Operators
and
Matrices
Volume 3, Number 2 (2009), 241–260

TRACE INEQUALITIES AND SPECTRAL SHIFT

ANNA SKRIPKA

(*Communicated by L. Rodman*)

Abstract. We derive monotonicity and convexity inequalities for traces of operator functions defined on self-adjoint elements of a semi-finite von Neumann algebra. Among tools involved in the proofs are a generalized Birman-Solomyak spectral averaging formula (obtained in the paper), a generalized Birman-Schwinger principle, and Koplienko's spectral shift function, a new, more straightforward, approach to which is developed in the paper.

1. Introduction

Monotonicity and convexity of (traces of) functions of an operator argument have been widely explored and have found a number of applications. However, in all cases, strong restrictions have been imposed either on the operator argument (see, e.g., [5]) or on the scalar function that gives rise to the operator function (see, e.g., [17]). In this paper, we continue studies of monotonicity and convexity of functionals $H \mapsto \tau[f(H)]$, defined on self-adjoint elements of (or affiliated with) a semi-finite von Neumann algebra $\mathscr{A}$ equipped with a normal faithful semi-finite trace τ, as well as functions $t \mapsto \tau[f(H(t))]$, where $[0,1] \ni t \mapsto H(t)$ is a path of self-adjoint operators in $\mathscr{A}$. For operator monotonicity and convexity, one can consult [13] and references cited therein. In our considerations, the self-adjoint operators H are allowed to be *non-sign-definite* (that is, they are not required to be positive or negative); the functions of an operator argument are mainly (but not always) defined by the spectral theorem. Discussing monotonicity or convexity of the functions $H \mapsto \tau[f(H)]$ of an operator argument given by the spectral theorem, we assume that the scalar function f is monotone or convex, respectively.

We obtain monotonicity and convexity results for the functional $\tau[f(\cdot)]$ defined on self-adjoint elements of a semi-finite von Neumann algebra $\mathscr{A}$ analogous to those of [12, 17] obtained in the particular case of $\mathscr{A} = \mathscr{B}(\mathscr{H})$ (see section 6). Our approach also employs spectral perturbation machinery, but it is more algebraic in nature than the one of [12, 17]. One of our main tools in the study of convexity of $H \mapsto \tau[f(H)]$ is a semi-finite von Neumann algebra analog of *Koplienko's spectral shift function* (KoSSF) (see section 4) and its symmetrized counterpart (see section 5). We obtain a new representation for KoSSF that immediately implies positivity of KoSSF (see section 5), with

Mathematics subject classification (2000): Primary 26A48, 26A51, 47A56; Secondary 47A55, 47C15.
Keywords and phrases: monotonicity, convexity, von Neumann algebra, spectral shift function, operator function.

Paper OaM-03-15

no appeal to double operator integration or Klein's convexity inequality (see section 6) that originally were used in [12, 17]. Positivity of the symmetrized KoSSF, which is a sum of two non-symmetrized KoSSFs, assures an algebraic direct proof of the monotonicity of the function $t \mapsto \tau[Vg(H_0+tV)]$ for a monotone function g and a τ-trace class operator V (see section 5).

Monotonicity of $t \mapsto \tau[Vg(H_0+tV)]$ is crucial in the perturbation theory approach to the convexity of operator functions; originally it was proved for τ the standard trace and g nonnegative by either double operator integration [3] or analytic function theory techniques [10]. We note that the relevance of the function $t \mapsto \tau[Vg(H_0+tV)]$ to convexity issues is rather natural. For instance, when the differentiation rule

$$\frac{d}{dt}\tau[f(H_0+tV)] = \tau[Vf'(H_0+tV)]$$

is applicable (see, e.g., [7, 14]), the function $t \mapsto \tau[Vf'(H_0+tV)]$ is increasing if and only if $t \mapsto \tau[f(H_0+tV)]$ is convex. In a wider setting, the function $t \mapsto \tau[f(H_0+tV)]$ can fail to be differentiable, but monotonicity of certain $t \mapsto \tau[Vg(H_0+tV)]$ can still be "responsible" for convexity of $t \mapsto \tau[f(H_0+tV)]$ (see section 6).

Our proofs engage Krein's trace formula and the Birman-Solomyak spectral averaging formula for the semi-finite von Neumann algebra analog of *Krein's spectral shift function* (KrSSF) [2, 6]. Monotonicity of $H \mapsto \tau[f(H)]$ (see section 6) follows from monotonicity of KrSSF with respect to a perturbation, which is proved by means of the version of the Birman-Schwinger principle for τ-trace class perturbations of dissipative operators [18, 33]. In the case of $\mathscr{A} = \mathscr{B}(\mathscr{H})$, monotonicity of KrSSF was proved for rank-one perturbations by analytic function theory techniques and then extended to trace class perturbations by approximation techniques. We also extend convexity results of [10] for the function $t \mapsto \tau[f(H(t))]$ to the case of a general trace τ. This involves the Birman-Solomyak formula for a general trace τ with averaging accomplished along a non-linear path of operators, which we prove by employing the Birman-Schwinger principle [18, 33] (see section 3). The spectral averaging formula discussed in section 3 is an extension of weaker versions of the Birman-Solomyak formula [1, 3, 10, 22, 32] that were obtained by double operator integration, analytic function theory, or perturbation determinant techniques.

Krein's and Koplienko's spectral shift functions can be transferred from the finite von Neumann algebra setting to the context of a unital C^* algebra $\mathscr{B}$ with a tracial state ϕ by means of the GNS-construction (see section 6). Similarly to the von Neumann algebra case, KrSSF and KoSSF can be applied to prove monotonicity and convexity results for the functional $H \mapsto \phi[f(H)]$ on $\mathscr{B}$ (see section 6), which reverses the approach of [4] of using monotonicity and convexity of $H \mapsto \phi[f(H)]$ to prove existence of KrSSF and KoSSF on $\mathscr{B}$, respectively.

We conclude our introduction with some notation and basic definitions on the theory of von Neumann algebras [7, 15, 30, 34]. Let $\mathscr{A}$ be a semi-finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$ and equipped with a normal semi-finite faithful trace τ. Let $\mathscr{L}_1(\mathscr{A},\tau)$ denote the τ-trace class ideal $\{A \in \mathscr{A} : \tau(|A|) < \infty\}$ and $\|A\|_1$ the τ-trace class norm $\tau(|A|)$ of $A \in \mathscr{L}_1(\mathscr{A},\tau)$. Let $\mathscr{A}^{sa}$ denote the set of self-adjoint operators in $\mathscr{A}$ and $\mathscr{A}_a^{sa}$ the set of self-adjoint operators affiliated with

$\mathscr{A}$. As usual, $E_H(\cdot)$ stands for the spectral measure of a self-adjoint operator H and $E_H(\lambda)$ for the spectral projection $E_H((-\infty,\lambda))$. Finally, let $\mathscr{P}$ denote the set of pairs $\{(H_0,V) : H_0 \in \mathscr{A}_a^{sa}, V = V^* \in \mathscr{L}_1(\mathscr{A},\tau)\}$.

