WEAK CONTRACTIONS AND TRACE CLASS PERTURBATIONS

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Abstract. An absolutely continuous contraction is said to be in the class $\mathbb{A}$ if it has isometric H^∞ functional calculus. We present evidence in favor of the conjecture that the class $\mathbb{A}$ is invariant under trace-class perturbations.

1. Introduction

The class $\mathbb{A}$ was defined in [6] to consist of those absolutely continuous contractions T on a Hilbert space for which the Sz.-Nagy—Foias functional calculus is isometric, i.e.,

$$\|u(T)\| = \|u\|_\infty \text{ for every } u \in H^\infty.$$

The work in this paper was motivated by the desire to understand the effect of trace-class perturbations on operators in the class $\mathbb{A}$. A related question was already mentioned in [6] (see Problem 10.9), and we formulated in an earlier note [7] the following conjecture.

CONJECTURE. *Assume that T and T' are absolutely continuous contractions, and $T - T'$ is a trace-class operator. Then T belongs to the class $\mathbb{A}$ if and only if T' does.*

The result of [7] verifies this statement in case T and T' are diagonalizable, in which case they are automatically of class C_{00} (i.e., $\lim_{n\to\infty}(\|T^n x\| + \|T^{*n} x\|) = 0$ for every vector x). In this paper we will show that the conjecture is also true in case $I - T^*T$ is a trace-class operator. For such operators, the C_{00} part does not contribute anything towards making the functional calculus an isometry; indeed, this C_{00} part is actually of class C_0, so that its functional calculus has nontrivial kernel [22, Theorem VIII.1.1]. Thus the context of this paper is quite removed from that of [7], and the methods we use need to be very different.

The result of [7] is essentially a measure theoretical observation. The main technical tool we use in this paper is a remarkable relationship between the characteristic operator function of the contraction T on the one hand, and the unitary part in the polar decomposition of T on the other. More precisely, if V is a unitary operator such that $V^*T = |T| = (T^*T)^{1/2}$, let us denote by

$$H(z) = (V + z)(V - z)^{-1}, \quad |z| < 1,$$

the Herglotz integral of the spectral measure of V, and set

$$B = (I + |T|)^{-1/2}(I - |T|)^{1/2}.$$

The analytic operator-valued function

$$\vartheta(z) = (BH(z)B + I)^{-1}(BH(z)B - I), \quad |z| < 1, \qquad (1)$$

is contractive, and its purely contractive part is (or, more accurately, *coincides* in the sense of Sz.-Nagy and Foias *with*) the characteristic function of T.

Different forms of this formula have appeared in the literature, though generally V is taken to be a more general (preferably small) perturbation of T. The applicability of our formula is perhaps more restrictive, but there seems to be additional information encoded in it.

We would like to take this opportunity to clarify the history of this type of formula. The first appearance of this calculation is in the work of M. S. Livšic [15]. In this paper, Livšic considers a partially isometric, completely nonunitary contraction T with defect indices $\mathfrak{d}_T = \mathfrak{d}_{T*} = 1$, introduces the concept of the characteristic function of T, and relates it with a unitary V such that $V - T$ has rank 1 via a formula essentially equivalent to (1). He extended this work to partial isometries with $\mathfrak{d}_T = \mathfrak{d}_{T*} < \infty$ in [16], and this is further extended by Yu. L. Smul'yan [19] to operators with infinite indices. The Livšic formula was independently rediscovered by L. de Branges [8] (see particularly Lemmas 2, 11 and their proofs). The notion of characteristic function is not used in this work. Instead, de Branges works with what is now called a de Branges space or, alternatively, the functional model associated with an inner and *-inner function which vanishes at zero. Therefore, his result applies to partially isometric contractions of class C_{00} with arbitrary defect indices. The case of C_{00} contractions with $\mathfrak{d}_T = \mathfrak{d}_{T*} = 1$ was yet again independently rediscovered by D. N. Clark [10]. In this work, T is not assumed to be partially isometric, and Clark also produces an explicit identification of the underlying Hilbert space with the space of square integrable functions relative to a measure on the circle. This measure is explicitly related to the characteristic function via a Herglotz integral. Clark's results were extended to arbitrary contractions by J. A. Ball [2] (see also [3]). Related results were also proved by P. A. Fuhrman [12] and C. Benhida and Timotin [4]. Some function theoretical implications of these formulas are explored in D. Sarason's monograph [18].

For the present work, the significance of (1) is that it provides a rather explicit method for calculating the spectral measure of V in terms of the characteristic function of T. In turn, under the hypothesis that $I - T^*T$ is of trace class, whether T belongs to the class $\mathbb{A}$ depends exclusively on the absolutely continuous part of V. Finally, the conjecture follows (under the trace-class hypothesis) from the Kato-Rosenblum invariance of the absolutely continuous spectrum under trace class perturbations.

The extreme case of a purely singular unitary V was considered earlier by P. Y. Wu [24]. He showed that for a contraction $T = V|T|$ with $\mathfrak{d}_T = \mathfrak{d}_{T*} < \infty$, the unitary operator V is purely singular if and only if T is the direct sum of a C_0 operator with a purely singular unitary. Our methods allow us to extend this result to operators T for which $I - T^*T$ is in the trace class.

The remainder of this paper is organized as follows. In Section 2 we review functional models and prove (1). In Section 3 we demonstrate a relationship between the spectral radius of operators of the form $u(T)$ and the characteristic function of T. This allows us to show in Section 4 that $T \in \mathbb{A}$ if and only if V has a large absolutely continuous part. Our main result is in Section 4, along with the extension mentioned above of the result of Wu. We conclude in Section 5 with a few examples and a discussion of the singular unitary dilations considered by Wu and K. Takahashi [25].

