Operators
and
Matrices
Volume 2, Number 2 (2008), 153–166

INERTIA THEOREMS BASED ON OPERATOR LYAPUNOV EQUATIONS

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(*communicated by R. Bhatia*)

Abstract. The well-known Carlson–Schneider inertia theorem for finite matrices, satisfying the Lyapunov equation with a semi-definite right-hand side, is extended to linear operators acting on an infinite dimensional Hilbert space. The proofs use extensively the theory of linear operators acting on indefinite inner product spaces. An application to stability problems of semigroups is also presented.

1. Introduction and main results

In 1962 D. Carlson and H. Schneider proved the following fundamental result (see [5]). For an $n\times n$ complex matrix A let $\pi(A), \nu(A)$, and $\delta(A)$ denote the number of eigenvalues, counting multiplicities, located in the open right halfplane $\mathbb{C}^+$, the open left halfplane $\mathbb{C}^-$, and on the imaginary axis $i\mathbb{R}$, respectively.

THEOREM 1.1. *Let $A \in \mathbb{C}^{n\times n}$ and let X be a Hermitian matrix such that*

$$AX + XA^* = W \geqslant 0.$$

(i) If $\delta(A) = 0$, then $\pi(X) \leqslant \pi(A)$ and $\nu(X) \leqslant \nu(A)$.
(ii) If X is invertible, then $\pi(A) \leqslant \pi(X)$ and $\nu(A) \leqslant \nu(X)$.
(iii) From (i) and (ii) it follows that if $\delta(A) = \delta(X) = 0$, then $\pi(X) = \pi(A)$ and $\nu(X) = \nu(A)$.

The main goal of this paper is to extend this result (and in particular part (iii)) to the case of linear operators acting on infinite dimensional Hilbert spaces (see Theorem 1.6 and Theorem 1.7 below).

To state our main results and to put them into a proper perspective we now introduce some appropriate notions and briefly describe the relevant developments in inertia theory.

Mathematics subject classification (2000): 47A62, 47B06, 47B50, 47D60, 47N70, 93A99, 15A24.
Key words and phrases: Spectrum localization, Lyapunov equation, inertia theorems, bisemigroup.
The research of the first author was partially supported by a visitor fellowship of the Netherlands Organization for Scientific Research (NWO).

Paper OAM-02-10

In what follows $\mathscr{H}$ is a complex Hilbert space, and $L(\mathscr{H})$ denotes the set of all linear bounded operators from $\mathscr{H}$ into $\mathscr{H}$. An operator $W \in L(\mathscr{H})$ will be called nonnegative if $\langle x, Wx\rangle \geqslant 0$ for all $x \in \mathscr{H}$ (which we will denote by $W \geqslant 0$), positive ($W > 0$) if for all nonzero $x \in \mathscr{H}$ $\langle x, Wx\rangle > 0$, and uniformly positive if for all nonzero $x \in \mathscr{H}$ $\langle x, Wx\rangle \geqslant \delta \|x\|^2$ for some positive number δ. Observe that in case $\mathscr{H}$ is finite dimensional the latter two concepts coincide. In an infinite dimensional space for an operator W to be uniformly positive is equivalent to being positive and invertible in $L(\mathscr{H})$. By $\sigma(A)$ we denote the spectrum of the operator A, be it bounded or unbounded.

Let $A(\mathscr{H} \to \mathscr{H})$ be a closed linear operator and let Γ be a simple, closed, rectifiable contour in the complex plane, such that Γ does not intersect the spectrum of A. Let $\sigma \subset \sigma(A)$ be an isolated part of spectrum of A, which means that both σ and $\sigma(A) \setminus \sigma$ are closed subsets of $\sigma(A)$, let σ be contained in the domain bounded by Γ, and let $\sigma(A)\sigma$ be outside the domain bounded by Γ. By

$$P_\sigma(A) = \frac{1}{2\pi i}\int_\Gamma (\lambda I - A)^{-1} \mathrm{d}\lambda$$

we denote the Riesz projection of A corresponding to σ. For an operator $A(\mathscr{H} \to \mathscr{H})$ with domain $D(A)$, a subspace $M \subset \mathscr{H}$ is called A-invariant if $A(M \cap D(A)) \subset M$. In that case $A|_M$ denotes the restriction of the operator A to $M \cap D(A)$.

Now let $\sigma \subset \mathbb{C}$ be closed. An A-invariant subspace M of $\mathscr{H}$ is called the spectral subspace of A associated with σ if

(i) $\sigma(A|_M) \subset \sigma$,
(ii) if N is any A-invariant subspace of $\mathscr{H}$ such that $\sigma(A|_N) \subset \sigma$, then $N \subset M$.

If σ is a bounded isolated subset of $\sigma(A)$, also called a bounded spectral set, then the range of the Riesz projection $P_\sigma(A)$, i.e. $\operatorname{Im} P_\sigma(A)$, is the spectral subspace of A associated with σ.