We recall that a von Neumann algebra $\mathscr{A}$ is a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ containing the identity operator and closed in the weak operator topology. A von Neumann algebra that admits a (not necessarily unique) faithful normal semi-finite trace is called semi-finite. A trace is a functional $\tau : \mathscr{A}^+ \mapsto [0,\infty]$ initially defined on the non-negative elements $\mathscr{A}^+$ of $\mathscr{A}$ and satisfying $\tau(\lambda A + \mu B) = \lambda\tau(A) + \mu\tau(B)$ for $A,B \in \mathscr{A}^+$, $\lambda,\mu \geqslant 0$ and $\tau(C^*C) = \tau(CC^*)$ for $C \in \mathscr{A}$. A trace is called faithful if it does not annihilate non-zero elements of $\mathscr{A}^+$, normal if $\tau(A_\alpha) \uparrow \tau(A)$ for each increasing net $\{A_\alpha\} \subset \mathscr{A}^+$ converging to A in the strong operator topology, semi-finite if for each $A \in \mathscr{A}^+$ there is a net (sequence when $\mathscr{H}$ is separable) of elements with finite traces which increases to A. A trace τ with the properties described above extends uniquely to a functional on the relative τ-trace class ideal $\mathscr{L}_1(\mathscr{A},\tau)$, with extended τ satisfying $\tau(AB) = \tau(BA)$ for $A \in \mathscr{L}_1(\mathscr{A},\tau)$, $B \in \mathscr{A}$. The algebra $\mathscr{B}(\mathscr{H})$ is a semi-finite von Neumann algebra, where the usual trace is a unique normal faithful semi-finite trace τ defined on $\mathscr{L}_1(\mathscr{B}(\mathscr{H}),\tau)$, the (usual) trace class ideal. If τ is finite on the whole $\mathscr{A}^+$ (and hence, on $\mathscr{A}$), then both τ and $\mathscr{A}$ are called finite; in this case, $\mathscr{L}_1(\mathscr{A},\tau) = \mathscr{A}$. The algebra of $n \times n$ matrices with the standard trace is a finite von Neumann algebra. An example of a finite von Neumann algebra with a unique normal faithful (finite) trace τ which contains a discrete Laplacian Δ with no point spectrum and such that $\tau[E_\Delta((-\infty,\lambda))]$ gives the value of the integrated density of states for Δ at point $\lambda \in \mathbb{R}$ is discussed in [31]. A self-adjoint operator H is said to be affiliated with $\mathscr{A}$ if all its spectral projections $E_H(\cdot)$ belong to $\mathscr{A}$.

2. Auxiliary lemmas

Many of the results in the paper will primarily be proved for operators in a finite von Neumann algebra and then extended to operators affiliated with a semi-finite algebra by means of standard approximation lemmas stated below.

LEMMA 2.1. ([28, Theorem VIII.20(b)]) *Let $\{H^{(n)}\}_{n=1}^\infty$ be a sequence of self-adjoint operators converging to a self-adjoint operator H in the strong resolvent sense. Then, for any bounded and continuous function g on $\mathbb{R}$, the sequence of $g(H^{(n)})$ converges to $g(H)$ in the strong operator topology.*

LEMMA 2.2. ([28, Theorem VIII.5(d)]) *Let g_n be a sequence of Borel functions, with $\sup_n \|g_n\|_\infty < \infty$, converging to a function g pointwise. Then, for every self-adjoint operator H, the sequence of $g_n(H)$ converges to $g(H)$ in the strong operator topology.*

LEMMA 2.3. *Let $(H_0,W) \in \mathscr{P}$ and $H = H_0 + W$. Then, for any function g continuous and bounded on $\mathbb{R}$,*

$$\text{s-lim}_{n\to\infty} g(H^{(n)}) = g(H),$$

where $H^{(n)} = P_nH_0P_n + W$, with $P_n = E_{H_0}((-n,n))$, $n \in \mathbb{N}$.

Proof. The sequence of bounded self-adjoint operators $\{H^{(n)}\}$ converges to the operator H in the strong resolvent sense and, therefore, the result is an immediate consequence of Lemma 2.1. $\square$

LEMMA 2.4. *Let $\{H^{(n)}\}_{n=1}^{\infty}$ be a sequence of operators in $\mathscr{A}_a^{sa}$ converging to $H \in \mathscr{A}_a^{sa}$ in the strong resolvent sense and let $V \in \mathscr{L}^1(\mathscr{A},\tau)$. Then, for any function g continuous and bounded on $\mathbb{R}$,*

$$\lim_{n\to\infty} \tau[Vg(H^{(n)})] = \tau[Vg(H)].$$

In particular, if $\{H^{(n)}\}_{n=1}^{\infty}$ is a sequence of operators in $\mathscr{A}$ converging to a bounded operator H in the strong operator topology, then

$$\lim_{n\to\infty} \tau[VH^{(n)}] = \tau[VH]. \tag{2.1}$$

Proof. In view of Lemma 2.1, the general assertion of Lemma 2.4 follows from its particular case (2.1). The proof of the latter can be found in [1, Lemma 2.5]. $\square$

REMARK 2.5. (i) A sequence of operators $\{H^{(n)}\}_{n=1}^{\infty}$ as in Lemma 2.3 satisfies the general assumption of Lemma 2.4.

(ii) Given a normal semi-finite trace τ, there exists an increasing sequence of orthogonal projections $\{Q_n\}_{n=1}^{\infty}$ converging to I in the strong operator topology, with $\tau(Q_n) < \infty$, $n \in \mathbb{N}$. For such $\{Q_n\}_{n=1}^{\infty}$, the sequence of $H^{(n)} = Q_nHQ_n$, with $H \in \mathscr{A}^{sa}$, satisfies the assumptions of Lemma 2.4.

The following fact is elementary, but, for convenience of references, it is stated in the format of a lemma.

LEMMA 2.6. *For P and Q orthogonal projections,*

$$Q - P = QP^{\perp} - Q^{\perp}P.$$

Proof. By applying a trivial representation $P = (Q+Q^{\perp})P$, one obtains

$$Q - P = Q - QP - Q^{\perp}P = QP^{\perp} - Q^{\perp}P. \qquad \square$$

3. Krein's spectral shift function

This section discusses results on KrSSF that we apply later in the study of monotonicity and convexity of functions of an operator argument.

Let $\mathscr{W}^1$ denote the Wiener class, that is, the set of continuous functions on $\mathbb{R}$ which can be represented as Fourier-Stieltjes transforms of finite Borel measures.

THEOREM 3.1. ([2, 6]) *Let $(H_0,V) \in \mathscr{P}$. Then, there exists a unique L^1-function $\xi_\tau(\lambda, H_0+V, H_0)$ satisfying Krein's trace formula*

$$\tau[f(H_0+V)-f(H_0)] = \int_{\mathbb{R}} f'(\lambda)\xi_\tau(\lambda, H_0+V, H_0)\, d\lambda, \tag{3.1}$$

for every $f \in C^1(\mathbb{R})$ with $f' \in \mathscr{W}^1$.

The function $\xi_\tau(\lambda, H_0+V, H_0)$ provided by Theorem 3.1 is called Krein's spectral shift function (KrSSF) associated with the pair of operators (H_0+V, H_0). Originally this function was introduced in the standard trace class setting in [19, 21]. For τ the standard trace, the requirement that $f \in B^1_{\infty,1}$ is sufficient for (3.1) to hold, while $f \in C^1(\mathbb{R})$ is not enough [24].

REMARK 3.2. Assume that τ is finite and let $(H_0,V) \in \mathscr{P}$. Then, for any absolutely continuous function f on $\mathbb{R}$ with $f' \in L^1(\mathbb{R})$, Krein's trace formula (3.1) holds and, for a.e. $\lambda \in \mathbb{R}$, KrSSF is the difference of the spectral distribution functions

$$\xi_\tau(\lambda, H_0, H_0+V) = \tau[E_{H_0+V}(\lambda)] - \tau[E_{H_0}(\lambda)].$$

In this case, Krein's trace formula can be proved by the integration by parts argument. The linear functional in (3.1) defined on L^1-functions f' is continuous and, hence, an L^∞-function $\xi_\tau(\lambda, H_0+V, H_0)$ appearing in the representation of the functional is unique. Another representation for KrSSF,

$$\xi_\tau(\lambda, H_0, H_0+V) = \tau\big[E_{H_0+V}(\lambda)E_{H_0}(\lambda)^\perp\big] - \tau\big[E_{H_0+V}(\lambda)^\perp E_{H_0}(\lambda)\big],$$

can be derived by applying Lemma 2.6 to the projections $P = E_{H_0}(\lambda)$ and $Q = E_{H_0+V}(\lambda)$.

LEMMA 3.3. ([2, 19]) *For $(H_0,V) \in \mathscr{P}$,*

$$\int_{\mathbb{R}} \xi_\tau(\lambda, H_0+V, H_0)\, d\lambda = \tau(V), \quad \int_{\mathbb{R}} |\xi_\tau(\lambda, H_0+V, H_0)|\, d\lambda \leqslant \|V\|_1.$$

Given a pair $(H_0, H_0+V) \in \mathscr{P}$ of bounded operators, KrSSF can be reconstructed from KrSSFs for finite trace truncations of the pair (H_0, H_0+V).

