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CHARACTERIZATION OF THE UNBOUNDED BICOMMUTANT OF $C_0(N)$ CONTRACTIONS

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Abstract. Recent results have shown that any closed operator A commuting with the backwards shift S^* restricted to $K_u^2 := H^2 \ominus uH^2$, where u is an inner function, can be realized as a Nevanlinna function of $S_u^* := S^*|_{K_u^2}$, $A = \varphi(S_u^*)$, where φ belongs to a certain class of Nevanlinna functions which depend on u. In this paper this is generalized to show that given any contraction T of class $C_0(N)$, that any closed (and not necessarily bounded) operator A commuting with the commutant of T is equal to $\varphi(T)$ where φ belongs to a certain class of Nevanlinna functions which depend on the minimal inner function m_T of T.

1. Introduction

Let u be an inner function and let $K_u^2 := H^2 \ominus uH^2$. Recall that the Nevanlinna class $\mathscr{N}$ in $\mathbb{D}$ is the class of functions $\varphi = \psi/\chi$ where $\psi, \chi \in H^\infty$ and χ is not the zero function. The Smirnov class $\mathscr{N}^+ \subset \mathscr{N}$ consists of all $\varphi = \psi/\chi \in \mathscr{N}$ for which χ is outer. As defined in [1], the local Smirnov class $\mathscr{N}_u^+$ consists of all $\varphi \in \mathscr{N}$ for which u, χ are relatively prime. As discussed in Sects. 3 and 5 of [1], any $\varphi \in \mathscr{N}_u^+$ has a unique canonical representation $\varphi = b/va$ where $a, b \in H^\infty$, a is an outer function such that $a(0) = 0$, $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$, v is inner and v, b and v, u are relatively prime. Given u and K_u^2, define the compression $S_u := P_u S|_{K_u^2}$, where S is the shift (multiplication by z) and P_u is the orthogonal projection of H^2 onto K_u^2. Since K_u^2 is invariant for the backwards shift S^*, S_u is the adjoint of $S_u^* := S^*|_{K_u^2}$.

Given any $\chi \in H^\infty$ such that χ, u are relatively prime, one can show that $\chi(S_u)$ is injective and has dense range so that $\chi(S_u)^{-1}$ can be realized as a densely defined and closed operator in K_u^2 [1] (actually, the results of [1] are expressed in terms of S_u^*, we restate them here in terms of S_u). Hence, as discussed at the end of Sect. 5 of the same paper, for any $\varphi \in \mathscr{N}_u^+$, one can naturally define $\varphi(S_u) = ((va)(S_u))^{-1} b(S_u)$ as a closed operator on a dense domain in K_u^2.

In [1], Sarason extends the results of Suárez in [2] to prove the following:

THEOREM 1. *(Sarason) A closed operator A densely defined in K_u^2 commutes with S_u if and only if $A = \varphi(S_u)$ where $\varphi \in \mathscr{N}_u^+$.*

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This is a natural extension of the following well-known fact, first established in [3]:

THEOREM 2. *(Sarason) A bounded operator B belongs to the commutant of S_u if and only if $B = h(S_u)$ for some $h \in H^\infty$.*

Recall that a contraction T is said to be of class C_0 if there is an H^∞ function v such that $v(T) = 0$. For any such contraction there is a minimal inner function $m_T \in H^\infty$ such that $m_T(T) = 0$ and m_T is a divisor of any $h \in H^\infty$ for which $h(T) = 0$. The multiplicity μ_T of T is the minimum cardinal number of a subset $\mathfrak{S} \subset \mathscr{H}$ such that $\bigvee_{n=0}^{\infty} T^n \mathfrak{S} = \mathscr{H}$.

In Sect. 4 of [4], Sz.-Nagy and Foias use their canonical Jordan model for any contraction T of class C_0 with $\mu_T, \mu_{T^*} < \infty$ to show that any element in the double commutant of T is a Nevanlinna function of T. Here the double commutant, $(T)''$, is defined, as usual, as the set of all bounded operators commuting with the commutant, $(T)'$, the set of all bounded operators commuting with T.

THEOREM 3. *(Sz.-Nagy, Foias) For any contraction T of type C_0 with finite multiplicities $\mu_T < \infty$ and $\mu_{T^*} < \infty$, all operators $A \in (T)''$ have the form $\varphi(T)$ where $\varphi \in \mathscr{N}_T$.*

They further show by example that there exist such contractions T for which there are $\varphi(T) \in (T)''$ where $\varphi \in \mathscr{N}$, $\varphi \notin H^\infty$ so that H^∞ functions of T do not exhaust the double commutant of T.