A point $\lambda_0 \in \sigma(A)$ is called a regular eigenvalue of A if λ_0 is an isolated point of $\sigma(A)$ and the corresponding spectral subspace $\operatorname{Im} P_{\{\lambda_0\}}(A)$ is finite dimensional (in $D(A)$, for the case when A is unbounded). For a nonempty subset $\Sigma \subset \mathbb{C}$, define

$$s(A;\Sigma) = \sum_{\lambda_0 \in \sigma(A)\cap\Sigma} \dim \operatorname{Im} P_{\{\lambda_0\}}(A)$$

if the intersection $\sigma(A) \cap \Sigma$ consists only of a finite number of regular eigenvalues of A. Otherwise, put $s(A;\Sigma) = \infty$.

The inertia $In(A) = (\pi(A), \nu(A), \delta(A))$ of an operator A with respect to the imaginary axis is defined by

$$\pi(A) = s(A;\mathbb{C}^+), \quad \nu(A) = s(A;\mathbb{C}^-), \quad \delta(A) = s(A;\{i\alpha | \alpha \in \mathbb{R}\}).$$

In this paper we deal with the inertia theory, based on the Lyapunov equation

$$AX + XA^* = W \tag{1}$$

with positive or uniformly positive operator W. In the framework of this theory the inertia of the operator A with respect to the imaginary axis is described in terms of the inertia of a self-adjoint solution X of (1). The first result of this nature is the celebrated Lyapunov Theorem, proved more than a century ago.

THEOREM 1.2. (Lyapunov) *Let A and W be $n \times n$ complex matrices such that W is positive definite.*

(i) If A is stable, that is $\nu(A) = n$, then the equation

$$AH + HA^* = W \tag{2}$$

has a unique negative definite solution H.

(ii) If there exists a negative definite matrix H satisfying (2)*, then A is stable.*

In the finite dimensional setting, for the case of positive right hand side W, the fundamental inertia result was obtained by A. Ostrowski and H. Schneider, and independently by M. Krein, in the early 60's (see [16], [13]), and reads as follows.

THEOREM 1.3. *Let $A \in \mathbb{C}^{n\times n}$. If X is a Hermitian matrix such that*

$$AX + XA^* = W \tag{3}$$

with W (uniformly) positive, then X is nonsingular and

$$\delta(A) = \delta(X) = 0, \tag{4a}$$

$$\pi(A) = \pi(X), \quad \nu(A) = \nu(X). \tag{4b}$$

Conversely, if $\delta(A) = 0$, then there exists a Hermitian matrix X such that the equations (3)*,* (4a) *and* (4b) *hold.*

A similar result was obtained by Taussky (see [20]).

The main result for the case of semidefinite right-hand side W is the theorem of Carlson and Schneider cited above (Theorem 1.1).

Further significant advances in the inertia theory were connected with the idea of controllability, which emerged from system theory and was introduced by Chen and Wimmer (see [6, 22]) and J. Snyders and M. Zakai (see [18]) into the study of inertia of matrices.

We define now the relevant notions. Observe that our definition of exact controllability differs from the one given in e.g., [7].

DEFINITION 1.1. Let $\mathscr{G}$ and $\mathscr{H}$ be Hilbert spaces. A pair of operators (A,B), where $A \in L(\mathscr{H})$ and $B \in L(\mathscr{G},\mathscr{H})$, is called almost exactly controllable if for some $p \in \mathbb{N}$ the linear set

$$\operatorname{Im}[B, AB, A^2B, \ldots, A^{p-1}B] = \sum_{j=0}^{p-1} \operatorname{Im} A^j B$$

is closed and has finite codimension in $\mathscr{H}$. If

$$\operatorname{Im}[B, AB, A^2B, \ldots, A^{p-1}B] = \mathscr{H}$$

for some p, then a pair (A,B) is called *exactly controllable*. A pair (A,B) is called *approximately controllable* if

$$\overline{\operatorname{Im}[B,AB,A^2B,A^3B\ldots]} = \mathscr{H}.$$

The following result has been proved for matrices in [6], [22].

THEOREM 1.4. *Let A and X be $n\times n$ complex matrices such that $X = X^*$,*

$$AX + XA^* = W \geqslant 0$$

and let the pair (A,W) be exactly controllable. Then

$$\delta(A) = \delta(X) = 0, \quad \pi(A) = \pi(X), \quad \nu(A) = \nu(X).$$

A more detailed exposition of the finite dimensional case can be found in the well-known book [14].

Now we turn to the infinite dimensional case. In this case the Lyapunov Theorem reads as follows (see [8]): The spectrum of an operator $A \in L(\mathscr{H})$ lies in the open left halfplane, that is $\sigma(A) \subset \mathbb{C}^-$, if and only if there exists a uniformly positive operator X such that $AX + XA^* = W$ is uniformly negative. Moreover, if $\sigma(A) \subset \mathbb{C}^-$, then for every uniformly negative W there exists a uniformly positive operator X, such that $AX + XA^* = W$.