LEMMA 3.4. *Let $(H_0,V) \in \mathscr{P}$, with H_0 bounded, and let $Q_n \uparrow I$ be a sequence of orthogonal projections in $\mathscr{A}$ with $\tau(Q_n) < \infty$. Let $[a,b]$ be a segment containing $\sigma(H_0) \cup \sigma(H_0+V)$. Then, for $f \in C[a,b]$,*

$$\int_{[a,b]} f(\lambda)\xi_\tau(\lambda, H_0, H_0+V)\, d\lambda = \lim_{n\to\infty} \int_{[a,b]} f(\lambda)\xi_\tau(\lambda, Q_nH_0Q_n, Q_nH_0Q_n + Q_nVQ_n)\, d\lambda.$$

Proof. It is straightforward to see that for a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_mx^m, \quad \text{with } m \geqslant 2 \text{ and } a_0, \ldots, a_m \in \mathbb{C},$$

$$\begin{aligned}\tau[f(H_0+V)-f(H_0)] &= a_1\tau(V)+\tau[a_2((H_0+V)^2-H_0^2)+\ldots+a_m((H_0+V)^m-H_0^m)]\\ &= a_1\tau(V)+\tau[Vp(H_0,V)]\end{aligned}$$

and

$$\tau[f(Q_nH_0Q_n+Q_nVQ_n)-f(Q_nH_0Q_n)] = a_1\tau(Q_nVQ_n)+\tau[Vp(Q_nH_0Q_n,Q_nVQ_n)],$$

where p is a polynomial in two variables. By Lemma 2.4, one has

$$\tau[Vp(H_0,V)] = \lim_{n\to\infty}\tau[Vp(Q_nH_0Q_n,Q_nVQ_n)]$$

and, thus,

$$\tau[f(H_0+V)-f(H_0)] = \lim_{n\to\infty}\tau[f(Q_nH_0Q_n+Q_nVQ_n)-f(Q_nH_0Q_n)]. \tag{3.2}$$

Combining (3.2) and Theorem 3.1 proves the lemma for f a polynomial. By Lemma 3.3, the L^1-norms of functions $\xi_\tau(\lambda,H_0,H_0+V)$ and $\xi_\tau(\lambda,Q_nH_0Q_n,Q_nH_0Q_n+Q_nVQ_n)$ are totally bounded by $\|V\|_1$ and, hence, the assertion of the lemma extends to the functions in $C[a,b]$. $\square$

REMARK 3.5. In the case of the standard trace τ and bounded H_0, taking $Q_n\uparrow I$ such that $\|[H_0,Q_n]\|_2\to 0$ suffices to prove existence of KrSSF [35].

Krein's spectral shift function can be transferred from a finite von Neumann algebra to a unital C^*-algebra with tracial state by means of the Gelfand-Naimark-Segal construction (cf. [27, 34]).

LEMMA 3.6. *Let $\mathscr{B}$ be a unital C^*-algebra with a tracial state ϕ. Let π be a $*$-homomorphism into a finite von Neumann algebra $\mathscr{A}$ possessing a faithful normal finite trace τ such that for every $A\in\mathscr{B}^{sa}$ and every continuous function $f:\sigma(A)\mapsto\mathbb{R}$, $\phi[f(A)]=\tau[f(\pi(A))]$. Then for H_0 and V in $\mathscr{B}^{sa}$ and for a.e. $\lambda\in\mathbb{R}$,*

$$\xi_\phi(\lambda,H_0,H_0+V)=\xi_\tau(\lambda,\pi(H_0),\pi(H_0+V)).$$

Proof. Application of Krein's trace formula to a C^1 function f implies

$$\begin{aligned}\phi[f(H_0+V)-f(H_0)] &= \tau[f(\pi(H_0+V))-f(\pi(H_0))]\\ &= \int_{\mathbb{R}} f'(\lambda)\xi_\tau(\lambda,\pi(H_0+V),\pi(H_0))\,d\lambda.\end{aligned}$$

By uniqueness of the Riesz representation for the continuous functional $\phi[f(H_0+V)-f(H_0)]$, which is defined on the space of continuous functions f', the function $\xi_\tau(\lambda,\pi(H_0),\pi(H_0+V))$ coincides with KrSSF for the pair (H_0,H_0+V). $\square$

We conclude this section with the Birman-Solomyak formula, where averaging is performed along a non-linear path of self-adjoint operators affiliated with a semi-finite von Neumann algebra. Preceding versions of this formula can be found in [3, 32, 11, 1, 22]. Our proof is adjustment of the one of [11, Theorem 4.3] to the case of a semi-finite von Neumann algebra with employment of the analog of the Birman-Schwinger principle for dissipative operators [18, 33] instead of theory of operator-valued Herglotz functions originally used in [11].

THEOREM 3.7. *Assume that $H_0 \in \mathscr{A}_a^{sa}$ and $[0,1] \ni s \mapsto V(s) \in \mathscr{L}_1(\mathscr{A},\tau)$ is a path of self-adjoint operators continuously differentiable in the norm $\|\cdot\|_1 + \|\cdot\|$. Then, for every $\lambda \in \mathbb{R}$ and $t \in [0,1]$,*

$$\int_0^t \tau\big[V'(s)E_{H_0+V(s)}(\lambda)\big]\,ds = \int_{-\infty}^{\lambda} \xi_\tau(s, H_0+V(t), H_0+V(0))\,ds. \tag{3.3}$$

Proof. Assume first that $V(s) \geqslant 0$. It was derived in the proof of the Birman-Schwinger principle for a semi-finite trace [33, Theorem 3.1] that

$$\int_{\mathbb{R}} \frac{\xi(\lambda, H_0+V(s), H_0)}{\lambda - z}\,d\lambda \\ = \tau\big[\log\big(I + V(s)^{1/2}(H_0 - zI)^{-1}V(s)^{1/2}\big) - \log(I)\big]. \tag{3.4}$$

Following the lines in the proof of [11, Lemma 4.2] with employment of [33, Lemma 2.5], one obtains

$$\frac{d}{ds}\tau\big[\log\big(I + V(s)^{1/2}(H_0 - zI)^{-1}V(s)^{1/2}\big)\big] \\ = \tau\big[V'(s)(H_0 + V(s) - zI)^{-1}\big], \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}.$$

Integrating on both sides of the equality above and comparing the result with (3.4) gives

$$\int_{\mathbb{R}} \frac{\xi(\lambda, H_0+V(s_2), H_0) - \xi(\lambda, H_0+V(s_1), H_0)}{\lambda - z}\,d\lambda \\ = \int_{s_1}^{s_2} \tau\big[V'(s)(H_0 + V(s) - zI)^{-1}\big]\,ds. \tag{3.5}$$

Decompose $V'(s)$ into

$$V'(s) = (V'(s))_+ - (V'(s))_-, \quad \text{with } 0 \leqslant (V'(s))_\pm \in \mathscr{L}_1(\mathscr{A},\tau).$$

Making consecutive use of the spectral theorem, the monotone convergence theorem, and the Fubini theorem in the second integral in (3.5) yields

$$\int_{\mathbb{R}} \frac{\xi(\lambda, H_0+V(s_2), H_0) - \xi(\lambda, H_0+V(s_1), H_0)}{(\lambda - \mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}\,d\lambda \\ = \int_{s_1}^{s_2}\int_{\mathbb{R}} \frac{d\tau\big[(V'(s))_+^{1/2}E_{H_0+V(s)}(\lambda)(V'(s))_+^{1/2}\big]}{(\lambda - \mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}\,ds \\ - \int_{s_1}^{s_2}\int_{\mathbb{R}} \frac{d\tau\big[(V'(s))_-^{1/2}E_{H_0+V(s)}(\lambda)(V'(s))_-^{1/2}\big]}{(\lambda - \mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}\,ds \\ = \int_{\mathbb{R}} \frac{1}{(\lambda - \mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}\int_{s_1}^{s_2} d\tau\big[V'(s)E_{H_0+V(s)}(\lambda)\big]\,ds.$$

By uniqueness of the Poisson transform, we conclude equality of the measures

$$\begin{aligned}\xi(\lambda,H_0+V(s_2),H_0+V(s_1))\,d\lambda &= \big(\xi(\lambda,H_0+V(s_2),H_0)-\xi(\lambda,H_0+V(s_1),H_0)\big)\,d\lambda\\ &= \int_{s_1}^{s_2}\tau\big[V'(s)E_{H_0+V(s)}(d\lambda)\big]\,ds.\end{aligned}$$

The case of a general, non-sign-definite, perturbation $V(s)$ can be treated literally the same way as in [11, Theorem 4.3]. $\square$

4. Koplienko's spectral shift function

By analogy with the studied case of a standard trace-class perturbation, we define Koplienko's spectral shift function (KoSSF) for the pair (H_0,H_0+V), with $(H_0,V)\in\mathscr{P}$, to be

$$\eta_\tau(\lambda,H_0,H_0+V)=\tau[VE_{H_0}(\lambda)]-\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V,H_0)\,dt,\quad \lambda\in\mathbb{R}. \tag{4.1}$$

KoSSF (4.1) enjoys a number of properties completely analogous to those known in the standard setting of the spectral perturbation theory.