Recall that a contraction T is said to be of class $C_0(N)$ if T^n, $(T^*)^n$ converge strongly to 0, and if $N = \mathfrak{d}_T = \mathfrak{d}_{T^*}$, where the deficiency index $\mathfrak{d}_T$ is defined as $\mathfrak{d}_T := \dim\left(\overline{(I - T^*T)\mathscr{H}}\right)$. A contraction T belongs to the class $C_0(N)$ if and only if it is unitarily equivalent to some $S(\Theta)$ where $S(\Theta)$ is the compression of the the shift on $H^2(\mathscr{H}_N)$, to the subspace $K^2(\Theta) := H^2(\mathscr{H}_N) \ominus \Theta H^2(\mathscr{H}_N)$. Here $\mathscr{H}_N$ is any N dimensional Hilbert space, $H^2(\mathscr{H}_N)$ is the Hardy space of functions on the unit disc which take values in the Hilbert space $\mathscr{H}_N$, and Θ is an $N \times N$ matrix valued inner function. The class $C_0(N)$ is contained in the class of C_0 contractions with finite multiplicities $\mu_T, \mu_{T^*} < \infty$. Indeed, if $T \in C_0(N)$ then $\mu_T, \mu_{T^*} \leqslant N$ ([5], [4] pg. 94). Moreover, if $T \in C_0(N)$, and Θ_T is the $N \times N$ matrix valued inner function such that T is unitarily equivalent to $S(\Theta_T)$, then m_T is equal to the quotient of $\det(\Theta_T)$ by the greatest common inner divisor of the minors of order $N-1$ of the matrix of Θ_T ([6], Chapter VI, Theorem 5.2).

Given a bounded operator B, a closed operator A (not necessarily bounded) will be said to commute with T provided $B : \mathrm{Dom}(A) \to \mathrm{Dom}(A)$ and $[A,B]f = (AB - BA)f = 0$ for all $f \in \mathrm{Dom}(A)$. This implies that AB is an extension of BA (in general a proper one) and will be written more concisely as $AB \supset BA$. Given a contraction T, we will say that a closed operator A belongs to the unbounded double commutant of T, $(T)''_{ub}$, if $AB \supset BA$ for any $B \in (T)'$. Note that $(T)'' \subset (T)''_{ub}$. Just as Sz.-Nagy and Foias used Sarason's original result, Theorem 2 to prove Theorem 3, in this paper we will perform the necessary modifications to the methods of [4] and use Sarason's new,

'unbounded' version, Theorem 1, of Theorem 2 to prove the following 'unbounded' analog of Theorem 3.

THEOREM 4. *Let T be a contraction of class $C_0(N)$. Then $A \in (T)''_{ub}$ if and only if $A = \varphi(T)$ for some $\varphi \in \mathscr{N}^+_{m_T}$.*

Note that our assumptions on T in Theorem 4 are stronger than those used by Sz.-Nagy and Foias in Theorem 3. We expect that there is a stronger version of Theorem 4 which holds for all T satisfying the conditions of Theorem 3, but this will not be proven here. The reason for the more restrictive assumption is that our proof will require the use of a lemma that states that any $T \in C_0(N)$ cannot be quasi-similar to its restriction to any proper invariant subspace, see Remark 2.0.2. If this lemma could be shown to hold for all contractions satisfying the conditions of Theorem 3, *i.e.* all C_0 contractions T with finite multiplicities $\mu_T, \mu_{T^*} < \infty$, then the methods used in this paper would imply that the conclusions of Theorem 4 hold for this more general class of contractions as well.

1.1. Contractions of class C_0 with finite multiplicity

In [4], a Jordan operator is defined as a contraction of the form:

$$S(u_1, u_2, ..., u_k) := S_{u_1} \oplus S_{u_2} \oplus ... \oplus S_{u_k}, \tag{1}$$

where each u_i is a non-constant inner function and each u_i is an inner divisor of u_{i-1} for $2 \leqslant i \leqslant k$. Clearly such an operator is of class $C_0(N)$ with minimal function u_1.

Recall that a bounded operator $X : \mathscr{H}_1 \to \mathscr{H}_2$ is called a quasi-affinity if it has dense range, and is injective, *i.e.* if it has a (possibly unbounded) inverse defined on a dense domain in $\mathscr{H}_2$. Given $T_i \in B(\mathscr{H}_i)$, $i = 1, 2$, T_1 is called a quasi-affine transform of T_2 if there exists a quasi-affinity X intertwining T_2 and T_1, $T_2X = XT_1$. This is denoted by $T_2 \succ T_1$. Note that $\succ$ is a transitive partial order, and that if $T_1 \succ T_2$ then $T_2^* \succ T_1^*$. If the T_i are of class C_0 and $T_1 \succ T_2$, then $m_{T_1} = m_{T_2}$, and $\mu_{T_1} \leqslant \mu_{T_2}$. In particular if $T_1 \succ T_2$ and $T_2 \succ T_1$, then T_1, T_2 are said to be quasi-similar, and $\mu_{T_1} = \mu_{T_2}$. For more details, we refer the reader to [4], or [6].

This next theorem of [4] shows that every contraction $T \in C_0$ with finite multiplicities has a canonical 'Jordan model'.