Some generalizations of the Theorem 1.3 were developed in [4] for the case of W being uniformly positive and the operators A, X, W acting in a Hilbert space. In [19] the operator equation $AX+XA^* = -C$ with nonnegative C, was studied in connection with the notion of approximate controllability, and various conditions for the existence of a nonnegative solution for this equation were investigated.

In the infinite dimensional setting Theorem 1.4 was generalized in [3] for the case when the pair (A,W) is exactly controllable. In [15] it was shown that if the assumption that the pair (A,W) is exactly controllable in Theorem 1.4 (and in [3]) is weakened to almost exactly controllability, then estimates for $|\pi(A) - \pi(X)|$, $|\nu(A) - \nu(X)|$ and also for $\delta(A)$ and $\delta(X)$ can be found.

The following important result was obtained in [17]. Observe that the Lyapunov equation here is different from the one in the earlier theorems; we chose to state the theorem exactly as it is stated in [17].

THEOREM 1.5. *Let A be a densely defined closed operator on a Hilbert space $\mathscr{H}$ with domain $D(A)$. Suppose that $\sigma = \sigma(A)\cap\mathbb{C}^+$ is a bounded spectral set of A, $\dim\operatorname{Im}P_\sigma(A) < \infty$, $H \in L(\mathscr{H})$ is a selfadjoint operator, such that $0 \notin \sigma_p(H)$, $\nu(H) < \infty$ and*

$$\langle (A^*H + HA)x, x\rangle \leqslant 0 \quad \textit{for all} \quad x \in D(A)$$

Then $\pi(A) \leqslant \nu(H)$.

The above Theorem generalizes part (ii) of Theorem 1.1. The following two theorems, which are the main results of the present paper, can be viewed as a full generalization of part (iii) of the Theorem 1.1 to the infinite dimensional setting, for the bounded and unbounded cases.

THEOREM 1.6. *Let $A, H \in L(\mathscr{H})$ be bounded linear operators, such that H is self-adjoint and invertible, $\nu(H) < \infty$, the spectrum of A does not contain eigenvalues which lie on the imaginary axis, and $\sigma(A) \cap \mathbb{C}^-$ is a spectral set. If*

$$A^*H + HA \geqslant 0 \tag{5}$$

then $\nu(H) = \nu(A)$.

Recall that a linear operator A is called boundedly invertible if there exists a bounded linear operator $B : \mathscr{H} \to \mathscr{H}$ such that $\operatorname{Im} B = D(A)$, $ABx = x$ for all $x \in \mathscr{H}$ and $BAx = x$ for all $x \in D(A)$. In that case we denote $B = A^{-1}$.

THEOREM 1.7. *Let $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ be a linear, densely defined closed operator with domain $D(A)$. Assume, in addition, that A is boundedly invertible, the spectrum of A does not contain eigenvalues which lie on the imaginary axis, and $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set. Suppose $H \in L(\mathscr{H})$ is a self-adjoint invertible operator such that $\nu(H) < \infty$ and*

$$\langle (A^*H + HA)x, x\rangle \geqslant 0, \qquad \forall x \in D(A). \tag{6}$$

Then $\nu(H) = \nu(A)$.

In the next section we give some background in the theory of indefinite inner products. Section 3 contains the proof of a lemma which is the basis for further proofs. In Section 4 we prove the main results of this paper. In Section 5 we give some background of the theory of strongly-continuous semigroups and application of our results. Section 6 is devoted to additional applications.

2. The Pontryagin space approach

The proof of the main theorem relies on facts from the theory of spaces with indefinite metric. In the following, we introduce several definitions and basic results. Gothic letters will denote general vector spaces with an indefinite inner product.

DEFINITION 2.1. Let $\mathfrak{V}$ be a complex vector space. A map $[\cdot,\cdot] : \mathfrak{V} \times \mathfrak{V} \longrightarrow \mathbb{C}$ satisfying:

(i) $[\alpha x_1 + \beta x_2, y] = \alpha\, [x_1, y] + \beta\, [x_2, y]$ for $x_1, x_2, y \in \mathfrak{V}$ and $\alpha, \beta \in \mathbb{C}$,

(ii) $[x, y] = \overline{[y, x]}$ for all $x, y \in \mathfrak{V}$,

is called an indefinite inner product or indefinite metric.

In contrast with spaces with definite inner product, indefinite inner product spaces may contain vectors x for which $[x, x] < 0$.

DEFINITION 2.2. A vector $x \in \mathfrak{V}$ is called positive, negative or neutral (with respect to $[\cdot,\cdot]$) depending on whether $[x, x] > 0$, $[x, x] < 0$ or $[x, x] = 0$, respectively. Positive (resp. negative) and neutral vectors are combined under the general term non-negative (resp. non-positive) vectors.