LEMMA 4.1. *Given $(H_0,V)\in\mathscr{P}$, $\lambda\mapsto\eta_\tau(\lambda,H_0,H_0+V)$ is a function of bounded variation with $\|\eta_\tau(\cdot,H_0,H_0+V)\|_\infty\leqslant 2\|V\|_1$.*

The following representation to be used in the proof of positivity of KoSSF is ensured by (4.1) and Theorem 3.7.

LEMMA 4.2. *Let $(H_0,V)\in\mathscr{P}$. Then*

$$\eta_\tau(\lambda,H_0,H_0+V)=\int_0^1\tau[VE_{H_0}(\lambda)-VE_{H_0+sV}(\lambda)]\,ds.$$

Alternatively, KoSSF can be defined as the unique L^∞-function satisfying Koplienko's trace formula.

THEOREM 4.3. *Let $(H_0,V)\in\mathscr{P}$ and $f\in C^1(\mathbb{R})$, with $f'\in\mathscr{W}^1$ an absolutely continuous function. Then,*

$$\tau\big[f(H_0+V)-f(H_0)-Vf'(H_0)\big]=\int_{\mathbb{R}}f''(\lambda)\eta_\tau(\lambda,H_0,H_0+V)\,d\lambda. \tag{4.2}$$

Proof. By Krein's trace formula,

$$\tau[f(H_0+V)-f(H_0)]=\int_{\mathbb{R}}f'(\lambda)\xi_\tau(\lambda,H_0+V,H_0)\,d\lambda. \tag{4.3}$$

Applying the spectral theorem and Lemma 2.4 yields

$$\tau[Vf'(H_0)]=\int_{\mathbb{R}}f'(\lambda)\,d\tau[VE_{H_0}(\lambda)]. \tag{4.4}$$

Combining (4.3) and (4.4) and then integrating by parts imply

$$\begin{aligned}
&\tau\big[f(H_0+V)-f(H_0)-Vf'(H_0)\big]\\
&\quad=-\int_{\mathbb{R}} f'(\lambda)\,d\left(\tau[VE_{H_0}(\lambda)]-\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V,H_0)\,dt\right)\\
&\quad=-\left[f'(\lambda)\left(\tau[VE_{H_0}(\lambda)]-\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V,H_0)\,dt\right)\right]\Bigg|_{-\infty}^{\infty} \qquad (4.5)\\
&\qquad+\int_{\mathbb{R}} f''(\lambda)\left(\tau[VE_{H_0}(\lambda)]-\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V,H_0)\,dt\right)d\lambda. \qquad (4.6)
\end{aligned}$$

Since f' is a Fourier-Stieltjes transform of a finite Borel measure, $\|f'\|_\infty$ is finite. By Lemma 3.3,

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V,H_0)\,dt=\tau(V)$$

and by Lemma 2.4,

$$\lim_{\lambda\to\infty}\tau[VE_{H_0+V}(\lambda)]=\tau(V).$$

Therefore, the expression in (4.5) equals 0. The integral in (4.6) exists since $f''\in L^1(\mathbb{R})$ and, by Lemma 4.1, $\eta_\tau(\cdot,H_0,H_0+V)$ is bounded. □

REMARK 4.4. If τ is finite and H_0 is bounded, then (4.2) holds for $f\in C^1[a,b]$ with f' absolutely continuous on $[a,b]$, where $[a,b]\supset\sigma(H_0)\cup\sigma(H_0+V)$.

COROLLARY 4.5. *Let $(H_0,V)\in\mathscr{P}$. Then $\eta_\tau(\cdot,H_0,H_0+V)$ is locally integrable. If, in addition, $H_0\in\mathscr{A}$, then*

$$\|\eta_\tau(\cdot,H_0,H_0+V)\|_1=\frac{1}{2}\tau(V^2).$$

Proof. The first assertion is an immediate consequence of Lemma 4.1. Applying Theorem 4.3 to $f(t)=t^2$ and then making use of nonnegativity of $\eta_\tau(\cdot,H_0,H_0+V)$ (see Lemma 5.6 in the next section) imply the second assertion. □

REMARK 4.6. Koplienko's trace formula

$$\begin{aligned}
&\tau\left[f(H_0+V)-f(H_0)-\frac{d}{dt}\Big(f(H_0+tV)\Big)\Big|_{t=0}\right] \qquad (4.7)\\
&\qquad=\int_{\mathbb{R}} f''(\lambda)\eta_\tau(\lambda,H_0,H_0+V)\,d\lambda,
\end{aligned}$$

with τ the standard trace and V a Hilbert-Schmidt operator, was originally proved in [16] for f a rational bounded function with non-real poles. For V in the trace class,

$\eta_\tau(\lambda,H_0,H_0+V)$ is given explicitly by (4.1); for V in the Hilbert-Schmidt class, existence of KoSSF has been established implicitly with employment of double operator integration and approximation techniques. The class of functions satisfying (4.7) was subsequently extended in [23, 25]. In all considered extensions of (4.7), the function $t \mapsto f(H_0+tV)$ has to be differentiable in the operator norm. It is known that for differentiability of the operator function $t \mapsto f(H_0+tV)$ the condition that f belong to the Besov space $B^1_{\infty,1}(\mathbb{R})$ is sufficient, while $f \in C^1(\mathbb{R})$ is not (see [26] and references cited therein). When V is in the τ-trace class, a double operator integral representation for the derivative of the differentiable function $t \mapsto f(H_0+tV)$ assures that $\tau\left[\frac{d}{dt}f(H_0+tV)\right] = \tau[Vf'(H_0+tV)]$ (see, e.g., [1]).

Repeating the approximation argument in the proof of Lemma 3.4, one can express KoSSF for (H_0,H_0+V) as a weak limit of KoSSFs for truncated operators.

LEMMA 4.7. *Let $(H_0,V) \in \mathscr{P}$, with H_0 bounded, and let $Q_n \uparrow I$ be a sequence of orthogonal projections in $\mathscr{A}$ with $\tau(Q_n) < \infty$. Let $[a,b]$ be a segment containing $\sigma(H_0) \cup \sigma(H_0+V)$. Then, for $f \in C[a,b]$,*

$$\begin{aligned}&\int_{[a,b]} f(\lambda)\eta_\tau(\lambda,H_0,H_0+V)\,d\lambda \\ &\qquad = \lim_{n\to\infty}\int_{[a,b]} f(\lambda)\eta_\tau(\lambda,Q_nH_0Q_n,Q_nH_0Q_n+Q_nVQ_n)\,d\lambda.\end{aligned}$$

Similarly to KrSSF, KoSSF can be transferred from a finite von Neumann algebra to a unital C^*-algebra with a tracial state by means of the Gelfand-Naimark-Segal construction.

LEMMA 4.8. *Assume the hypothesis of Lemma 3.6. Then for H_0 and V in $\mathscr{B}^{sa}$ and for a.e. $\lambda \in \mathbb{R}$,*

$$\eta_\phi(\lambda,H_0,H_0+V) = \eta_\tau(\lambda,\pi(H_0),\pi(H_0+V)).$$

5. Symmetrized Koplienko's spectral shift function

In this section, we obtain a helpful representation for KoSSF, which implies a direct proof of positivity of KoSSF. We also introduce a symmetrized KoSSF, whose positivity is used to give an algebraic proof of monotonicity of the function $t \mapsto \tau[Vg(H_0+tV)]$ for a monotone function g and $(H_0,V) \in \mathscr{P}$. The symmetrized KoSSF complements Krein's spectral shift function in a sense that the first one measures, in a certain way, how far the spectrum of an operator shifts while the second one what portion of the spectrum shifts under a perturbation.