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2.   Functional Models

We begin by recalling one of the main results in the theory of functional models. For a contraction $T \in \mathscr{L}(\mathfrak{H})$ acting on the complex, separable Hilbert space $\mathfrak{H}$, we denote by $D_T = (I - T^*T)^{1/2}$ the defect operator, by $\mathfrak{D}_T = (D_T\mathfrak{H})^-$ the defect space, and by $\mathfrak{d}_T = \dim(\mathfrak{D}_T)$ the defect index of T. One defines an analytic function $\vartheta_T : \mathbb{D} \to \mathscr{L}(\mathfrak{H})$ on the unit disk $\mathbb{D}$ by setting

$$\vartheta_T(z) = -T + zD_{T^*}(I - zT^*)^{-1}D_T, \quad z \in \mathbb{D}.$$

The function ϑ_T is contractive, i.e. $\|\vartheta_T(z)\| \leqslant 1$ for all $z \in \mathbb{D}$. Conversely, given Hilbert spaces $\mathfrak{F}, \mathfrak{G}$, and a contractive analytic function $\vartheta : \mathbb{D} \to \mathscr{L}(\mathfrak{F}, \mathfrak{G})$, one constructs a contraction $S(\vartheta) \in \mathscr{L}(\mathfrak{H}(\vartheta))$ as follows. First one defines a measurable function $\Delta : \mathbb{T} = \partial\mathbb{D} \to \mathscr{L}(\mathfrak{F})$ by

$$\Delta(\zeta) = (I - \vartheta(\zeta)^*\vartheta(\zeta))^{1/2}, \quad \zeta \in \mathbb{T},$$

and the space

$$\mathfrak{H}(\vartheta) = \mathfrak{K}_+ \ominus \{\vartheta v \oplus \Delta v : v \in H^2(\mathfrak{F})\},$$

where $\mathfrak{K}_+ = H^2(\mathfrak{G}) \oplus (\Delta L^2(\mathfrak{F}))^-$. Multiplication by the variable on $\mathfrak{K}_+$ is an isometric operator U_+ such that $U_+^*\mathfrak{H}(\vartheta) \subset \mathfrak{H}(\vartheta)$, and the operator $S(\vartheta)$ is defined by the requirement that

$$S(\vartheta)^* = U_+^*|\mathfrak{H}(\vartheta).$$

The operator $S(\vartheta)$ is usually known as the functional model associated with ϑ.

Now, every contractive analytic function ϑ can be written as a direct sum $\vartheta = \vartheta_0 \oplus \vartheta_1$, $\vartheta_j \in \mathscr{L}(\mathfrak{F}_j, \mathfrak{G}_j)$, such that ϑ_0 is purely contractive (i.e., $\|\vartheta_0(0)f\| < \|f\|$ for all $f \in \mathfrak{F}_0 \setminus \{0\}$), while ϑ_1 is a constant unitary operator. It is easy to see that $S(\vartheta_0)$ is unitarily equivalent to $S(\vartheta)$. Similarly, every contraction $T \in \mathscr{L}(\mathfrak{H})$ can be decomposed as $T = T_0 \oplus T_1$, $T_j \in \mathscr{L}(\mathfrak{H}_j)$, such that T_1 is unitary, and T_0 has no nontrivial unitary restriction; T_0 is said to be completely nonunitary or cnu.

It is proved in [22, Chapter VI] that $S(\vartheta_T)$ is unitarily equivalent to the cnu summand T_0 of T. Analogously, the purely contractive parts of $\vartheta_{S(\vartheta)}$ and ϑ must coincide; two operator-valued analytic functions are said to coincide if one is obtained from the other via multiplication on both sides by constant unitary operators.

Thus, the theory of functional models establishes a bijective correspondence between unitary equivalence classes of cnu contractions, and coincidence classes of purely contractive operator-valued analytic functions. The purely contractive part of ϑ_T is called the characteristic function of T, and is denoted by Θ_T. Thus the function $\Theta_T : \mathbb{D} \to \mathscr{L}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$ is given by $\Theta_T(z) = \vartheta_T(z)|\mathfrak{D}_T$. Implicit in this statement is the fact that T maps $\mathfrak{D}_T^\perp$ isometrically onto $\mathfrak{D}_{T^*}^\perp$.

We will now prove (1) under somewhat more general assumptions. We start with two simple results, whose proofs are easy computations. We use the notation $\Re X = (X + X^*)/2$ for the real part of an operator X.

LEMMA 2.1. *Consider a contraction $C \in \mathscr{L}(\mathfrak{E})$, and the analytic function $H_C : \mathbb{D} \to \mathscr{L}(\mathfrak{E})$ defined by*

$$H_C(z) = (I + zC^*)(I - zC^*)^{-1}, \quad z \in \mathbb{D}.$$

We have then

$$\Re H_C(z) = (I - \overline{z}C)^{-1}(I - |z|^2 CC^*)(I - zC^*)^{-1} \geqslant 0$$

for all $z \in \mathbb{D}$.

LEMMA 2.2. *Given elements A, B in a Banach algebra, $I + B^2 A$ is invertible if and only if $I + BAB$ is invertible. When this is the case, we have*

$$B(I + BAB)^{-1} = (I + B^2 A)^{-1} B.$$

In the following result we will use a decomposition $T = V|T| = V(T^*T)^{1/2}$ with V a partial isometry. This is not necessarily the unique polar decomposition of T, but the action of V is uniquely determined on $\ker(T)^\perp$, which it maps isometrically onto $\ker(T^*)^\perp$. Also observe that $V|\mathfrak{D}_T^\perp = T|\mathfrak{D}_T^\perp$ maps $\mathfrak{D}_T^\perp$ isometrically onto $\mathfrak{D}_{T^*}^\perp$, and thus $V^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$ for all such partial isometries U.