We denote the set of all positive (negative) vectors by $\mathfrak{V}^+$ ($\mathfrak{V}^-$) and the set of all neutral vectors by $\mathfrak{V}^0$, i.e.

$$\begin{aligned} \mathfrak{V}^+ &= \{x \in \mathfrak{V} \mid [x,x] > 0\}, \\ \mathfrak{V}^- &= \{x \in \mathfrak{V} \mid [x,x] < 0\}, \\ \mathfrak{V}^0 &= \{x \in \mathfrak{V} \mid [x,x] = 0\}. \end{aligned}$$

A subspace $\mathfrak{W} \subseteq \mathfrak{V}$ is called positive (non-negative), negative (non-positive) or neutral if $\mathfrak{W} \subseteq \mathfrak{V}^+ \cup \{0\}$ ($\mathfrak{W} \subseteq \mathfrak{V}^+ \cup \mathfrak{V}^0$), $\mathfrak{W} \subseteq \mathfrak{V}^- \cup \{0\}$ ($\mathfrak{W} \subseteq \mathfrak{V}^- \cup \mathfrak{V}^0$) or $\mathfrak{W} \subseteq \mathfrak{V}^0$, respectively. A positive subspace $\mathfrak{M}$ is called maximal if for every positive subspace $\mathfrak{M}_1 \supseteq \mathfrak{M}$ we have $\mathfrak{M}_1 = \mathfrak{M}$. Maximal negative, maximal non-positive, maximal non-negative and maximal neutral subspaces are defined in a similar way. Suppose now, that the space $\mathfrak{V}$ admits a decomposition into the direct sum of positive and negative subspaces: $\mathfrak{V} = \mathfrak{N}^+ \dot{+} \mathfrak{N}^-$. Suppose also that the subspaces $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are orthogonal with respect to the indefinite inner product $[\cdot,\cdot]$. In this case the decomposition is called canonical and denoted by

$$\mathfrak{V} = \mathfrak{N}^+ [\dot{+}] \mathfrak{N}^-.$$

A space $\mathfrak{V}$, in which $\mathfrak{N}^+$ and $\mathfrak{N}^-$ form Hilbert spaces with respect to the inner products $[\cdot,\cdot]$ and $-[\cdot,\cdot]$, respectively, is called a Krein space. In the special case where $\kappa = \dim(\mathfrak{N}^-)$ is finite, $\mathfrak{V}$ is called a Pontryagin space and denoted by Π_κ. On a Krein space we can define a definite inner product as follows: for $x = x_+ + x_-$ and $y = y_+ + y_-$, with $x_+, y_+ \in \mathfrak{N}^+$ and $x_-, y_- \in \mathfrak{N}^-$, we define $\langle x, y \rangle = [x_+, y_+] - [x_-, y_-]$. With this inner product the space becomes a Hilbert space, whose norm gives the Krein space its topology and defines its bounded linear operators.

It is known (see [1]) that in a Krein space $\mathfrak{V}$, the dimension of any non-positive subspace $\mathfrak{M} \subset \mathfrak{V} = \mathfrak{N}^+ [\dot{+}] \mathfrak{N}^-$ does not exceed the dimension of $\mathfrak{N}^-$, and if $\mathfrak{M}$ is a maximal non-positive subspace of a Krein space $\mathfrak{V}$, then $\dim \mathfrak{M} = \dim \mathfrak{N}^-$. So the non-positive subspaces of a Pontryagin space Π_κ have dimension at most κ, and exactly κ if and only if they are maximal non-positive.

DEFINITION 2.3. A linear operator A with dense domain $D(A)$, acting in a Krein space $\mathfrak{V}$ is called dissipative, if

$$\Im [Ax, x] \geqslant 0$$

for all $x \in D(A)$. A is called strictly dissipative if

$$\Im [Ax, x] > 0$$

and uniformly dissipative if there exists $\delta > 0$ such that

$$\Im [Ax, x] \geqslant \delta [x, x]$$

for all $0 \neq x \in D(A)$.

Of special interest are invariant subspaces of operators that are dissipative with respect to the indefinite inner product. In this regard, we cite the following result.

THEOREM 2.1. *Let A be a bounded dissipative operator acting in a Pontryagin space Π_κ. Then:*

(i) There exists a maximal non-negative (non-positive) subspace $\mathfrak{M}^+$ $(\mathfrak{M}^-)$ which is A-invariant and $\sigma(A|_{\mathfrak{M}^+}) \subseteq \overline{\mathbb{C}_+}$ $(\sigma(A|_{\mathfrak{M}^-}) \subseteq \overline{\mathbb{C}_-})$,

(ii) If A is a strictly dissipative operator, then there exists unique maximal non-negative (non-positive) subspace $\mathfrak{M}^+$ $(\mathfrak{M}^-)$ which is A-invariant.
Moreover, this subspace is positive (negative) and $\mathfrak{M}^+ \dotplus \mathfrak{M}^- = \Pi_\kappa$.