Given two operators A,B in $\mathscr{A}_a^{sa}$, with $B-A \in \mathscr{L}_1(\mathscr{A},\tau)$, we define a symmetrized Koplienko's spectral shift function by

$$\gamma_\tau(\lambda,A,B) = \eta_\tau(\lambda,A,B) + \eta_\tau(\lambda,B,A), \quad \lambda \in \mathbb{R}. \tag{5.1}$$

LEMMA 5.1. *Let A,B be operators in $\mathscr{A}_a^{sa}$, with $(A,B-A)\in\mathscr{P}$. Then, for any $\lambda\in\mathbb{R}$,*

$$\gamma_\tau(\lambda,A,B)=\tau[(B-A)(E_A(\lambda)-E_B(\lambda))] \tag{5.2}$$

$$=\tau\big[(B-A)E_A(\lambda)E_B(\lambda)^\perp\big]+\tau\big[(A-B)E_A(\lambda)^\perp E_B(\lambda)\big]. \tag{5.3}$$

Proof. The representation (5.2) follows from (4.1), (5.1), and the property of the ξ-function $\xi(\lambda,A,B)=-\xi(\lambda,B,A)$. Then by Lemma 2.6 applied to $Q=E_A(\lambda)$ and $P=E_B(\lambda)$ we also have (5.3). $\square$

It follows from (5.2) that

$$\gamma_\tau(\lambda,B,A)=\gamma_\tau(\lambda,A,B)=\gamma_\tau(0,A-\lambda I,B-\lambda I).$$

As one can see from (5.3), the quantity $\gamma_\tau(\lambda,A,B)$ describes the net shift of the spectrum of A (or B) across point λ when the perturbation $B-A$ (or $A-B$) is applied. In the particular case when A and B are matrices diagonalizable in the same basis, $\gamma_\tau(0,B,A)$ measures the sum of the distances between the corresponding sign-different eigenvalues of A and B.

In the case of a finite trace τ and bounded operators A and B, KoSSF admits representations similar to (5.2) and (5.3) for the symmetrized KoSSF.

LEMMA 5.2. *Let τ be finite and let A,B be operators in $\mathscr{A}^{sa}$. Then, for any $\lambda\in\mathbb{R}$,*

$$\eta_\tau(\lambda,A,B)=\tau[(\lambda I-B)(E_B(\lambda)-E_A(\lambda))]$$

$$=\tau\big[(\lambda I-B)E_B(\lambda)E_A(\lambda)^\perp\big]+\tau\big[(B-\lambda I)E_B(\lambda)^\perp E_A(\lambda)\big].$$

Proof. Clearly,

$$\tau[(\lambda I-B)(E_B(\lambda)-E_A(\lambda))] \tag{5.4}$$

$$=\lambda\tau[E_B(\lambda)-E_A(\lambda)]\ -\tau[BE_B(\lambda)-BE_A(\lambda)],$$

with the second summand equal to

$$\tau[(B-A)E_A(\lambda)]-\tau[BE_B(\lambda)-AE_A(\lambda)]. \tag{5.5}$$

Applying the spectral theorem and integrating by parts yields

$$\tau[BE_B(\lambda)-AE_A(\lambda)]=\int_{-\infty}^{\lambda}t\,d\tau[E_B(t)-E_A(t)] \tag{5.6}$$

$$=\big(t\tau[E_B(t)-E_A(t)]\big)\big|_{-\infty}^{\lambda}-\int_{-\infty}^{\lambda}\tau[E_B(t)-E_A(t)]\,dt,$$

with

$$\big(t\tau[E_B(t)-E_A(t)]\big)\big|_{-\infty}^{\lambda}=\lambda\tau[E_B(\lambda)-E_A(\lambda)]. \tag{5.7}$$

Combining (5.4)–(5.7) and comparing the result with (4.1) completes the proof of the lemma. $\square$

The main results of this section are the theorem and two its consequences stated below. They will be used in the next section in the proof of monotonicity and convexity results.

THEOREM 5.3. *Let $(H_0,V)\in\mathscr{P}$. Let g be a function continuous, bounded, and decreasing on $\mathbb{R}$. Then*

$$t\mapsto\tau[Vg(H_0+tV)]$$

is decreasing.

COROLLARY 5.4. *For any $(H_0,V)\in\mathscr{P}$ and $\lambda\in\mathbb{R}$, the function*

$$[0,1]\ni t\mapsto\tau[VE_{H_0+tV}(\lambda)]$$

is decreasing.

COROLLARY 5.5. *For any $(H_0,V)\in\mathscr{P}$ and every value of the spectral parameter $\lambda\in\mathbb{R}$, the symmetrized KoSSF and the representative of KoSSF given by* (4.1) *are nonnegative.*

The proofs of the main results proceed in several steps.

In the particular case of a finite trace and bounded operators, Lemma 5.2 immediately implies positivity of KoSSF.

LEMMA 5.6. *Let τ be finite and let $A,B\in\mathscr{A}^{sa}$. Then, for every $\lambda\in\mathbb{R}$,*

$$\eta_\tau(\lambda,A,B)\geqslant 0.$$

Proof. By the spectral theorem,

$$(\lambda I-B)E_B((-\infty,\lambda))=\int_{(-\infty,\lambda)}(\lambda-t)\,dE_B(t)\geqslant 0,$$
$$(B-\lambda I)E_B([\lambda,\infty))=\int_{[\lambda,\infty)}(t-\lambda)\,dE_B(t)\geqslant 0,$$

and hence,

$$\tau\big[(\lambda I-B)E_B(\lambda)E_A(\lambda)^\perp\big]\geqslant 0\quad\text{and}\quad\tau\big[(B-\lambda I)E_B(\lambda)^\perp E_A(\lambda)\big]\geqslant 0.$$

Combining the latter inequalities with Lemma 5.2 completes the proof. $\square$

Applying Lemma 4.7 extends the result of Lemma 5.6 to a wider setting.

LEMMA 5.7. *Let $A,B\in\mathscr{A}^{sa}$ satisfy $(A,B-A)\in\mathscr{P}$. Then, for a.e. $\lambda\in\mathbb{R}$, $\eta_\tau(\lambda,A,B)\geqslant 0$ and $\gamma_\tau(\lambda,A,B)\geqslant 0$.*

To complete the proofs of Theorem 5.3, Corollary 5.4, and Corollary 5.5 in their most general setting, we need the two lemmas below.

LEMMA 5.8. *For any* $(H_0,V)\in\mathscr{P}$ *and* $t_1,t_2,\lambda\in\mathbb{R}$,

$$\gamma_\tau(\lambda,H_0+t_1V,H_0+t_2V)=(t_2-t_1)\big(\tau[VE_{H_0+t_1V}(\lambda)]-\tau[VE_{H_0+t_2V}(\lambda)]\big).$$

Proof. The result follows from (5.2) by letting $A=H_0+t_1V$ and $B=H_0+t_2V$. $\square$

LEMMA 5.9. *Let* $A,B\in\mathscr{A}^{sa}$. *For* $g\in C^\infty(\mathbb{R})$,

$$\tau[(B-A)(g(B)-g(A))]=\int_{\mathbb{R}}g'(\lambda)\gamma_\tau(\lambda,A,B)\,d\lambda.$$

Proof. It is a straightforward consequence of definition (5.1) and Theorem 4.3 applied to $f\in C_0^\infty(\mathbb{R})$ with $f'\big|_{[a,b]}=g$, where $[a,b]\supset\sigma(A)\cup\sigma(B)$. $\square$

Now we are ready to complete the proofs of Theorem 5.3, Corollary 5.4, and Corollary 5.5.