THEOREM 2.3. *Consider a contraction $T \in \mathscr{L}(\mathfrak{H})$, and a partial isometry $V \in \mathscr{L}(\mathfrak{H})$ such that $T = V|T|$. Define an analytic function $H : \mathbb{D} \to \mathscr{L}(\mathfrak{H})$ by*

$$H(z) = (I + zV^*)(I - zV^*)^{-1}, \quad z \in \mathbb{D},$$

and set $B = (I + |T|)^{-1/2}(I - |T|)^{1/2}$. Then the operator $BH(z)B + I$ is invertible for all $z \in \mathbb{D}$, and we have

$$V^*\vartheta_T(z) = (BH(z)B + I)^{-1}(BH(z)B - I), \quad z \in \mathbb{D}. \tag{2}$$

Proof. In terms of the decomposition $T = V|T|$ we have

$$D_T = (I - |T|^2)^{1/2}, \quad D_{T^*} = I - VV^* + V(I - |T|^2)^{1/2}V^*.$$

We obtain therefore

$$\begin{aligned} V^*\vartheta_T(z) &= -V^*T + zV^*D_{T^*}(I - zT^*)^{-1}D_T \\ &= -|T| + zV^*V(I - |T|^2)^{1/2}V^*(I - zT^*)^{-1}(I - |T|^2)^{1/2} \end{aligned} \tag{3}$$

for all $z \in \mathbb{D}$. Observe now that $V^*V|T|^k V^* = |T|^k V^*$ for all $k \geqslant 0$; indeed, this is true when $k = 0$ because V is a partial isometry, while for $k \geqslant 1$ it is a consequence of the fact that $V^*V = I$ on the support of $|T|$. We can then replace $|T|^k$ by an arbitrary continuous function of $|T|$ in this relation. In particular,

$$V^*V(I - |T|^2)^{1/2}V^* = (I - |T|^2)^{1/2}V^*.$$

Therefore (3) can be rewritten as

$$V^*\vartheta_T(z) = -|T| + zD_T V^*(I - zT^*)^{-1}D_T, \quad z \in \mathbb{D}. \tag{4}$$

Note now that

$$zV^*(I - zV^*)^{-1} = \frac{1}{2}(H(z) - I), \quad I - |T| = 2(I + B^2)^{-1}B^2, \quad D_T = 2(I + B^2)^{-1}B,$$

as can be seen by simple calculations. Next we observe that

$$
\begin{aligned}
I - zT^* &= I - zV^* + (I - |T|)zV^* \\
&= (I + (I - |T|)zV^*(I - zV^*)^{-1})(I - zV^*) \\
&= (I + (I + B^2)^{-1}B^2(H(z) - I))(I - zV^*) \\
&= (I + B^2)^{-1}(I + B^2 H(z))(I - zV^*).
\end{aligned}
$$

The first lemma above shows that $H(z)$, and hence $BH(z)B$, has nonnegative real part, and therefore $I + BH(z)B$ is invertible. The second lemma then implies that $I + B^2 H(z)$ is also invertible for all $z \in \mathbb{D}$, and then (4) yields

$$
\begin{aligned}
V^*\vartheta_T(z) &= -|T| + D_T zV^*(I - zV^*)^{-1}(I + B^2 H(z))^{-1}(I + B^2)D_T \\
&= -|T| + D_T \frac{1}{2}(H(z) - I)(I + B^2 H(z))^{-1}2B \\
&= -|T| + D_T(H(z) - I)B(I + BH(z)B)^{-1},
\end{aligned}
$$

where we used the identity in the lemma with $H(z)$ in place of A. We deduce the equality

$$
\begin{aligned}
V^*\vartheta_T(z)(I + BH(z)B) &= -|T|(I + BH(z)B) + D_T(H(z) - I)B \\
&= (I + B^2)^{-1}[(B^2 - I)(I + BH(z)B) + 2B(H(z) - I)B] \\
&= BH(z)B - I,
\end{aligned}
$$

which is equivalent to the one in the statement. $\quad\square$

It is easy to see that the purely contractive part of $V^*\vartheta_T$ acts on the space $\mathfrak{D}_T$, and it is given by equation (2) provided that $BH(z)B$ and I are interpreted as operators on this space. Clearly, $V^*\vartheta_T(z)h = -h$ for $h \in \mathfrak{D}_T^\perp$.

Formula (2) yields the equality $I - V^*\vartheta_T(z) = 2(BH(z)B + I)^{-1}$, so that in particular $I - V^*\vartheta_T(z)$ is invertible for all $z \in \mathbb{D}$, and

$$BH(z)B = (I + V^*\vartheta_T(z))(I - V^*\vartheta_T(z))^{-1}, \quad z \in \mathbb{D}. \tag{5}$$

Note that the function ϑ_T coincides with $V^*\vartheta_T$ when V is a unitary operator; such a unitary operator can only be chosen when $\ker(T)$ and $\ker(T^*)$ have the same dimension. Assume that this condition is satisfied, V is chosen to be unitary, and E is the spectral measure of V. Using the notation

$$\mu_h(\sigma) = \langle E(\sigma)h, h \rangle$$

for the scalar measure corresponding with a vector $h \in \mathfrak{H}$, (5) yields the following formula for the Herglotz integral of μ_{Bh}:

$$\langle (I + V^*\vartheta_T(z))(I - V^*\vartheta_T(z))^{-1}h, h \rangle = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu_{Bh}(\zeta), \quad z \in \mathbb{D}, h \in \mathfrak{H}. \quad (6)$$

Taking real parts we obtain the Poisson integral of μ_{Bh}:

$$\|\Delta(z)(I - V^*\vartheta_T(z))^{-1}h\|^2 = \int_{\mathbb{T}} \Re \frac{\zeta + z}{\zeta - z} \, d\mu_{Bh}(\zeta). \quad (7)$$

It may seem surprising that $I - V^*\vartheta_T(z)$ is invertible for all z. This fact can also be proved directly as follows.

LEMMA 2.4. *Let* $\vartheta : \mathbb{D} \to \mathscr{L}(\mathfrak{H})$ *be a contractive analytic function such that* $\vartheta(0) \geqslant 0$. *Then* $I + \vartheta(z)$ *is invertible for every* $z \in \mathbb{D}$.