Here we denote by $\mathbb{C}_-$ and $\mathbb{C}_+$ the open lower and upper half planes, respectively.

In fact, the above theorem is true for every maximal dissipative operator. For the notion of maximal dissipative operator and more details see [12].

Let $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let H be an invertible, bounded selfadjoint operator. Then the inner product defined by

$$[x, y] = \langle Hx, y \rangle \tag{7}$$

is indefinite. Sometimes, $[\cdot,\cdot]$ is called H-inner product, and the corresponding notions are called similarly, e.g. an operator dissipative with respect to H-inner product is called H-dissipative. If in addition, H has precisely κ negative regular eigenvalues (counting multiplicities), then the space $\mathscr{H}$ is a Pontryagin space with respect to $[\cdot,\cdot]$, i.e. $(\mathscr{H}, [\cdot,\cdot]) = \Pi_\kappa$. Here we should note, that when H is taken to be an arbitrary self-adjoint operator, then the induced space endowed with indefinite metric is sometimes called the H-space. Indefinite inner product spaces were used in several works on inertia, see, for example, [10, 15, 17]. It is instructive to restate Theorem 1.3 in terms of basic notions arising in the theory of indefinite inner product spaces for the finite dimensional case.

THEOREM 2.2. *Let H be an invertible self-adjoint operator acting on a finite dimensional Hilbert space $\mathscr{H}$, and let $[\cdot,\cdot]$ be the indefinite inner product defined by (7). Assume that $A \in L(\mathscr{H})$ is a strictly H-dissipative operator, that is*

$$\Im\,[Ax, x] > 0, x \in \mathscr{H} \setminus \{0\}.$$

Then the dimension of a maximal non-negative (non-positive) subspace is equal to the the dimension of the spectral subspace of A corresponding to the upper (lower) half plane.

We will need the following result in the proof of one of the main theorems. The proof of it can be found in [1].

THEOREM 2.3. *Let H be selfadjoint invertible operator and let $\mathfrak{H}$ be the corresponding H-space. Suppose that A is a closed, H-dissipative operator, that is $\frac{1}{i}\langle (HA - A^*H)x, x\rangle \geqslant 0$ for $x \in D(A)$. Let $\sigma = \sigma(A) \cap \mathbb{C}_-$ be a bounded spectral set of A, and let P_σ denote the corresponding Riesz projection. Then the subspace $\operatorname{Im} P_\sigma$ is non-positive and A-invariant.*

3. The Basic Lemma

In this section we denote by Π_κ the Pontryagin space, which arises from the bounded, self-adjoint and invertible operator H, acting on a Hilbert space $(\mathscr{H}, \langle \cdot, \cdot \rangle)$, with a finite number of negative eigenvalues with finite multiplicities, whose sum equals to κ. That is, Π_κ is the space $\mathscr{H}$, endowed with the indefinite inner product given by the formula

$$[x, y] = \langle Hx, y \rangle .$$

The following result is important in the proof of the main theorems.

LEMMA 3.1. *Let A be a bounded, dissipative operator on the Pontryagin space Π_κ with no real eigenvalues. Then there is a κ-dimensional maximal nonpositive A-invariant subspace $\mathfrak{M}$ such that $\sigma(A|_{\mathfrak{M}})$ is in the open lower half plane.*

Proof. As A is dissipative in the indefinite inner product induced by H, we have

$$\frac{1}{i}(HA - A^*H) = D \geqslant 0.$$

For $\delta > 0$ consider $A(\delta) = A + i\delta H^{-1}$. One easily checks that

$$\frac{1}{i}(HA(\delta) - A(\delta)^*H) = D + 2\delta I > 0.$$

Hence $A(\delta)$ is strictly and uniformly dissipative. Therefore, the spectrum of $A(\delta)$ in the open lower half plane is a finite set of regular eigenvalues, and the spectral subspace corresponding to the open lower half plane is a maximal H-nonpositive subspace (see Paragraph 2 of Chapter 2 in [1], also Section 2). So there is an $A(\delta)$-invariant maximal H-nonpositive subspace $\mathfrak{M}(\delta)$.