Proof of Theorem 5.3. It follows from Lemma 5.7 and Lemma 5.9 that $t\mapsto\tau[Vg(H_0+tV)]$ is decreasing when H_0 is bounded and $g\in C^\infty(\mathbb{R})$ is bounded and decreasing. By Lemma 2.2 and Lemma 2.4, the function $t\mapsto\tau[Vg(H_0+tV)]$ is decreasing for g continuous, bounded, and decreasing. Next, if H_0 is unbounded, let $P_n=E_{H_0}((-n,n))$, $n\in\mathbb{N}$, and let $H^{(n)}(t)=P_nH_0P_n+tV$, $H(t)=H_0+tV$, $t\in[0,1]$. Applying Lemma 2.3 yields

$$\text{s-}\lim_{n\to\infty}g(H^{(n)}(t))=g(H(t)),$$

and hence by Lemma 2.4,

$$\tau[Vg(H(t_1))]\geqslant\tau[Vg(H(t_2))],$$

for $t_1\leqslant t_2$. $\square$

Proof of Corollary 5.4. For each $\lambda\in\mathbb{R}$, there is a sequence of real-analytic decreasing functions g_λ approximating $\chi_{(-\infty,\lambda)}$ and such that $g_\lambda(H_0+tV)$ approaches $E_{H_0+tV}((-\infty,\lambda))$ in the strong operator topology (see, e.g., [10, proof of Theorem 1.8]). Applying Theorem 5.3 and Lemma 2.4 completes the proof. $\square$

Proof of Corollary 5.5. Combining Corollary 5.4 and Lemma 5.8 proves positivity of the symmetrized KoSSF; combining Lemma 4.2 and Corollary 5.4 proves positivity of KoSSF. $\square$

REMARK 5.10. Positivity of KoSSF for a trace class perturbation V was proved by employing the Birman-Solomyak spectral averaging formula [3, 11] and for V in the Hilbert-Schmidt class by approximating from the finite rank perturbation case [16].

6. Monotonicity and convexity

In this section, we utilize the results of the preceding sections to prove monotonicity and convexity of some functions of an operator argument.

As it is shown in the theorem below, KrSSF is monotone with respect to the perturbation.

THEOREM 6.1. *Let* $(H_0,V)\in\mathscr{P}$*, with* $V\geqslant 0$*. Then for a.e.* $\lambda\in\mathbb{R}$*,*

$$\xi_\tau(\lambda,H_0+V,H_0)\geqslant 0.$$

Proof. It follows from the Birman-Schwinger principle in semi-finite von Neumann algebras [33] that for a.e. $\lambda\in\mathbb{R}$,

$$\xi_\tau(\lambda,H_0+V,H_0)=\lim_{\varepsilon\to 0^+}\xi_\tau\big(I,I+V^{1/2}(H_0-\lambda I-\mathrm{i}\varepsilon I)^{-1}V^{1/2}\big).$$

Here the ξ-index $\xi_\tau\big(I,I+V^{1/2}(H_0-\lambda I-\mathrm{i}\varepsilon I)^{-1}V^{1/2}\big)$ equals

$$\frac{1}{\pi}\tau\big[\arg\big(I+V^{1/2}(H_0-\lambda I-\mathrm{i}\varepsilon I)^{-1}V^{1/2}\big)-\arg(I)\big], \tag{6.1}$$

where $\arg(\cdot)$ is the principal branch of the argument with cut along the negative imaginary semi-axis. The expression in (6.1) is nonnegative, so $\xi_\tau(\lambda,H_0+V,H_0)$ is nonnegative as well. □

As an application of Krein's trace formula, we obtain monotonicity of the functional $\tau[f(\cdot)]$.

COROLLARY 6.2. *Let* $(H_0,V)\in\mathscr{P}$*, with* $V\geqslant 0$*. Then,*

$$\tau[f(H_0+V)-f(H_0)]\geqslant 0$$

holds for any increasing function f *satisfying Krein's trace formula* (3.1)*. If, in addition,* τ *is finite, then*

$$\tau[f(H_0+V)]\geqslant\tau[f(H_0)]$$

holds for every increasing continuous bounded function f *on* $\mathbb{R}$*.*

Proof. The general statement of the corollary follows from Theorem 6.1. The particular case can be proved by consecutive approximations of a continuous function f by functions of class C^1 in the case of a bounded H_0 and then of an unbounded H_0 by its bounded truncations. These approximations are justified by Lemma 2.2 and Lemma 2.3, respectively. □

REMARK 6.3. The assumption of Corollary 6.2 that f be bounded can be dropped when $\mathscr{A}$ is finite and H_0 and V are (bounded) elements of $\mathscr{A}^{sa}$.

As an analog of a geometric characterization of convexity of a function of a scalar argument in terms of the tangent lines to its graph, we have Klein's convexity inequality

$$\tau[f(H_0+V)-f(H_0)-Vf'(H_0)] \geqslant 0 \tag{6.2}$$

for certain functions f of a self-adjoint operator argument.

THEOREM 6.4. *Let* $(H_0,V) \in \mathscr{P}$. *Then,* (6.2) *holds for any* f *satisfying Koplienko's trace formula* (4.2) *with* f' *increasing. If, in addition,* τ *is finite, then the inequality* (6.2) *holds for any bounded* $f \in C^1(\mathbb{R})$ *with* f' *increasing and bounded.*

Proof. The general statement of the theorem follows from Corollary 5.5. For a finite τ, bounded H_0 and $f \in C^2(\mathbb{R})$, (6.2) follows from Remark 4.4 and Corollary 5.5. Any smooth convex function f on a compact set can be approximated uniformly by a sequence $\{f_n\}$ of convex functions in C^2 such that f' is approximated uniformly by $\{f_n'\}$. For a finite τ and bounded H_0, this approximation extends (6.2) to the class of bounded smooth convex functions f (see Lemma 2.2). To conclude the proof of (6.2), we approximate an unbounded operator H_0 by a sequence of bounded ones as in Lemma 2.3. □

REMARK 6.5. Klein's convexity inequality in the finite dimensional case was discussed in [29] and in the case when τ is a tracial state on a unital C^* algebra in [27]. For τ the standard trace, and H_0 and V bounded self-adjoint operators, V in the Hilbert-Schmidt class, the inequality (6.2) holds for all convex functions f in $C^\infty(\mathbb{R})$, which follows from positivity of KoSSF [16]. This observation (see, e.g., [12]) also builds on Koplienko's trace formula [16].

REMARK 6.6. Along with Theorem 4.3, (6.2) yields

$$\int_{\mathbb{R}} g(\lambda)\eta_\tau(\lambda,H_0,H_0+V)\,d\lambda \geqslant 0,$$

for any $0 \leqslant g \in L^1(\mathbb{R})$, which implies positivity of KoSSF for almost every value of the spectral parameter λ, while techniques of section 5 guarantee positivity of the representative of KoSSF given by (4.1) for every $\lambda \in \mathbb{R}$.

REMARK 6.7. Let $\mathscr{B}$ be a unital C^*-algebra with a tracial state ϕ. Monotonicity of the functional $\phi[f(\cdot)]$ on $\mathscr{B}^{sa}$ (obtained in [27]) was employed in [4] to prove existence of KrSSF on $\mathscr{B}^{sa}$. By Lemma 3.6, existence and monotonicity of KrSSF on $\mathscr{B}$ follow immediately from existence and monotonicity of KrSSF on a finite von Neumann algebra, respectively. Therefore, monotonicity of $\phi[f(\cdot)]$ also follows from Krein's trace formula. Klein's convexity inequality was applied in [4] in the proof of existence of KoSSF on $\mathscr{B}^{sa}$. By Lemma 4.8, existence of KoSSF on $\mathscr{B}$ follows immediately from existence of KoSSF on a finite von Neumann algebra. Consequently, Klein's convexity inequality on $\mathscr{B}$ also follows from Koplienko's trace formula.

Convexity of the functional $V \mapsto \tau[f(H_0+V)-f(H_0)]$, with f convex, can be proved by utilizing results of this section, sections 3, and 5 and ideas of the proof of [17, Theorem 1 and Corollary 2].

LEMMA 6.8. *Let $H_0 \in \mathscr{A}_a^{sa}$. Then for any $\lambda \in \mathbb{R}$, the function*

$$V \mapsto \int_{-\infty}^{\lambda} \xi(t, H_0+V, H_0)\,dt$$

is concave and the function

$$V \mapsto \tau[VE_{H_0}(\lambda)] - \int_{-\infty}^{\lambda} \xi(t, H_0+V, H_0)\,dt$$

is convex on the set of self-adjoint elements of $\mathscr{L}_1(\mathscr{A}, \tau)$.

Proof. Follows from Theorem 3.7 and Corollary 5.4. □

REMARK 6.9. Concavity of the integral spectral shift function with respect to the perturbation generalizers the important property of a finite Hermitian matrix H that the sum $S(H)$ of the negative eigenvalues of H is concave with respect to H (cf. [20]). In the standard trace class setting, concavity of the integral spectral shift function with respect to the perturbation was discussed in [10, 17].