Proof. The Schwarz lemma implies that an analytic function $g : \mathbb{D} \to \overline{\mathbb{D}}$ maps the disk $|z| \leqslant r$ to the disk bounded by the circle $(z - g(0))/(1 - \overline{g(0)}z) = r$. In particular, if $g(0) \geqslant 0$, then $\Re g(z) \geqslant -|z|$ for all z. Applying this observation to the functions $g_h(z) = \langle \vartheta(z)h, h \rangle$ with $\|h\| = 1$, we deduce that $\Re(I + \vartheta(z)) \geqslant 1 - |z|$, and invertibility follows. $\square$

3. Functional Calculus and Spectrum

As mentioned in the introduction, a contraction $T \in \mathscr{L}(\mathfrak{H})$ is said to be absolutely continuous provided that $T = T_0 \oplus T_1$, with T_0 completely nonunitary, and T_1 a unitary operator with absolutely continuous spectral measure (relative to normalized arclength measure m on $\mathbb{T}$). The reason for this terminology is that a contraction is absolutely continuous if and only if its minimal unitary dilation U has absolutely continuous spectrum. For such contractions, the polynomial functional calculus $u \mapsto u(T)$ can be extended to arbitrary elements of the algebra H^∞ of bounded analytic functions defined in $\mathbb{D}$. The operator $u(T)$ is simply the compression to $\mathfrak{H}$ of $u(U)$, and $u(U)$ is defined using the spectral measure of U. The absolutely continuous operator T belongs to the class $\mathbb{A}$ if the map $u \mapsto u(T)$ is an isometry from H^∞ into $\mathscr{L}(\mathfrak{H})$. It is known [6, Proposition 7.3] that for T in the class $\mathbb{A}$, the essential norm $\|u(T)\|_e$ is also equal to $\|u\|_\infty$ for every $u \in H^\infty$. Moreover, H^∞ is a function algebra in the technical sense that the norm of each element is equal to its spectral radius. We deduce that for $T \in \mathbb{A}$ and $u \in H^\infty$ we have

$$|u(T)|_{sp} = \|u(T)\| = \|u(T)\|_e,$$

where $|X|_{\mathrm{sp}}$ denotes the spectral radius of X. It is well-known (see, for instance, [11]) that

$$\sigma(u(T)) \supset u(\sigma(T) \cap \mathbb{D}), \quad u \in H^\infty.$$

It follows that $T \in \mathbb{A}$ provided that $\sigma(T) \supset \mathbb{D}$ or, more generally, if $\sigma(T) \cap \mathbb{D}$ is a dominating set in the sense of [9].

We will describe some necessary, and some sufficient, conditions for a cnu contraction to belong to $\mathbb{A}$ in terms of characteristic functions. As mentioned in the preceding section, every cnu contraction is unitarily equivalent to a functional model. We will therefore consider a contraction $T = S(\vartheta) \in \mathscr{L}(\mathfrak{H}(\vartheta))$, where $\vartheta : \mathbb{D} \to \mathscr{L}(\mathfrak{F}, \mathfrak{G})$ is a purely contractive analytic function, and we will continue to use the notation Δ, U_+, $\mathfrak{K}_+$ from the preceding section. We need to introduce some further notation to facilitate the statement of a form of the commutant lifting theorem. We will denote by $H^\infty(\mathscr{L}(\mathfrak{G}))$ the Banach algebra of all bounded analytic functions $F : \mathbb{D} \to \mathscr{L}(\mathfrak{G})$. The elements in this algebra determine (by pointwise action) operators on $H^2(\mathfrak{H})$ which commute with the operator of multiplication by the variable. The operator determined by a function F will also be denoted by F. We will denote by $L^\infty(\mathscr{L}(\mathfrak{G}, \mathfrak{F}))$ the Banach algebra of weak operator measurable functions $F : \mathbb{T} \to \mathscr{L}(\mathfrak{G}, \mathfrak{H})$ which are essentially bounded (relative to m). The elements in this algebra determine operators in $\mathscr{L}(L^2(\mathfrak{G}), L^2(\mathfrak{F}))$, again by pointwise application. These operators can be restricted to $H^2(\mathfrak{G})$ to yield elements in $\mathscr{L}(H^2(\mathfrak{G}), L^2(\mathfrak{F}))$.

LEMMA 3.1. *Consider a cnu contraction* $T = S(\vartheta)$. *For every operator* $X \in \{T\}'$, *there exist functions* $X_{11} \in H^\infty(\mathscr{L}(\mathfrak{G}))$, $X_{21} \in L^\infty(\mathscr{L}(\mathfrak{G}, \mathfrak{F}))$, $X_{22} \in L^\infty(\mathscr{L}(\mathfrak{F}))$, *and* $\Psi \in H^\infty(\mathscr{L}(\mathfrak{F}))$ *with the following four properties:*

1. $X_{11}\vartheta = \vartheta\Psi$,
2. $X_{21}\vartheta + X_{22}\Delta = \Delta\Psi$,
3. *the range of* $X_{22}(\zeta)$ *is contained in* $(\Delta(\zeta)\mathfrak{F})^-$ *for almost every* $\zeta \in \mathbb{T}$,
4. $X = P_{\mathfrak{H}(\vartheta)}\widehat{X}|\mathfrak{H}(\vartheta)$, *where*

$$\widehat{X} = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{bmatrix}.$$

Conversely, given functions X_{11}, X_{21}, X_{22} *and* Ψ *satisfying conditions (1-3), the operator* X *defined by (4) belongs to the commutant* $\{T\}'$. *The operator* X *is equal to zero if and only if there exists a function* $\Phi \in H^\infty(\mathscr{L}(\mathfrak{G}, \mathfrak{F}))$ *such that*

$$X_{11} = \vartheta\Phi, \quad X_{21} = \Delta\Phi, \quad and \quad X_{22} = 0.$$

Proof. As mentioned above, this is merely a formulation of the commutant lifting theorem [22]. □