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be the matrix representation of A with respect to the canonical decomposition $\Pi_\kappa = \Pi_\kappa^+[\dot{+}]\Pi_\kappa^-$. It can be shown (see [12]) that the following equation can be used to characterize the A-invariant maximal nonpositive subspaces

$$A_{12} + A_{11}K = K(A_{22} + A_{21}K), \tag{8}$$

where K is a contractive operator acting from Π_κ^- to Π_κ^+. Let $K(\delta)$ be the angular operator corresponding to the subspace $\mathfrak{M}(\delta)$. Since the closed unit ball $B(\Pi_\kappa^-, \Pi_\kappa^+)$ is compact with respect to the weak operator topology, there exists a sequence $\{\delta_i\}_{i=1}^\infty$ such that $\delta_i \downarrow 0$ and the sequence $K(\delta_i)$ converges to some $K \in B(\Pi_\kappa^-, \Pi_\kappa^+)$. Let $\mathfrak{M}$ be the κ-dimensional, maximal non-positive subspace corresponding to operator K. Since the subspaces $\mathfrak{M}(\delta_i)$ are $A(\delta_i)$-invariant, it follows that $\mathfrak{M}$ is an A-invariant subspace. Indeed, from the equation (8) it follows that

$$A_{12}(\delta_i) + A_{11}(\delta_i)K(\delta_i) = K(\delta_i)(A_{22}(\delta_i) + A_{21}(\delta_i)K(\delta_i)).$$

Note that $A_{kj}(\delta_i)$ converges to A_{kj} in the uniform operator topology. Since $\|K(\delta_i)\| \leqslant 1$, it follows that $A_{21}(\delta_i)K(\delta_i)$ converges in the weak operator topology to $A_{21}K$. As all these operators act in the finite dimensional space Π_κ^- this convergence holds also in the uniform operator topology. The same is true for the sequence $A_{22}(\delta_i)+A_{21}(\delta_i)K(\delta_i)$. Hence the right hand side $K(\delta_i)(A_{22}(\delta_i)+A_{21}(\delta_i)K(\delta_i))$ of the above equation converges in the weak operator topology to $K(A_{22}+A_{21}K)$. In the same way, the left-hand side converges in the weak operator toplogy to $A_{12}+A_{11}K$. We finally get that $A_{12}+A_{11}K = K(A_{22}+A_{21}K)$. Thus, we conclude that the subspace $\mathfrak{M}$ defined by means of operator K is A-invariant. (This argument is the same as the one used in [12], page 85.) It can easily be shown that the A-invariant maximal H-nonpositive subspace $\mathfrak{M}$ is such that $\sigma(A|_{\mathfrak{M}})$ is in the closed lower half plane. Now suppose that λ_0 is a point in $\sigma(A|_{\mathfrak{M}})$ that is on the real line. Since $\mathfrak{M}$ is finite dimensional, it has to be an eigenvalue of $A|_{\mathfrak{M}}$, and hence an eigenvalue of A. But A does not have any point spectrum on the real line by assumption, and so it follows that $\sigma(A|_{\mathfrak{M}})$ is actually in the open lower half plane. □

4. Proof of the main theorems

We are ready now to prove one of the main results of this paper.

Proof of Theorem 1.6. Since H is bounded, selfadjoint, and invertible, and since $\kappa = \nu(H) < \infty$, the indefinite inner product given by

$$[x,y] = \langle Hx, y\rangle \qquad \text{for all} \quad x,y \in \mathscr{H}$$

makes $\mathscr{H}$ into a Pontryagin space Π_κ.

From the Lyapunov equation (5) it follows, that the operator $B = iA$ is dissipative in Π_κ. Also it does not have eigenvalues on the real line. Applying Lemma 3.1 to iA, we conclude that there exists a κ-dimensional, A-invariant, maximal H-nonpositive subspace $\mathfrak{M}$, such that $\sigma(A|_{\mathfrak{M}})$ is entirely in $\mathbb{C}^-$. On the other hand, since $\sigma = \sigma(A)\cap\mathbb{C}^-$ is a spectral set, by Theorem 2.3 $\operatorname{Im} P_\sigma$ is a non-positive A-invariant subspace in Π_κ. From the maximality of $\mathfrak{M}$, it follows that $\dim \operatorname{Im} P_\sigma = \kappa$. □

Next, we prove the second of our main results.

Proof of Theorem 1.7. Since A is a densely defined, closed operator, we recall (see for example [11]) that A is boundedly invertible if and only if A^* is boundedly invertible and in this case $(A^*)^{-1} = (A^{-1})^*$. From (6) and the fact that A is boundedly invertible it follows that

$$\langle (A^{-1})^*(A^*H+HA)A^{-1}x, x\rangle \geqslant 0.$$

for every $x \in \mathscr{H}$. Hence

$$(A^{-1})^*H + HA^{-1} \geqslant 0.$$

Since $\sigma(A)\cap\mathbb{C}^-$ is a bounded spectral set it follows that $\sigma(A^{-1})\cap\mathbb{C}^-$ is a spectral set as well. Also for every nonzero $\lambda \in \mathbb{C}$ if λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is eigenvalue of A^{-1}. The converse is also true. Since bounded invertibility of A implies that 0 is not an eigenvalue of either A or A^{-1}, we conclude that operator A^{-1} does not have eigenvalues on imaginary axis. So, applying Theorem 1.6 to the operator A^{-1}, we conclude that

$$\nu(H) = \nu(A^{-1}).$$

But $\nu(A^{-1}) = \nu(A)$, and the theorem follows. $\square$

5. Applications to Semigroups

Strongly continuous semigroups are important, since many natural phenomena can be described using them.