THEOREM 6.10. *Let $[0,1] \ni s \mapsto H_0 + V(s)$ be an operator concave (convex) path, with $\cup_{s\in[0,1]}(H_0, V(s)) \subset \mathscr{P}$. Then,*

$$s \mapsto \tau[f(H_0+V(s)) - f(H_0)] \tag{6.3}$$

is concave (convex) for any absolutely continuous concave (convex) function f satisfying Krein's trace formula and such that f' is bounded on $\mathbb{R}$. If, in addition, τ is finite, then the function $s \mapsto \tau[f(H_0+V(s))]$ is concave (convex) for any bounded continuous concave (convex) function f.

Proof. Step 1. Assume that f is concave.

Assume first that $V(s) = sV$, for some $V \in \mathscr{L}_1(\mathscr{A}, \tau)$. By Lemma 6.8, for any nonnegative, bounded, and nonincreasing function g on $\mathbb{R}$, the function

$$s \mapsto \int_{\mathbb{R}} g(\lambda)\xi(\lambda, H_0+V(s), H_0)\,d\lambda \tag{6.4}$$

is concave. If $f' \geqslant 0$, then Krein's trace formula (3.1) and concavity of the function in (6.4) with $g = f'$ imply concavity of the function in (6.3).

Assume now that $s \mapsto V(s)$ is an arbitrary concave function. If $f' \geqslant 0$, then by Corollary 6.2,

$$\begin{aligned}\tau\big[f\big(H_0+V(\alpha s_1+(1-\alpha)s_2)\big)-f(H_0)\big]&\\ \geqslant \tau\big[f\big(\alpha(H_0+V(s_1))+(1-\alpha)(H_0+V(s_2))\big)-f(H_0)\big],&\end{aligned} \tag{6.5}$$

for $\alpha, s_1, s_2 \in [0,1]$. Applying concavity of the function (6.3) along the linear path

$$t \mapsto W(t) = t(H_0 + V(s_1)) + (1-t)(H_0 + V(s_2))$$

joining $W(0) = H_0 + V(s_2)$ and $W(1) = H_0 + V(s_1)$ yields

$$\begin{aligned}
&\tau\big[f\big(\alpha(H_0+V(s_1)) + (1-\alpha)(H_0+V(s_2))\big) - f(H_0)\big] \\
&\quad = \tau\big[f\big(W(\alpha\cdot 1 + (1-\alpha)\cdot 0)\big) - f(H_0)\big] \\
&\quad \geqslant \alpha\tau[f(W(1)) - f(H_0)] + (1-\alpha)\tau[f(W(0)) - f(H_0)] \\
&\quad = \alpha\tau\big[f\big(H_0+V(s_1)\big) - f(H_0)\big] + (1-\alpha)\tau\big[f\big(H_0+V(s_2)\big) - f(H_0)\big].
\end{aligned}$$

In the case of an arbitrary (not necessarily nonnegative) f', the proof completes by applying the result to a nondecreasing function $f + ct$, with $c > 0$.

Step 2. Assume that f is convex. Its derivative f' can represented as a difference of a constant function and a non-increasing bounded function g. Applying the theorem to an antiderivative of g (which satisfies Krein's trace formula since f and every constant function do) completes the proof of convexity of the function in (6.3).

Step 3. Let τ be finite. Approximating a continuous concave (convex) function f by smooth concave (convex) functions extends the result of the theorem to the case of bounded continuous f and bounded H_0 (see Lemma 2.2). Applying Lemma 2.3 extends the result to the case of an unbounded operator H_0. $\square$

Concavity of the function in (6.3) implies concavity of the functional $\tau[f(\cdot)]$ on the elements of $\mathscr{A}^{sa}$ that differ by elements in $\mathscr{L}_1(\mathscr{A},\tau)$.

COROLLARY 6.11. *Let $(H_0,V_1),(H_0,V_2) \in \mathscr{P}$. Then for any $\alpha \in [0,1]$ and f an absolutely continuous concave function satisfying Krein's trace formula* (3.1) *and such that f' is bounded,*

$$\begin{aligned}
&\tau[f(\alpha(H_0+V_1) + (1-\alpha)(H_0+V_2)) - f(H_0)] \\
&\qquad \geqslant \alpha\tau[f(H_0+V_1) - f(H_0)] + (1-\alpha)\tau[f(H_0+V_2) - f(H_0)].
\end{aligned} \tag{6.6}$$

If, in addition, τ is finite, then

$$\tau[f(\alpha A + (1-\alpha)B)] \geqslant \alpha\tau[f(A)] + (1-\alpha)\tau[f(B)] \tag{6.7}$$

holds for any $A,B \in \mathscr{A}_a^{sa}$, $\alpha \in [0,1]$, and f a bounded continuous concave function on $\mathbb{R}$.

Proof. Let $V(s) = V_1 + s(V_2 - V_1)$, $s \in [0,1]$. Then by Theorem 6.10,

$$\begin{aligned}
&\tau[f(\alpha(H_0+V(0)) + (1-\alpha)(H_0+V(1))) - f(H_0)] \\
&\qquad \geqslant \alpha\tau[f(H_0+V(0)) - f(H_0)] + (1-\alpha)\tau[f(H_0+V(1)) - f(H_0)].
\end{aligned} \tag{6.8}$$

Comparing the left and right hand sides of (6.6) and (6.8) completes the proof. $\square$

Adjusting the proof of [17, Corollary 2] to the case of a general trace implies concavity of the integral KrSSF with respect to a concave path of perturbations.

THEOREM 6.12. *Let* $[0,1]\ni s\mapsto H_0+V(s)$ *be an operator path, with* $\cup_{s\in[0,1]}(H_0, V(s))\subset\mathscr{P}$. *Assume that* $s\mapsto V(s)$ *is operator concave. Then, for any* $\lambda\in\mathbb{R}$, *the function*

$$s\mapsto\int_{-\infty}^{\lambda}\xi(t,H_0+V(s),H_0)\,dt$$

is concave.

Another result on convexity of the function $t\mapsto\tau[f(H(t))]$ is adaptation of [10, Theorem 3.6] to the case of a general trace.

THEOREM 6.13. *Let* $H_0\in\mathscr{A}_a^{sa}$ *and let* $[0,1]\ni s\mapsto V(s)\in\mathscr{L}_1(\mathscr{A},\tau)$ *be a path of self-adjoint operators twice continuously differentiable in the norm* $\|\cdot\|_1+\|\cdot\|$ *such that* $V''(s)\leqslant 0$. *Let* f *be a concave (convex)* C^1*-function with* $f'\in\mathscr{W}^1$ *an absolutely continuous function. Assume, in addition, that* H_0 *is bounded from below and* $f'(\lambda)=o(1)$, *as* $\lambda\to\infty$. *Then*

$$s\mapsto\tau[f(H_0+V(s))-f(H_0)]$$

is concave (convex).

Proof. Applying Krein's trace formula and then integrating by parts imply

$$\tau[f(H_0+V(s))-f(H_0)]=-\int_{\mathbb{R}}f''(\lambda)\int_{-\infty}^{\lambda}\xi_\tau(t,H_0+V(s),H_0)\,dt\,d\lambda. \tag{6.9}$$

Therefore, concavity (convexity) of the function in (6.9) amounts to the concavity of

$$s\mapsto\int_{-\infty}^{\lambda}\xi(t,H_0+V(s),H_0)\,dt,$$

which is a consequence of Theorem 3.7 and decrease of the function

$$t\mapsto\tau[V'(t)E_{H_0+V(t)}((-\infty,\lambda))].$$

The latter can be proved by following the lines in the proof of [10, Theorem 1.8] and replacing the standard trace with τ. $\square$

REMARK 6.14. In the case of operators H defined on a finite dimensional Hilbert space and τ the standard trace, proofs of convexity of $H\mapsto\tau[f(H)]$ were based on the analysis of the eigenvalues (see, e.g., [12]).