Given a function $u \in H^\infty$, the preceding lemma shows that $u(T)$ is invertible if and only if there exist functions X_{ij}, Ψ and Φ satifying conditions (1-3) in the statement, and in addition

$$uX_{11} - I = \vartheta\Phi, \tag{8}$$

$$uX_{21} = \Delta\Phi, \tag{9}$$

and

$$uX_{22} = I. \tag{10}$$

Let us set $\omega = \omega_T = \{\zeta \in \mathbb{T} : \Delta(\zeta) \neq 0\}$, and note that X_{21} and X_{22} are always zero on the complement of ω. Therefore, if X_{11} and Φ can be found to satisfy (8), then X_{21}, X_{22}, and Ψ are uniquely determined as

$$X_{21} = \frac{1}{u}\Delta\Phi, \quad X_{22} = \frac{1}{u}I, \quad \text{and} \quad \Psi = \frac{1}{u}(I + \Phi\vartheta).$$

In order for X_{22} to be bounded, it is necessary that u be essentially bounded away from zero on ω, and in this case the function X_{21} is bounded as well. We summarize these observations in the following statement, first proved in [23, Proposition 5.2].

PROPOSITION 3.2. *Consider a cnu contraction* $T = S(\vartheta)$, *a function* $u \in H^\infty$, *and a scalar* $\lambda \in \mathbb{C}$. *The operator* $\lambda - u(T)$ *is invertible if and only if the following conditions are satisfied:*
1. *the function* $\lambda - u$ *is essentially bounded away from zero on* ω,
2. *there exist functions* $X \in H^\infty(\mathscr{L}(\mathfrak{G}))$ *and* $\Phi \in H^\infty(\mathscr{L}(\mathfrak{F}, \mathfrak{G}))$ *with the following properties:*
    (a) $(u - \lambda)X - \vartheta\Phi = I$,
    (b) the meromorphic function $(I + \Phi\vartheta)/(u - \lambda)$ *is bounded in* $\mathbb{D}$.

Proof. The function X plays the role of X_{11} in the above discussion, where we replace the function u by $u - \lambda$. Condition (b) amounts to the existence of the bounded analytic function Ψ. $\square$

The invertibility test in the preceding proposition is easily extended to absolutely continuous contractions. Indeed, if $T = T_0 \oplus T_1$ with T_0 cnu and T_1 unitary with spectral measure E, then the spectrum of $u(T_1)$ is simply the E-essential range of the function u. It will be convenient to introduce the notation Σ_{T_1} for the support of E; this is a set of minimum arclength measure such that $E(\Sigma_{T_1}) = I$. We will also use the more general notation

$$\Sigma_T = \omega_{T_0} \cup \Sigma_{T_1}.$$

These sets are well defined up to a set of arclength measure equal to zero.

We obtain now a first test for $T \in \mathbb{A}$.

PROPOSITION 3.3. *Consider an absolutely continuous contraction* T, *and a function* $u \in H^\infty$. *We have then*

$$\|u(T)\| \geqslant |u(T)|_{sp} \geqslant \text{ess sup}\{|u(\zeta)| : \zeta \in \Sigma_T\}.$$

In particular, if $m(\Sigma_T) = 1$, *it follows that* $T \in \mathbb{A}$.

Proof. By the preceding proposition, the spectrum of $u(T_0)$ contains the essential range of $u|\omega_{T_0}$, while the spectrum of $u(T_1)$ is precisely the essential range of $u|\Sigma_{T_1}$. The proposition follows immediately from these observations since $u(T) = u(T_0) \oplus u(T_1)$. $\square$

This result has a converse in one case which is important for us. Let us recall from [22] that a contractive analytic function $\vartheta \in H^\infty(\mathscr{L}(\mathfrak{F}, \mathfrak{G}))$ is said to have *a scalar multiple* $f \in H^\infty \setminus \{0\}$ if there exists a contractive analytic function $\tau \in H^\infty(\mathscr{L}(\mathfrak{G}, \mathfrak{F}))$ such that

$$\vartheta(z)\tau(z) = f(z)I_\mathfrak{G}, \quad \text{and} \quad \tau(z)\vartheta(z) = f(z)I_\mathfrak{F} \tag{11}$$

for every $z \in \mathbb{D}$. When this happens, the spaces $\mathfrak{F}$ and $\mathfrak{G}$ must have the same dimension, so that ϑ coincides with a function in $H^\infty(\mathscr{L}(\mathfrak{F}))$.

THEOREM 3.4. *Assume that T is an absolutely continuous contraction such that the characteristic function Θ_T has a scalar multiple. Then $T \in \mathbb{A}$ if and only if $m(\Sigma_T) = 1$.*

Proof. In view of Proposition 3.3, it suffices to show that $T \notin \mathbb{A}$ if $m(\mathbb{T} \setminus \Sigma_T) > 0$. Assume that this last condition is satisfied, let $\vartheta \in H^\infty(\mathscr{L}(\mathfrak{F}))$ coincide with Θ_T, and let $f \in H^\infty \setminus \{0\}$ be a scalar multiple of ϑ. We can then find a function $u \in H^\infty$ which is a multiple of f, such that $\|u\|_\infty = 1$ and $|u(\zeta)| \leqslant 1/2$ for almost every $\zeta \in \Sigma_T$. Indeed, such a function can be obtained as the product of f with an appropriate outer function. We will show that $T \notin \mathbb{A}$ by proving that $|u(T)|_{\mathrm{sp}} < 1 = \|u\|_\infty$. Since $|u(T)|_{\mathrm{sp}} \leqslant 1$, it will suffice to show that $\lambda - u(T)$ is invertible for every $\lambda \in \mathbb{T}$. If we use the decomposition $T = T_0 \oplus T_1$ into cnu and unitary parts, we have

$$\|u(T_1)\| = \text{ess } \sup\{|u(\zeta)| : \zeta \in \Sigma_{T_1}\} \leqslant \frac{1}{2},$$

so it suffices to show that $\lambda - u(T_0)$ is invertible. Fix then $\lambda \in \mathbb{T}$, and observe that

$$\inf\{|u(z) - \lambda| + |f(z)| : z \in \mathbb{D}\} > 0.$$

The scalar corona theorem implies the existence of functions $x, \varphi \in H^\infty$ such that

$$x(u - \lambda) - \varphi f = 1.$$

Fix a function τ satisfying (11), and set

$$X = xI, \quad \Phi = \varphi\tau.$$

Condition (1) of Proposition 3.2 is clearly satisfied because $\omega_{T_0} \subset \Sigma_T$, and X and Φ satisfy part (a) of condition (2). Part (b) is also satisfied because ϑ and Φ commute, and therefore

$$\frac{I + \Phi\vartheta}{u - \lambda} = \frac{I + \vartheta\Phi}{u - \lambda} = X.$$