DEFINITION 5.1. A family $\{T(t)\}_{t\geqslant 0}$ of bounded linear operators acting in a Hilbert space $\mathscr{H}$ is called a strongly continuous semigroup (usually denoted by C_0-semigroup) if the functional equations

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t,s \geqslant 0, \\ T(0) = I, \end{cases}$$

hold, and the map $t \mapsto T(t)$ is continuous in the strong operator topology.

DEFINITION 5.2. The (infinitesimal) generator $A : D(A) \subseteq \mathscr{H} \to \mathscr{H}$ of a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ is the operator defined by

$$Ax = \lim_{h\downarrow 0} \frac{(T(h)x - x)}{h},$$

with

$$D(A) = \left\{ x \in \mathscr{H} \,\middle|\, \lim_{h\downarrow 0} \frac{(T(h)x - x)}{h} \text{ exists} \right\}.$$

The generator of a strongly continuous semigroup is a closed, densely defined linear operator that determines the semigroup uniquely, see [9].

DEFINITION 5.3. Let $\{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup on a Hilbert space $\mathscr{H}$. The semigroup is

- strongly stable if for all $x \in \mathscr{H}$, $\lim_{t\to\infty} \|T(t)x\| = 0$,
- (uniformly) exponentially stable if there exists an $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\, \|T(t)\| = 0.$$

Clearly, exponential stability implies strong stability, and these two types of stability are equivalent in the finite dimensional case, while in the general case for strongly continuous semigroups exponential stability is equivalent to uniform stability, i.e. to $\lim_{t\to\infty} \|T(t)\| = 0$, but not to strong stability, see Section V.1 in [9].

Even when the semigroup $\{T(t)\}_{t\geqslant 0}$ is not uniformly exponentially stable, we may be able to decompose the underlying space $\mathscr{H}$ into a direct sum $\mathscr{H}_+ \oplus \mathscr{H}_-$ such that $\{T(t)\}_{t\geqslant 0}$ is "forward" exponentially stable on $\mathscr{H}_+$ and "backward" exponentially stable on $\mathscr{H}_-$. If the above decomposition is possible, the semigroup is called hyperbolic:

DEFINITION 5.4. A semigroup $\{T(t)\}_{t\geqslant 0}$ on a Hilbert space $\mathscr{H}$ is said to be hyperbolic, if the space $\mathscr{H}$ decomposes into $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, where $\mathscr{H}_+$ and $\mathscr{H}_-$ are $T(t)$-invariant closed non-trivial subspaces; the restriction $\{T_+(t)\}_{t\geqslant 0}$ of the semigroup $\{T(t)\}_{t\geqslant 0}$ to $\mathscr{H}_+$ is exponentially stable, while the restriction $\{T_-(t)\}_{t\geqslant 0}$ to $\mathscr{H}_-$ extends to a strongly continuous group on $\mathscr{H}_-$, defined by $T_-(t) = T(-t)^{-1}$ for $t < 0$.

The decomposition of the space $\mathscr{H}$ is equivalent to the existence of a bounded projection P such that $\operatorname{Ker} P = \mathscr{H}_+$ and $\operatorname{Im} P = \mathscr{H}_-$. In particular, if the image of P is finite dimensional, then the restriction of $T(t)$ to $\mathscr{H}_-$ automatically extends to a continuous group.

Another important notion is that of bisemigroup.

DEFINITION 5.5. $\{E(t)\}_{t\in\mathbb{R}} \subset L(\mathscr{H})$ is a bisemigroup if there exists a nontrivial bounded projection operator P, called the separating projection, such that the restriction $\{E(t)|_{\operatorname{Ker} P}\}_{t\geqslant 0}$ is a strongly continuous semigroup on $\operatorname{Ker} P$, while the restriction $\{-E(-t)|_{\operatorname{Im} P}\}_{t\geqslant 0}$ is a strongly continuous semigroup on $\operatorname{Im} P$, and, moreover, both these semigroups have a negative exponential growth bound (in other words, they are both exponentially stable semigroups, see[2]).

Let A_+, A_- be the closed, densely defined linear operators on $\operatorname{Ker} P$ and $\operatorname{Im} P$ respectively, such that A_+ is the generator of the semigroup $\{E(t)|_{\operatorname{Ker} P}\}_{t\geqslant 0}$, and $-A_-$ is the generator of the semigroup $\{-E(-t)|_{\operatorname{Im} P}\}_{t\geqslant 0}$. The operator A with domain $D(A) = D(A_+) \oplus D(A_-)$ defined by $A(x_+ + x_-) = A_+x_+ - A_-x_-$, where $x_+ \in D(A_+)$ and $x_- \in D(A_-)$, is called the generator of the bisemigroup. The generator of the bisemigroup has the following spectral property: the resolvent set of A contains a strip $\{z \in \mathbb{C} \mid |\operatorname{Re} z| < \delta\}$ about the imaginary axis, see [2].