THEOREM 5. *(Sz.-Nagy, Foias) Let T be any contraction of class C_0 with finite multiplicities $\mu_T, \mu_{T^*} < \infty$. Then T is quasi-similar to a Jordan model operator $S(m_1, ..., m_k)$ where $k = \mu_T = \mu_{T^*}$, and $m_1 = m_T$.*

We will also need the following lemma from [4].

LEMMA 1. *(Sz.-Nagy , Foias) Let m, m' be non-constant inner functions, with m an inner divisor of m', $m' = mq$, q inner. Then $\mathscr{H}^0 := qH^2 \ominus m'H^2 = K^2_{m'} \ominus K^2_q$ is invariant for $S_{m'}$ and $S^0 := S_{m'}|_{\mathscr{H}^0}$ is unitarily equivalent to S_m.*

Moreover, $R : K_m^2 \to \mathscr{H}^0$, $Rh = qh$ *is a unitary transformation onto* $\mathscr{H}^0$ *such that* $S^0 R = RS_m$.

The operator $Q := R^{-1} q(S_{m'}) : K_{m'}^2 \to K_m^2$, *where* $q(S_{m'}) : K_{m'}^2 \to \mathscr{H}^0$ *is such that* $QK_{m'}^2 = K_m^2$ *and* $S_m Q = QS_{m'}$.

2. Proof of Theorem 4

Recall, as defined in [6], for any contraction T, $\mathscr{N}_T$ is the class of Nevanlinna functions $\varphi = \psi/\chi$ such that $\chi(T)$ is injective and has dense range. The class of all such χ is denoted by K_T^∞. For all $\varphi \in \mathscr{N}_T$, one can define $\varphi(T) = \chi^{-1}(T)\psi(T)$.

2.0.1. Remark

By Theorem 1.1, Chapter 4 of [6], if $\varphi = \psi/\chi \in \mathscr{N}_T$, then $\varphi(T) = \chi^{-1}(T)\psi(T)$ is a closed operator on a dense domain in $\mathscr{H}$, and $\varphi(T) \in (T)''_{ub}$.

For a contraction of class C_0 the following lemma shows that this definition of $\mathscr{N}_T$ coincides with $\mathscr{N}_{m_T}^+$.

LEMMA 2. *For a contraction of class* C_0, $\mathscr{N}_T = \mathscr{N}_{m_T}^+$.

Proof. This lemma is a consequence of known results. If $\varphi \in \mathscr{N}_{m_T}^+$, let $\varphi = b/va$ be its canonical decomposition with $b, v, a \in H^\infty$, a, outer, and v an inner function relatively prime to both m_T and b (see the Introduction, Sect. 1, and [1]). By Proposition 3.1, Chapter 3, of [6], any outer function belongs to $K^\infty(T)$, the class of H^∞ functions χ for which $\chi(T)$ is injective and has dense range. Secondly, by Proposition 4.7 (b), Chapter 3, of [6], an inner function v belongs to K_T^∞ if and only if v and m_T are relatively prime. Also by earlier results of the same section ([6], pg. 121), K_T^∞ is multiplicative. It follows that $va \in K^\infty(T)$ so that $\varphi = b/va \in \mathscr{N}_T$ and $\mathscr{N}_{m_T}^+ \subset \mathscr{N}_T$.

If $\phi \in \mathscr{N}_T$, then ϕ has the canonical decomposition $\phi = b/va$ where v is inner and a is outer, and v, b are relatively prime. Using the results mentioned above, and the uniqueness of the canonical decomposition, it is not difficult to show that if v, m_T are not relatively prime, then $va \notin K_T^\infty$, and that $\phi \notin \mathscr{N}_T$. Hence v, m_T are relatively prime so that $\mathscr{N}_T \subset \mathscr{N}_{m_T}^+$. $\square$

PROPOSITION 1. *Let* T *be a contraction of class* C_0 *with* $\mu_T, \mu_{T^*} < \infty$. *Then if* $A \in (T)''_{ub}$, *there is a* $\varphi \in \mathscr{N}_T^+$ *and a dense domain* $\mathfrak{D} \subset \mathscr{H}$ *such that* $Af = \varphi(T)f$ *for all* $f \in \mathfrak{D}$.

The following lemma is needed in the proof of Proposition 1.

LEMMA 3. *(Sarason) If* A *is a densely defined operator commuting with every* H^∞ *function of* S_u *then* A *is closable, and* $\overline{A}$ *commutes with* S_u.

The above lemma has not been published before. It will appear, along with its proof, in an upcoming paper by D. Sarason.

The proof of the above proposition follows the proof of Theorem 3 very closely. We will partially sketch the unchanged portions of the proof, and indicate where the methods of [4] are modified.

