010, Article ID 198608, (2010), 7 pp.

[31] S. STEVIĆ, *On operator P^g_φ from the logarithmic Bloch-type space to the mixed-norm space on unit ball*, Appl. Math. Comput. **215** (11) (2010), 4248–4255.

[32] S. STEVIĆ, *On a new product-type operator on the unit ball*, J. Math. Inequal. **16** (4) (2022), 1675–1692.

[33] S. STEVIĆ, *Metrical boundedness and compactness of a new operator between some spaces of analytic functions*, Axioms **12** (9), Article No. 851, (2023), 15 pp.

[34] S. STEVIĆ, *Note on the norm of linear combinations of a class of linear operators*, J. Math. Inequal. **19** (2) (2025), 637–647.

[35] S. STEVIĆ AND A. K. SHARMA, *Composition operators from the space of Cauchy transforms to Bloch and the little Bloch-type spaces on the unit disk*, Appl. Math. Comput. **217** (24) (2011), 10187–10194.

[36] V. A. TRENOGIN, B. M. PISAREVSKIY, AND T. S. SOBOLEVA, *Zadachi i Uprazhneniya po Funktsional'nomu Analizu*, Izdanie vtoroe, FIZMATLIT, Moskva, 2005, (in Russian).

[37] W. YANG AND X. ZHU, *Generalized weighted composition operators from area Nevanlinna spaces to Bloch-type spaces*, Taiwan J. Math. **16** (3) (2012), 869–883.

[38] X. ZHU, *Generalized weighted composition operators on weighted Bergman spaces*, Numer. Funct. Anal. Opt. **30** (7–8) (2009), 881–893.

[39] X. ZHU, *Generalized weighted composition operators from Bloch spaces into Bers-type spaces*, Filomat **26** (6) (2012), 1163–1169.

[40] X. ZHU, *Generalized weighted composition operators from H^∞ to the logarithmic Bloch spaces*, Filomat **30** (14) (2016), 3867–3874.

[41] X. ZHU, Q. HU, AND D. QU, *Polynomial differentiation composition operators from Besov-type spaces into Bloch-type spaces*, Math. Methods Appl. Sci. **47** (1) (2024), 147–168.

(Received October 14, 2025)

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