We conclude that $\lambda - u(T_0)$ is invertible, as claimed. $\square$

We conclude this section with a simpler criterion for $T \in \mathbb{A}$.

PROPOSITION 3.5. *An absolutely continuous contraction T belongs to $\mathbb{A}$ if either of the following two conditions is satisfied:*

1. *$\partial_T \neq \partial_{T^*}$;*
2. *$I - T^*T$ is a compact operator and $\dim\ker(T) \neq \dim\ker(T^*)$.*

Proof. If (1) holds, then $\Theta_T(z)$ cannot be invertible for any $z \in \mathbb{D}$, and therefore $\sigma(T) = \overline{\mathbb{D}}$ by the characterization of the spectrum in terms of the characteristic function [22]. If (2) holds, then T is either essentially unitary with nonzero index, in which case its Fredholm spectrum is $\mathbb{D}$, or it is a nununitary essential isometry, in which case $\mathbb{D}$ is contained in the essential spectrum. Thus, under either hypothesis we have $\sigma(T) = \overline{\mathbb{D}}$, and this implies that $T \in \mathbb{A}$. $\square$

4. Polar Decomposition and the Class $\mathbb{A}$

In this section we will only consider contractions T such that $I - T^*T$ is a trace class operator. We will denote by $\mathfrak{S}_p$ the Schatten p-class of compact operators, so that $\mathfrak{S}_1$ is the trace class.

Let us recall that, in the terminology of [22, Chapter VIII], a contraction T is called a *weak contraction* if $I - T^*T \in \mathfrak{S}_1$ and $\sigma(T) \neq \overline{\mathbb{D}}$.

LEMMA 4.1. *Assume that $T \in \mathscr{L}(\mathfrak{H})$ satisfies $I - T^*T \in \mathfrak{S}_1$, and $V \in \mathscr{L}(\mathfrak{H})$ a unitary operator such that $T = V|T|$.*

1. *We have $1 - |T| = I - V^*T \in \mathfrak{S}_1$ and $V - T \in \mathfrak{S}_1$.*
2. *The function $I - V^*\vartheta_T$ has a scalar multiple.*
3. *If T is completely nonunitary, then $\mathfrak{D}_T$ is *-cyclic for V, i.e. $\mathfrak{H} = \bigvee_{n \in \mathbb{Z}} V^n \mathfrak{D}_T$.*
4. *If, in addition, T is a weak contraction, then ϑ_T also has a scalar multiple.*

Proof. Part (1) follows from the identities

$$I - |T| = (I - T^*T)(I + |T|)^{-1}, \quad V - T = V(I - |T|).$$

The existence of the unitary V implies that $\ker(T)$ and $\ker(T^*)$ have the same finite dimension, and this implies that both D_T and D_{T^*} belong to $\mathfrak{S}_2$. Thus

$$D_{T^*}(I - zT^*)^{-1}D_T \in \mathfrak{S}_1, \quad z \in \mathbb{D},$$

so that

$$\frac{1}{2}(I - V^*\vartheta_T) = I + \Psi,$$

where $\Psi(z) \in \mathfrak{S}_1$ for all $z \in \mathbb{D}$. Since $I - V^*\vartheta_T(z)$ is invertible for all $z \in \mathbb{D}$, it follows that the function $\det(I + \Psi)$ is not zero, and hence it is a scalar multiple of $I + \Psi$ and $I - V^*\vartheta$; see the discussion in [5, Section VI.2].

In order to verify (3), consider the reducing space

$$\mathfrak{M} = \mathfrak{H} \ominus \bigvee_{n \in \mathbb{Z}} V^n \mathfrak{D}_T$$

for V. Since $\mathfrak{M}$ is orthogonal to $\mathfrak{D}_T$, $V|\mathfrak{M} = T|\mathfrak{M}$, and hence $\mathfrak{M} = \{0\}$ if T is cnu.

Finally, the fact that ϑ_T has a scalar multiple is equivalent to the fact that Θ_T has a scalar multiple, and this is known [22, Theorem VIII.1.1] for weak contractions. $\square$

The preceding lemma allows us to transform information about T into information about V. Let us write $V = V_c \oplus V_s$, with V_c absolutely continuous and V_s singular (relative to arclength measure). We will use the notation $\Sigma_V = \Sigma_{V_c}$.

THEOREM 4.2.   *Let T be a cnu contraction with $I - T^*T \in \mathfrak{S}_1$, and let V be a unitary operator such that $T = V|T|$. Then $\Sigma_V = \omega_T$ up to a set of m-measure zero.*

Proof. Let $u \in H^\infty \setminus \{0\}$ be a scalar multiple of $I - V^* \vartheta_T$, so that we have

$$\varphi(z)(I - V^* \vartheta_T(z)) = (I - V^* \vartheta_T(z))\varphi(z) = u(z)I, \quad z \in \mathbb{D},$$

for some function $\varphi \in H^\infty(\mathscr{L}(\mathfrak{H}))$. Equivalently, $\varphi(z)/u(z) = (I - V^* \vartheta_T(z))^{-1}$ for all $z \in \mathbb{D}$ for which $u(z) \neq 0$. Functions in $H^\infty(\mathscr{L}(\mathfrak{H}))$ have radial limits in the strong operator topology, and we deduce that $(I - V^* \vartheta_T(z))^{-1}$ must also have such radial limits. Now,

$$\frac{\varphi(\zeta)}{u(\zeta)}(I - V^* \vartheta_T(\zeta)) = (I - V^* \vartheta_T(\zeta))\frac{\varphi(\zeta)}{u(\zeta)} = I$$

for almost every $\zeta \in \mathbb{T}$. Therefore the radial limit of $(I - V^* \vartheta_T(z))^{-1}$ at ζ is precisely the inverse of $I - V^* \vartheta_T(\zeta)$.