Proof. By Theorem 5, T is quasi-similar to $S := S(m_1,...,m_N)$ acting on $\mathscr{G} := K^2_{m_1} \oplus ... \oplus K^2_{m_N}$ where $N = \mu_T < \infty$ and $m_1 = m_T$. For convenience we will denote $K^2_j := K^2_{m_j}$ and $S_j := S_{m_j}$. Let $X : \mathscr{G} \to \mathscr{H}$ and $Y : \mathscr{H} \to \mathscr{G}$ be the quasi-affinities such that

$$TX = XS \text{ and } SY = YT. \tag{2}$$

By Lemma 1, for each $j = 1,...,N$ there is a subspace $\mathscr{H}^0_j \subset K^2_1$ which is invariant for S_1, and such that $S^0_j := S_1|_{\mathscr{H}^0_j}$ is unitarily equivalent to S_j,

$$S^0_j R_j = R_j S_{m_j}, \tag{3}$$

with each $R_j : K^2_j \to \mathscr{H}^0_j$ unitary, $j = 1,...,N$. Observe that $\mathscr{H}^0_1 = K^2_{m_1}$. Also by the same lemma there exist bounded operators $Q_j : K^2_{m_1} \to K^2_{m_j}$ such that

$$S_j Q_j = Q_j S_1, \tag{4}$$

where each Q_j is onto $K^2_{m_j}$.

Now consider the operators

$$S^0 := S^0_1 \oplus ... \oplus S^0_N \text{ on } \mathscr{G}^0 := \mathscr{H}^0_1 \oplus ... \oplus \mathscr{H}^0_N, \tag{5}$$

and

$$\hat{S} := \oplus^N_{j=1} S_1 \text{ on } \hat{\mathscr{G}} := \oplus^N_{j=1} K^2_1. \tag{6}$$

If we let $R := \oplus^N_{j=1} R_j$ and $Q := \oplus^N_{j=1} Q_j$ it follows from (3) and (4) that R is unitary, that

$$S^0 R = RS, \ \ R\mathscr{G} = \mathscr{G}^0, \tag{7}$$

and that

$$SQ = Q\hat{S}, \quad Q\hat{\mathscr{G}} = \mathscr{G}. \tag{8}$$

According to Lemma 1, $Q_j = R_j^{-1} q_j(S_1)$ for the inner function q_j where $m_1 = m_j q_j$, $1 \leqslant j \leqslant N$, and $H^0_j = K^2_1 \ominus K^2_{q_j}$. It follows that $\mathrm{Ker}(Q_j) = K^2_{q_j}$ so that $K^2_1 = H^0_j \oplus \mathrm{Ker}(Q_j)$ for each $1 \leqslant j \leqslant N$ and $\hat{\mathscr{G}} = \mathscr{G}^0 \oplus \mathrm{Ker}(Q)$.

Using the quasi-affinity Y, (7), and the fact that $\hat{S}|_{\mathscr{G}^0} = S^0$ yields

$$\hat{S}RY = RYT. \tag{9}$$

Now choose any $W \in (\hat{S})'$. By (2), (8) and (9), it follows that

$$\begin{aligned} T(XQWRY) &= XSQWRY = XQ\hat{S}WRY = XQW\hat{S}RY \\ &= (XQWRY)T, \end{aligned} \tag{10}$$

so that $XQWRY \in (T)'$. Hence for $A \in (T)''_{ub}$, one has $A(XQWRY) \supset (XQWRY)A$. Let $C := RYXQ$ and define $B := RYAXQ$ on $\mathrm{Dom}(B) := \mathfrak{D}_1 \oplus \mathrm{Ker}(Q) \subset \mathscr{G}^0 \oplus \mathrm{Ker}(Q) = \hat{\mathscr{G}}$

where $\mathfrak{D}_1 := \{f \in \mathscr{G}^0 |\ f = RYg;\ g \in \mathrm{Dom}(AXQRY)\}$. The linear manifold $\mathfrak{D}_1$ is clearly dense in $\mathscr{G}^0$ since $I \in (\hat{S})'$ implies $XQRY \in (T)'$ so that $A(XQRY) \supset (XQRY)A$. This, along with the facts that $\mathrm{Dom}(A)$ is dense in $\mathscr{H}$ and $RY : \mathscr{H} \to \mathscr{G}^0$ is a quasi-affinity imply that $\mathfrak{D}_1$ is dense in $\mathscr{G}^0$, and hence that $\mathrm{Dom}(B)$ is dense in $\hat{\mathscr{G}}$. Observe that the maps B, C obey

$$BWC \supset CWB. \tag{11}$$

Furthermore, both B and C commute with $\hat{S}$, $[\hat{S}, C] = 0$ and $B\hat{S} \supset \hat{S}B$. Indeed,

$$\begin{aligned} \hat{S}B &= \hat{S}(RY)AXQ = (RY)T(AXQ) \subset RYA(TX)Q \\ &= RYA(XS)Q = (RYAXQ)\hat{S} = B\hat{S}, \end{aligned} \tag{12}$$

the same follows for the bounded operator C when A is replaced by I. Using the same argument as above, and the fact that $A \in (T)''_{ub}$ further shows that $h(\hat{S})B \subset Bh(\hat{S})$ for any $h \in H^\infty$.