Lyapunov stability results can be successfully applied to strongly continuous semigroups. For a bounded generator A one can characterize the stability of a semigroup using the generalized Lyapunov theorem, as the stability can be restated in terms of spectral properties of A. This is also the case for the infinitesimal generator A. It is known (see [7]) that the strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ generated by A is exponentially stable if and only if there exists a positive operator $Q \in L(\mathscr{H})$ such that

$$\langle (QA + A^*Q)x, x\rangle < 0$$

for all $0 \neq x \in D(A)$. One can also find an inertia-like result in [9], which states that a strongly continuous semigroup on a Hilbert space is exponentially stable if and only if $\mathbb{C}^+$ is contained in the resolvent set of generator A, and the resolvent of A is bounded in $\mathbb{C}^+$.

Next we turn our attention to an application to the theory of bisemigroups. As an immediate application of Theorem 1.7 we have the following theorem.

THEOREM 5.1. *Let A be the generator of a bisemigroup. Assume that $\sigma(A) \cap \mathbb{C}^-$ is a bounded spectral set. Suppose further that there exists an invertible self-adjoint operator $H \in L(\mathscr{H})$ with $\nu(H) < \infty$ such that* (6) *holds. Then $\nu(A) = \nu(H)$, and in particular, A generates a hyperbolic semigroup.*

Proof. All that is to prove are the spectral properties of A. Clearly, since A is the generator of a bisemigroup there is a strip around the imaginary axis in the resolvent set of A. This shows that A is invertible, and no point on the imaginary axis is an eigenvalue of A. □

6. Additional Applications

In this section we shall investigate several applications of our main result. First, we treat the case where $H = I - K$, with K compact.

THEOREM 6.1. *Let $A \in L(\mathscr{H})$ be such that the spectrum of A does not contain eigenvalues on the imaginary axis and $\sigma(A) \cap \mathbb{C}^-$ is a spectral set. Let $K = K^* \in L(\mathscr{H})$ be compact with $1 \notin \sigma(K)$. If*

$$A^*(I-K) + (I-K)A \geqslant 0$$

then $\nu(I-K) = \nu(A)$.

Proof. The spectrum of operator $I - K$ consists of a finite or infinite number of regular eigenvalues, with, possibly, the limit point at 1. Thus, there is only a finite number of such points lying in the left half plane. That is $\nu(I-K) < \infty$. Since all the conditions of Theorem 1.6 are satisfied, the conclusion holds true. □

In the second application, consider again the equation

$$A^*H + HA = W \geqslant 0.$$

Since $W \geqslant 0$, we can write $W = B^*B$. Now, we assume approximate controllability of the pair (A^*, B^*) and relax the condition on the spectrum of A. Under the assumptions we made, we deduce a Chen–Wimmer type result.

THEOREM 6.2. *Let $A, H \in L(\mathscr{H})$ be bounded linear operators, such that $\sigma(A) \cap \mathbb{C}^-$ is a spectral set, H is self-adjoint and invertible and $\nu(H) < \infty$. If*

$$A^*H + HA = B^*B \tag{9}$$

and the pair (A^*, B^*) is approximately controllable, then $\nu(H) = \nu(A)$.

Proof. We will show that approximate controllability of the pair (A^*, B^*) guarantees that A does not have eigenvalues on the imaginary axis. Then the conditions of Theorem 1.6 are satisfied, and hence the conclusion is also true. First assume, to the contrary, that $Ax = i\lambda x$ for $x \neq 0$ and some $\lambda \in \mathbb{R}$. From the above equation it follows that

$$\|Bx\|^2 = \langle B^*Bx, x\rangle = \langle HAx, x\rangle + \langle Hx, Ax\rangle = (i\lambda - i\lambda)\langle Hx, x\rangle = 0$$

and hence $Bx = 0$, i.e. $x \in \operatorname{Ker} B$. Also

$$B(A)^k x = (i\lambda)^k Bx = 0.$$

It follows that $x \in \bigcap_{k \geqslant 0} \operatorname{Ker} B(A)^k$, and then the approximate controllability of (A^*, B^*) implies that $x = 0$. This is a contradiction, so A has no eigenvalues on the imaginary axis. From the Theorem 1.6 it follows that $\nu(H) = \nu(A)$ and the theorem follows.

$\square$

REMARK 6.3. In the last theorem we have assumed that the operator H is invertible. We can relax this condition and assume that $0 \notin \sigma_c(H)$. Using the same technique as in the proof of Theorem 6.2 we can show, that 0 is not an eigenvalue of H.

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(Received January 21, 2007)

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