Use now the notation of Section 2, and take radial limits in (7) to obtain for each $h \in \mathfrak{H}$

$$\|\Delta(\zeta)(I - V^* \vartheta_T(\zeta))^{-1}h\|^2 = \frac{d\mu_{Bh}}{dm}(\zeta) \tag{12}$$

almost everywhere on $\mathbb{T}$; here $\Delta(\zeta) = (I - \vartheta_T(\zeta)^* \vartheta_T(\zeta))^{1/2}$, and this function is equal to zero almost everywhere on $\mathbb{T} \setminus \omega_T$. By part (3) of the preceding lemma, the vectors $\{D_T h : h \in \mathfrak{H}\}$ form a *-cyclic set for V. We conclude that for the spectral measure E_c of V_c we have $E_c(\mathbb{T} \setminus \omega_T) = 0$, and therefore $\sigma_V \subset \omega_T$ (up to a set of m-measure zero). To conclude the proof we consider the set $\omega_0 = \omega_T \setminus \sigma_T$. Since $I - V^* \vartheta_T(\zeta)$ is invertible for almost every ζ, equation (12) implies that $\Delta(\zeta)k = 0$ almost everywhere on ω_0 for every $k \in \mathfrak{H}$. This implies that $\Delta(\zeta) = 0$ almost everywhere on ω_0 (because the space $\mathfrak{H}$ is separable), and therefore $m(\omega_0) = 0$. $\square$

The particular case of Theorem 4.2 in which $m(\Sigma_V) = 0$ yields the following extension of [24, Theorem 2.1].

THEOREM 4.3. *Let T be a weak contraction, and V a unitary operator such that $T = V|T|$. The following are equivalent:*
1. *V is purely singular; and*
2. *$T = T_0 \oplus T_1$, where T_0 is an operator of class C_0 and T_1 is a purely singular unitary operator.*

Proof. Decompose $T = T_0 \oplus T_1$ into its cnu and unitary components. We can also write $V = V_0 \oplus T_1$, where V_0 is a unitary operator such that $T_0 = V_0|T_0|$. Assertion (1) is equivalent to $m(\Sigma_{V_0}) = 0$, while (2) is equivalent to $m(\omega_{T_0}) = 0$; cf. [22, Theorem VIII.2.1]. The result follows because $m(\Sigma_{V_0}) = m(\omega_{T_0})$ by Theorem 4.2. $\square$

Whether T is in $\mathbb{A}$ can now also be determined in terms of V.

THEOREM 4.4. *Let the absolutely continuous contraction $T \in \mathcal{L}(\mathfrak{H})$ satisfy $I - T^*T \in \mathfrak{S}_1$, and assume that $T = V|T|$ for some unitary operator $V \in \mathcal{L}(\mathfrak{H})$. Then $T \in \mathbb{A}$ if and only if $m(\Sigma_V) = 1$.*

Proof. We consider first the case of a weak contraction T. As mentioned above, in this case ϑ_T has a scalar multiple, and therefore Theorem 3.4 applies: $T \in \mathbb{A}$ if and only if $m(\Sigma_T) = 1$. It will suffice to show that $\Sigma_T = \Sigma_V$ up to a set of measure zero. To do this, we write the decomposition $T = T_0 \oplus T_1$ into cnu and unitary parts, and the corresponding decomposition $V = V_0 \oplus T_1$. The desired equality follows immediately from Theorem 4.2 applied to T_0 and V_0.

Assume now that T is not a weak contraction, i.e. $\sigma(T) = \overline{\mathbb{D}}$. In this case we know that $T \in \mathbb{A}$ from the first paragraph in Section 3, and we must show that $m(\Sigma_V) = 1$ as well. Assume to the contrary that $m(\mathbb{T} \setminus \Sigma_V) > 0$. As shown in [17], this hypothesis implies that $\sigma(V + K) \setminus \sigma(V)$ contains only isolated Fredholm eigenvalues if $K \in \mathfrak{S}_1$; in particular $\sigma(V + K) \not\supset \mathbb{D}$ for such K. This contradicts the fact that

$$\sigma(V + (T - V)) = \sigma(T) \supset \mathbb{D},$$

while $T - V \in \mathfrak{S}_1$. $\square$

Our main result on perturbations of operators in $\mathbb{A}$ is now easy to prove.

THEOREM 4.5. *Let $T, T' \in \mathcal{L}(\mathfrak{H})$ be absolutely continuous contractions such that $I - T^*T \in \mathfrak{S}_1$ and $T - T' \in \mathfrak{S}_1$. Then $T \in \mathbb{A}$ if and only if $T' \in \mathbb{A}$.*

Proof. Observe first that $I - T'^*T'$ must also belong to $\mathfrak{S}_1$. In case $\sigma(T) = \sigma(T') = \overline{\mathbb{D}}$ both T and T' belong to $\mathbb{A}$. Therefore it will suffice to prove the theorem under the additional assumption that one of the two contractions is weak. As the difference $T - T'$ is compact and T is semi-Fredholm, we have the equality of Fredholm indices

$$\dim \ker(T) - \dim \ker(T^*) = \dim \ker(T') - \dim \ker(T'^*).$$

One of these differences is zero, and hence the other is zero as well. We deduce that there exist unitary operators V, V' such that $T = V|T|$ and $T' = V'|T'|$. Note that

$$V - V' = (V - T) + (T - T') + (T' - V)$$

belongs to $\mathfrak{S}_1$, and the Kato-Rosenblum theorem implies that $\Sigma_V = \Sigma_{V'}$ up to a set of measure zero. In particular, $m(\Sigma_V) = 1$ if and only if $m(\Sigma_{V'}) = 1$, and the desired conclusion follows immediately from the preceding theorem. $\square$

The preceding result can be viewed as an instance of the following invariance result for the set Σ_T.