Since B, C are linear transformations from $\hat{\mathscr{G}}$ into $\mathscr{G}^0 \subset \hat{\mathscr{G}}$, they can be viewed as operators on $\hat{\mathscr{G}}$. Since W, C are bounded we can consider their matrix representations $W = [W_{ij}]$, $C = [C_{ij}]$, $i, j = 1, ..., N$ with respect to the decomposition $\hat{\mathscr{G}} = \oplus_{j=1}^N K_1^2$.

We would like to write B as such a matrix. However, since A and hence B is in general unbounded, we need to check that such a matrix representation of B is valid. For example it could be that the domain of B does not contain any vectors of the form $f = f_1 \oplus 0 ... \oplus 0$ in which case it would not be possible to write B as a matrix with respect to the decomposition $\hat{\mathscr{G}} = \oplus_{j=1}^N K_1^2$.

Since $W \in (\hat{S})'$, is arbitrary, it can be chosen to be any matrix $W = [W_{ij}]$, $1 \leqslant i, j \leqslant N$, such that its entries $W_{ij} \in (S_1)'$. Now observe that the range of $C = RYXQ$ is dense in $\mathscr{G}^0$. This follows from (7), (8) and the fact that both X, Y are quasi-affinities. Choosing $W_j := E_{1j}$, where E_{ij} are the matrix units with respect to the decomposition $\hat{\mathscr{G}} = \oplus_{j=1}^N K_1^2$, and using $BWC \supset CWB$ (see equation (11)), it follows that any vector $\oplus_{j=1}^N \delta_{ij} f_1$ where $\oplus_{j=1}^N f_j \in C\mathrm{Dom}(B)$ belongs to $\mathrm{Dom}(B)$ for each $1 \leqslant i \leqslant N$. Since $\mathrm{Ran}(C)$ is dense in $\mathscr{G}^0 = K_1^2 \oplus \mathscr{H}_2^0 ... \oplus \mathscr{H}_N^0$, and $\mathrm{Dom}(B)$ is dense in $\hat{\mathscr{G}}$, it follows that the set of all such f_1, where $\oplus_{j=1}^N f_j \in C\mathrm{Dom}(B)$ is a dense linear manifold in K_1^2. In conclusion,

$$\mathrm{Dom}(B') := \{\oplus_{j=1}^N f_j \in \mathrm{Dom}(B) |\ \oplus_{j=1}^N \delta_{ij} f_j \in \mathrm{Dom}(B);\ 1 \leqslant i \leqslant N\} \subset \mathrm{Dom}(B) \tag{13}$$

defines a dense linear manifold in $\hat{\mathscr{G}} = \oplus_{j=1}^N K_1^2$. Let $B' := B|_{\mathrm{Dom}(B')}$. It follows that $B' = [B'_{ij}]$ where each B'_{ij} is a densely defined linear operator in K_1^2. Explicitly, $\mathrm{Dom}(B'_{ij}) = \{f \in K_1^2 \mid \oplus_{k=1}^N \delta_{jk} f \in \mathrm{Dom}(B')\}$, and given $f \in \mathrm{Dom}(B'_{ij})$, $B'_{ij} f = P_i B' f$ where P_i projects $\hat{\mathscr{G}} = \oplus_{k=1}^N K_1^2$ onto the ith copy of K_1^2.

We must also check that $WC : \mathrm{Dom}(B') \to \mathrm{Dom}(B')$ for any $W \in (\hat{S})'$ so that we still have

$$CWB' \subset B'WC, \tag{14}$$

instead of equation (11). To do this it suffices to verify that given any vector $\hat{f} := (f, 0, ..., 0) \in \mathrm{Dom}(B')$ that $WC\hat{f} \in \mathrm{Dom}(B')$. If we choose for example $W = 1$, equation (11) implies that $C : \mathrm{Dom}(B) \to \mathrm{Dom}(B)$ so that $C\hat{f} = \hat{g} = (g_1, ..., g_N) \in \mathrm{Dom}(B)$.

More generally, $WC\hat{f} = (\sum_{j=1}^{N} W_{1j}g_j, \sum W_{2j}g_j, ..., \sum W_{Nj}g_j)$. To show that $WC\hat{f} \in \mathrm{Dom}(B')$ we need to show that each $\hat{g}_i = \oplus_{j=1}^{N} \delta_{ij} \sum_{k=1}^{N} W_{ik}g_k$ belongs to $\mathrm{Dom}(B)$, $1 \leqslant i \leqslant N$. For example consider the vector $\hat{g}_1 := (\sum_{j=1}^{N} W_{1j}g_j, 0, ..., 0)$. This is clearly equal to $W^{(1)}C\hat{f}$ where $W^{(1)} = [\delta_{1i}W_{ij}]$ is the matrix obtained by taking the first row of the matrix representation of W and setting all remaining entries to 0. Since $W \in (\hat{S})'$, each W_{ij} commutes with S_1, and so it follows that we also have $W^{(1)} \in (\hat{S})'$. Hence $W^{(1)}C : \mathrm{Dom}(B) \to \mathrm{Dom}(B)$, so that $W^{(1)}C\hat{f} = \hat{g}_1 \in \mathrm{Dom}(B)$. Using similar arguments for the other entries, it follows that each $\hat{g}_i$; $1 \leqslant i \leqslant N$ belongs to $\mathrm{Dom}(B)$ so that $WC\hat{f} \in \mathrm{Dom}(B')$. We conclude that $WC : \mathrm{Dom}(B') \to \mathrm{Dom}(B')$. This shows that (14) holds, $B'WC \supset CWB'$. Also it is clear that since $h(\hat{S}) : \mathrm{Dom}(B') \to \mathrm{Dom}(B')$, and the matrix representation of $h(\hat{S}) = h(S_1) \oplus ... \oplus h(S_1)$ is diagonal with respect to the decomposition $\hat{G} = \oplus_{j=1}^{N} K_1^2$, we also still have that $B'h(\hat{S}) \supset h(\hat{S})B'$ for any $h \in H^\infty$.