In the case of τ a tracial state on a unital C^*-algebra, convexity of the functional $\tau[f(\cdot)]$ on the self-adjoint elements of the algebra was obtained in [27] by continuous variational techniques [8, 9]. For τ a normal semi-finite faithful trace on a semi-finite von Neumann algebra $\mathscr{A}$, convexity of $\tau[f(\cdot)]$, with $f(0)=0$, on sign-definite τ-measurable operators affiliated with $\mathscr{A}$ was proved in [5] by invoking the notion of spectral dominance. In the latter case, convexity of $\tau[f(\cdot)]$ was shown to be equivalent to Jensen's inequality for traces [5]. The arguments of [5] did not involve pairing of elements that differ by a τ-trace class operator, and, in particular, did not involve KrSSF.

Restricting $\tau[f(\cdot)]$ to the positive elements of $\mathscr{A}_a^{sa}$ in [5] allowed the functional $\tau[f(\cdot)]$ to attain the value of ∞ or $-\infty$.

In the particular case of τ the standard trace and $\mathscr{A}$ the algebra of bounded linear operators $\mathscr{B}(\mathscr{H})$ on a separable Hilbert space $\mathscr{H}$, convexity of the functional $V \mapsto \tau[f(H_0+V)-f(H_0)]$ was also obtained on the trace class non-sign-definite self-adjoint operators V, with the additional requirement that $f(H_0+V)-f(H_0)$ be in the trace class, where H_0 is a self-adjoint operator. An operator $f(H_0+V)-f(H_0)$ is in the trace class if, for instance, f is a C^1-function whose derivative is in the Wiener class. For bounded operators, convexity of $V \mapsto \tau[f(H_0+V)-f(H_0)]$ for a convex $f \in C^\infty(\mathbb{R})$ was carried over from the finite-dimensional case by standard approximation arguments (see, e.g., [12]). The case of unbounded operators was treated in [17] by spectral perturbation theory techniques, including Krein's trace formula [19] and the Birman-Solomyak spectral averaging formula [3, 11] for KrSSF [19]. Similar techniques were applied in the proof of convexity of functions $[0,1] \ni t \mapsto \tau[f(H(t))-f(H(0))]$ [10, 17].

Acknowledgement. The author is thankful to Konstantin A. Makarov for useful discussions and to the anonymous referee for constructive criticism.

REFERENCES

[1] N. A. AZAMOV, A. L. CAREY, P. G. DODDS, F. A. SUKOCHEV, *Operator integrals, spectral shift, and spectral flow,* Canad. J. Math., to appear.

[2] N. A. AZAMOV, P. G. DODDS, F. A. SUKOCHEV, *The Krein spectral shift function in semifinite von Neumann algebras*, Integr. Equ. Oper. Theory, **55** (2006), 347–362.

[3] M. SH. BIRMAN, M. Z. SOLOMYAK, *Remarks on the spectral shift function,* Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 3–26; English transl. in J. Soviet Math. **3**, 4 (1975), 408–419.

[4] K. N. BOYADZHIEV, *Mean value theorems for traces,* Math. Japon. **38** (1993), 217–224.

[5] L. G. BROWN, H. KOSAKI, *Jensen's inequality in semi-finite von Neumann algebras,* J. Operator Theory, **23** (1990), 3–29.

[6] R. W. CAREY, J. D. PINCUS, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta Math., **138** (1977), 153–218.

[7] J. Dixmier, *Von Neumann Algebras*, North-Holland, Amsterdam, 1981.

[8] T. FACK, *Sur la notion de valeur caractéristique,* J. Operator Theory, **7** (1982), 307–333.

[9] T. FACK, H. KOSAKI, *Generalized s-numbers of τ-measurable operators,* Pacific J. Math., **123**, 2 (1986), 269–300.

[10] F. GESZTESY, K. A. MAKAROV, A. K. MOTOVILOV, *Monotonicity and concavity properties of the spectral shift function.* Stochastic processes, physics and geometry: new interplays, II (Leipzig, 1999), 207–222, CMS Conf. Proc., **29**, Amer. Math. Soc., Providence, RI, 2000.

[11] F. GESZTESY, K. A. MAKAROV, S. N. NABOKO, *The spectral shift operator,* in J. Dittrich, P. Exner, and M. Tater (eds.) "Mathematical Results in Quantum Mechanics", Operator Theory: Advances and Applications, Vol. 108, Birkhäuser, Basel, 1999, 59–90.

[12] F. GESZTESY, A. PUSHNITSKI, B. SIMON, *On the Koplienko Spectral shift function, I. Basics,* Zh. Mat. Fiz. Anal. Geom., **4**, 1 (2008), 63–107.

[13] F. HANSEN, G. K. PEDERSEN, *Jensen's operator inequality,* Bull. London Math. Soc., **35** (2003), 553–564.

[14] B. FUGLEDE AND R. V. KADISON, *Determinant theory in finite factors*, Ann. Math., **55** (1952), 520–530.

[15] R. V. KADISON AND J. R. RINGROSE, *Fundamentals of the Theory of Operator Algebras Vol. II,* Academic Press, Orlando, FL, 1986.

[16] L. S. KOPLIENKO, *Trace formula for nontrace-class perturbations,* Sibirsk. Mat. Zh., **25**, 5 (1984), 6–21 (Russian). English translation: Siberian Math. J., **25**, 5 (1984), 735–743.

[17] V. KOSTRYKIN, *Concavity of eigenvalue sums and the spectral shift function,* J. Funct. Anal., **176**, 1 (2000), 100–114.

[18] V. KOSTRYKIN, K. A. MAKAROV, A. SKRIPKA, *The Birman-Schwinger principle in von Neumann algebras of finite type,* J. Funct. Anal., **247** (2007), 492–508.

[19] M. G. KREIN, *On a trace formula in perturbation theory*, Matem. Sbornik, **33** (1953), 597–626 (Russian).

[20] E. H. LIEB, H. SIEDENTOP, *Convexity and concavity of eigenvalue sums,* J. Statist. Phys., **63** (1991), 811–816.

[21] I. M. LIFSHITS, *On a problem of the theory of perturbations connected with quantum statistics*, Uspehi Matem. Nauk, **7** (1952), 171–180 (Russian).

[22] K. A. MAKAROV, A. SKRIPKA, *Some applications of the perturbation determinant in finite von Neumann algebras,* Canad. J. Math., to appear.

[23] H. NEIDHARDT, *Spectral shift function and Hilbert-Schmidt perturbation: extensions of some work of L.S. Koplienko,* Math. Nachr., **138** (1988), 7–25.

[24] V.V. PELLER, *Hankel operators in the perturbation theory of unbounded self-adjoint operators.* Analysis and partial differential equations, Lecture Notes in Pure and Applied Mathematics, 122, Dekker, New York, 1990, 529–544.

[25] V. V. PELLER, *An extension of the Koplienko-Neidhardt trace formulae,* J. Funct. Anal., **221** (2005), 456–481.

[26] V. V. PELLER, *Multiple operator integrals and higher operator derivatives,* J. Funct. Anal., **233** (2006), 515–544.

[27] D. PETZ, *Spectral scale of self-adjoint operators and trace inequalities,* J. Math. Anal. Appl., **109** (1985), 74–82.

[28] M. REED, B. SIMON, *Methods of Modern Mathematical Physics I, Functional Analysis,* 2nd. ed., Academic Press, New York, 1980.

[29] D. RUELLE, *Statistical Mechanics. Rigorous Results,* Bejamin, New York, 1969.

[30] S. SAKAI, *C^*-algebras and W^*-algebras,* Ergebn. Math. und ihrer Grenzgeb., 60, Springer-Verlag, New York-Heilderberg, 1971.

[31] M. A. SHUBIN, *Discrete magnetic Laplacian,* Comm. Math. Phys. **164**, 2 (1994), 259–275.

[32] B. SIMON, *Spectral averaging and the Krein spectral shift,* Proc. Amer. Math. Soc., **126**, 5 (1998), 1409–1413.

[33] A. SKRIPKA, *On properties of the ξ-function in semi-finite von Neumann algebras,* Integr. Equ. Oper. Theory, **62**, 2 (2008), 247–267.

[34] M. TAKESAKI, *Theory of operator algebras. I,* Springer-Verlag, New York-Heidelberg, 1979.

[35] D. VOICULESCU, *On a trace formula of M. G. Krein,* Operator Theory: Adv. Appl., **24** (1987), 329–332.

(Received October 16, 2008)

Anna Skripka
Department of Mathematics
Texas A&M University
College Station, TX 77843
USA
e-mail: askripka@math.tamu.edu