THEOREM 4.6. *Let $T, T' \in \mathcal{L}(\mathfrak{H})$ be two absolutely continuous contractions such that $I - T^*T \in \mathfrak{S}_1$, $\dim \ker(T) = \dim \ker(T^*)$, and $T - T' \in \mathfrak{S}_1$. Then $\Sigma_T = \Sigma_{T'}$ up to a set of m-measure zero.*

Proof. Note that we also have $\dim\ker(T') = \dim\ker(T'^*)$, and therefore $T = V|T|$ and $T' = V'|T'|$ for some unitary operators V and V' such that $V - V' \in \mathfrak{S}_1$. By the Kato-Rosenblum theorem, it will suffice therefore to show that $\Sigma_T = \Sigma_V$ and $\Sigma_{T'} = \Sigma_{V'}$. Write then $T = T_1 \oplus T_2$ with T_1 cnu and T_2 unitary, and consider the corresponding decomposition $V = V_1 \oplus T_2$. It suffices then to show that $\omega_{T_1} = \Sigma_{V_1}$, and this equality follows from Theorem 4.2. $\square$

5. Miscellaneous Examples

One may inquire whether the methods presented here will eventually yield our conjecture. Unfortunately, several of the ingredients fail in general. Consider, for instance, a diagonalizable operator $T \in \mathbb{A}$, and its polar decomposition $T = V|T|$. In this case the operator V itself is diagonalizable, hence purely singular: $m(\Sigma_V) = 0$. Thus one cannot hope to characterize the class $\mathbb{A}$ in terms of polar decompositions.

In the diagonalizable case, it is still true that T is of class C_{00}. There exist however operators $T = V|T|$ of class C_{00} for which V is absolutely continuous, thus violating the equality $\omega_T = \Sigma_V$. An example is obtained as follows. Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis in $\mathfrak{H}$, and define a weighted shift T by the requirement that

$$Te_n = \left(\frac{|n|+1}{|n|+2}\right)e_{n+1}, \quad n \in \mathbb{Z}.$$

For this T, the unitary V is a bilateral shift. Thus $\Sigma_V = \mathbb{T}$, while $\omega_T = \emptyset$.

We would like to conclude with a discussion of unitary operators on $\mathfrak{H} \oplus \mathfrak{K}$ of the form

$$V = \begin{bmatrix} T & X \\ Y & Z \end{bmatrix},$$

in other words, unitary dilations of T in the sense of Halmos [13] (as opposed to the power dilations considered by Sz.-Nagy [21]). As seen, for instance, in [1], the operators X, Y, Z can be written as

$$X = D_{T^*}B, \quad Y = AD_T, \quad Z = -AT^*B + W,$$

where $A, B^* \in \mathscr{L}(\mathfrak{H}, \mathfrak{K})$ and $W \in \mathscr{L}(\mathfrak{K})$ are partial isometries such that

$$A^*A = P_{\mathfrak{D}_T}, BB^* = P_{\mathfrak{D}_{T^*}}, W^*W = I - B^*B, \text{ and } WW^* = I - AA^*.$$

Since V is unitary, the operator

$$\widetilde{T} = \begin{bmatrix} T & 0 \\ AD_T & 0 \end{bmatrix}$$

is partially isometric, and it has the decomposition $\widetilde{T} = V|\widetilde{T}|$, with

$$|\widetilde{T}| = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Easy calculations show that

$$D_{\widetilde{T}} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, D_{\widetilde{T}^*} = \begin{bmatrix} D_{T^*}^2 & -TD_T A^* \\ -AD_T T^* & WW^* + AT^*TA^* \end{bmatrix},$$

$$(I - z\widetilde{T}^*)^{-1} D_{\widetilde{T}} = \begin{bmatrix} 0 & z(I - zT^*)^{-1} D_T A^* \\ 0 & I \end{bmatrix},$$

and finally

$$\vartheta_{\widetilde{T}}(z) = \begin{bmatrix} -T & zD_{T^*} \vartheta_T(z) A^* \\ -AD_T & -zAT^* \vartheta_T(z) A^* + zWW^* \end{bmatrix}$$

for $z \in \mathbb{D}$. Therefore

$$V^* \vartheta_{\widetilde{T}}(z) = \begin{bmatrix} I & 0 \\ 0 & zB^* \vartheta_T(z) A^* + zW^* \end{bmatrix}, \quad z \in \mathbb{D}.$$

In this formula, ϑ_T could be replaced by Θ_T if we view A as an element in $\mathscr{L}(\mathfrak{D}_T, \mathfrak{K})$ and B as an element in $\mathscr{L}(\mathfrak{K}, \mathfrak{D}_{T^*})$. This formula, along with Theorem 2.3, gives us in principle the spectral measure of V. As seen earlier in this paper, it may be difficult to exploit this formula in the absence of certain scalar multiples. The existence of such scalar multiples is guaranteed if $\dim(\mathfrak{K}) < \infty$, which is only possible when $\mathfrak{d}_T = \mathfrak{d}_{T^*} < \infty$. In this case we deduce easily that V is purely singular if and only if T is the orthogonal sum of an operator of class $C_0(N)$ ($N = \mathfrak{d}_T$) with a singular unitary operator. The fact that an operator of class $C_0(N)$ has a singular unitary dilation was first proved in [25]. Wu and Takahashi also prove a converse of this result: if a contraction with finite defect indices has a singular unitary dilation (even with $\mathfrak{K}$ of infinite dimension), then the operator is the direct sum of a $C_0(N)$ operator with a singular unitary. Our methods do not yield this implication.

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