Since C commutes with $\hat{S}$ it follows that the matrix entries C_{ij} of C are bounded operators commuting with S_1. By Theorem 2, it follows that there are H^∞ functions c_{ij} such that $C_{ij} = c_{ij}(S_1)$. Similarly, since $B'h(\hat{S}) \supset h(\hat{S})B'$ for any $h \in H^\infty$ and the matrix representation of $h(\hat{S})$ is diagonal, $h(\hat{S}) = h(S_1) \oplus ... \oplus h(S_1)$, it follows that $B'_{ij}h(S_1) \supset h(S_1)B'_{ij}$. By Lemma 3, each B'_{ij} has a closure $\overline{B'_{ij}}$ commuting with S_1, and by Theorem 1 and Lemma 2, there exist Nevanlinna functions $\varphi_{ij} = b_{ij}/\beta_{ij} \in \mathscr{N}_{m_1}^+ = \mathscr{N}_{m_T}^+ = \mathscr{N}_T$ such that $\overline{B'_{ij}} = \beta_{ij}^{-1}(S_1)b_{ij}(S_1)$ where each $\beta_{ij}^{-1}(S_1)$ is densely defined in K_1^2.

Since $B'WC \supset WCB'$ for any $W \in (\hat{S})'$, choose W so that $W_{ij} = I$ for $(i,j) = (k,1)$ and all other entries 0. It follows that there is a dense set $\mathscr{D} \subset K_1^2$ such that

$$\beta_{ik}(S_1)^{-1}b_{ik}(S_1)c_{1j}(S_1)f = c_{ik}(S_1)\beta_{1j}^{-1}(S_1)b_{1j}(S_1)\phi \quad \forall\, f \in \mathscr{D}. \tag{15}$$

Hence if $d_{ij} := \beta_{ij}c_{ij}$, then $b_{ik}(S_1)d_{1j}(S_1)f = d_{ik}(S_1)b_{1j}(S_1)f \quad \forall\, f \in \mathscr{D}$. Since $d_{ij}, c_{ij} \in H^\infty$ and $\mathscr{D}$ is dense it follows that

$$b_{ik}(S_1)d_{1j}(S_1) = d_{ik}(S_1)b_{1j}(S_1). \tag{16}$$

Since C has dense range in $\mathscr{G}^0 = K_1^2 \oplus_{j=2}^{N} \mathscr{H}_j^0$, elements of the form $\sum_{j=1}^{N} c_{1j}(S_1)g_j$ where each $g_j \in K_1^2$ are dense in K_1^2. Furthermore, since each $\beta_{ij}(S_1)$ has dense range in K_1^2, it follows that elements of the form $\sum_{j=1}^{N} d_{1j}(S_1)g_j$, $g_j \in K_1^2$ are dense in K_1^2.

The proof now proceeds as in [4], pgs. 109–111, and leads to the conclusion that there are $v, w \in H^\infty$ such that v and $m_1 = m_T$ are relatively prime and

$$b_{ik}(S_1)v(S_1) - d_{ik}(S_1)w(S_1) = (b_{ik}v)(S_1) - (\beta_{ik}c_{ik}w)(S_1) = 0, \;\; 1 \leqslant i,k \leqslant N. \tag{17}$$

Hence, for all $f \in \mathrm{Ran}(\beta_{ik}(S_1))$ it follows that

$$\left(\beta_{ik}^{-1}(S_1)b_{ik}(S_1)v(S_1) - c_{ik}(S_1)w(S_1)\right)f = \left(\overline{B'_{ik}}v(S_1) - C_{ik}w(S_1)\right)f = 0. \tag{18}$$

In the above note that each $(b_{ik}v)(S_1)$ maps $\mathrm{Dom}(\beta^{ik}(S_1)^{-1}) \subset \mathrm{Dom}(\overline{B'_{ik}})$ to itself. It follows that

$$\begin{aligned} 0 &= \left(\overline{B'}v(\hat{S}) - Cw(\hat{S})\right)f \\ &= RY\left(AXQv(\hat{S}) - XQw(\hat{S})\right)f \quad \forall\, f \in \mathrm{Dom}(B'). \end{aligned} \tag{19}$$

Since R is unitary, and Y is injective,

$$0 = (Av(T) - w(T))XQf \quad \forall f \in \mathrm{Dom}(B'). \tag{20}$$

Here, note that $f \in \mathrm{Dom}(B')$, and $B'f = RYAXQf$ so that $XQf \in \mathrm{Dom}(A)$. Since $Av(T) \supset v(T)A$, we can now conclude that

$$(v(T)A - w(T))g = 0, \tag{21}$$

for all $g \in XQ\mathrm{Dom}(B')$. Since $\mathrm{Dom}(B')$ is dense in $\hat{\mathscr{G}}$ we have $\overline{XQ\mathrm{Dom}(B')} = \overline{XQ\hat{\mathscr{G}}} = \overline{X\mathscr{G}} = \mathscr{H}$. Hence there is a dense linear manifold of vectors $\mathfrak{D} := XQ\mathrm{Dom}(B')$ such that for all $f \in \mathfrak{D}$, $(v(T)A - w(T))f = 0$, and hence $Af = v(T)^{-1}w(T)f = \varphi(T)f$ for all $f \in \mathfrak{D} \subset \mathrm{Dom}(\varphi(T))$ where $\varphi \in \mathscr{N}_T$ is a Nevanlinna function. $\square$

The main result, Theorem 4 will now follow from the above proposition once it is established that given any $\varphi \in \mathscr{N}_T$, the closed densely defined operator $\varphi(T)$ has no proper closed restrictions or extensions.

PROPOSITION 2. *Given any contraction T of class $C_0(N)$, and any $\varphi \in \mathscr{N}_T$, the closed operator $\varphi(T)$ has no proper closed densely defined extensions or restrictions belonging to $(T)''_{ub}$.*

This proposition is a consequence of the Jordan model for T and the following lemma.

LEMMA 4. *Given an inner function u and $\varphi \in \mathscr{N}_u^+$, the closed operator $\varphi(S_u)$ has no proper closed densely defined extension or restriction commuting with S_u.*

The proof of this lemma follows immediately from the first part of the proof of Lemma 5.7 of [1].

2.0.2. Remark

The proof of Proposition 2 relies on the above lemma, as well as the following two facts taken from [4]. First, if T, T' are contractions of class C_0 with finite multiplicities, then $T \succ T'$ implies T, T' have the same Jordan models. Conversely T, T' having the same Jordan model implies they are in fact quasi-similar. Furthermore, if $T \in C_0(N)$; $N \geqslant 1$, then the restriction $T|_S$ to a proper invariant subspace $S \subset \mathscr{H}$ cannot have the same Jordan model as T. That is, T cannot be quasi-similar to its restriction to a proper invariant subspace. These facts are contained in Corollaries $1-2$ of [4].

Proof of proposition 2. Let S_T be the Jordan model of T on $\mathscr{G} := K_1^2 \oplus \ldots \oplus K_N^2$, and as in the proof of Proposition 1, let X, Y be the quasi-affinities such that $X : \mathscr{G} \to \mathscr{H}$, $Y : \mathscr{H} \to \mathscr{G}$ and $TX = XS_T$, $S_T Y = YT$.

Now consider $\varphi(T)$, and suppose that R is a densely defined proper closed restriction of $\varphi(T)$ such that $R \in (T)''_{ub}$, and let $\Gamma_\varphi \supsetneqq \Gamma_R$ denote the graphs of these two operators. Note that $\Gamma_\varphi, \Gamma_R \subset \mathscr{H} \oplus \mathscr{H}$ are invariant for $T \oplus T$, and since $XY \in (T)'$ it follows that Γ_φ, Γ_R are also invariant for $XY \oplus XY$.

It is straightforward to verify that $\overline{(Y\oplus Y)\Gamma_R}$ and $\overline{(Y\oplus Y)\Gamma_\varphi}$ are invariant for $S_T\oplus S_T$. I claim that $\overline{(Y\oplus Y)\Gamma_\varphi}\supsetneqq\overline{(Y\oplus Y)\Gamma_R}$. Suppose to the contrary that $\overline{(Y\oplus Y)\Gamma_\varphi}=\overline{(Y\oplus Y)\Gamma_R}$, so that $\overline{(XY\oplus XY)\Gamma_\varphi}=\overline{(XY\oplus XY)\Gamma_R}\subset\Gamma_R\subsetneqq\Gamma_\varphi$. It is clear that both of these subspaces are invariant for $T\oplus T$. Let $\Pi_1:=T\oplus T|_{\Gamma_\varphi}$ and $\Pi_2:=T\oplus T|_{\overline{(XY\oplus XY)\Gamma_R}}$.

Since $T\oplus T$ is a contraction of class $C_0(2N)$ on $\mathscr{H}\oplus\mathscr{H}$, and $\Gamma_\varphi\subset\mathscr{H}\oplus\mathscr{H}$ is invariant for $T\oplus T$, it follows from Lemma 3.1, Chapter *IX* of [6] that $\Pi_1:=T\oplus T|_{\Gamma_\varphi}$ is a contraction of class $C_0(N')$, $N'\leqslant 2N$.

Moreover since by assumption $XY\oplus XY:\Gamma_\varphi\to\overline{(XY\oplus XY)\Gamma_\varphi}=\overline{(XY\oplus XY)\Gamma_R}$, it follows that $XY\oplus XY$ is a quasi-affinity such that $\Pi_2(XY\oplus XY)=(XY\oplus XY)\Pi_1$. This shows that $\Pi_2\succ\Pi_1$. By Corollary 1, pg 91 of [4] (see Remark 2.0.2 above), the Π_i are quasi-similar. Since Π_1 is the restriction of Π_2 to the non-trivial invariant subspace $\Gamma_R\subset\Gamma_\varphi$, this contradicts Corollary 2, pg 92 of [6] (again, see Remark 2.0.2 above), that no contraction of class $C_0(N)$ can be quasi-similar to its restriction to a proper invariant subspace. This contradiction proves that $\overline{(Y\oplus Y)\Gamma_R}\subsetneqq\overline{(Y\oplus Y)\Gamma_\varphi}$. As remarked earlier, $\overline{(Y\oplus Y)\Gamma_\varphi}\subset\Gamma(\varphi(S_T))$ is invariant for $S_T\oplus S_T$. Hence $\Gamma(R'):=\overline{(Y\oplus Y)\Gamma_R}\subsetneqq\Gamma(\varphi(S_T))$ is the graph of a densely defined closed operator R' which is a non-trivial proper restriction of $\varphi(S_T)$, and which commutes with S_T. Since $\varphi(S_T)=\varphi(S_1)\oplus\ldots\oplus\varphi(S_N)$, it follows that $R'=R'_1\oplus\ldots\oplus R'_N$ where each R'_i is a closed restriction of $\varphi(S_i)$ commuting with S_i. Since R' is proper, one of the R'_i must be a proper closed restriction of $\varphi(S_i)$ commuting with S_i. This contradicts Lemma 4, and proves that $\varphi(T)$ has no proper closed restriction which belongs to $(T)''_{ub}$.

Conversely, if $(T)''_{ub}\ni A\supsetneqq\varphi(T)$ is a proper closed extension of $\varphi(T)$, then $A^*\subsetneqq\varphi(T)^*=\widetilde{\varphi}(T^*)$, where $\widetilde{\varphi}(z):=\overline{\varphi(\overline{z})}\in\mathscr{N}^+_{T^*}$ (see part (v) of Theorem 1.1 in Chapter IV of [6]). Since $T\in C_0(N)$ implies that $T^*\in C_0(N)$, the above arguments show that this is not possible. $\square$

We now have collected all the ingredients needed in the proof of Theorem 4 which we restate below for convenience.

THEOREM 4. *Let T be a contraction of class $C_0(N)$. Then $A\in(T)''_{ub}$ if and only if $A=\varphi(T)$ for some $\varphi\in\mathscr{N}_T=\mathscr{N}^+_{m_T}$.*

Proof. By Proposition 1, there is a $\varphi\in\mathscr{N}^+_{m_T}$ and a dense domain of vectors $\mathfrak{D}\subset\mathrm{Dom}(A)\cap\mathrm{Dom}(\varphi(T))$ such that $Af=\varphi(T)f$ for all $f\in\mathfrak{D}$. Let $\mathrm{Dom}(A'):=\{f\in\mathrm{Dom}(A)\cap\mathrm{Dom}(\varphi(T))\mid Af=\varphi(T)f\}\supset\mathfrak{D}$, and define $A':=A|_{\mathrm{Dom}(A')}$. Then A' is a closed restriction of A: if $(f_n)_{n\in\mathbb{N}}\subset\mathrm{Dom}(A')$ is such that $f_n\to f$ and $A'f_n\to g$, then $\varphi(T)f_n=Af_n\to g$ so that $g=\varphi(T)f=Af$ by the fact that $A,\varphi(T)$ are closed. This proves that $f\in\mathrm{Dom}(A')$ and that A' is closed. Furthermore, $A'\in(T)''_{ub}$. To see this, consider arbitrary $W\in(T)'$ and $f\in\mathrm{Dom}(A')\subset\mathrm{Dom}(A)$. Since $W:\mathrm{Dom}(A)\to\mathrm{Dom}(A)$, $WA\subset AW$ and $W\varphi(T)\subset\varphi(T)W$, it follows that $AWf=WAf=WA'f=W\varphi(T)f=\varphi(T)Wf$. This shows that $AWf=\varphi(T)Wf$ for all $f\in\mathrm{Dom}(A')$, so that $Wf\in\mathrm{Dom}(A')$, $WA'f=A'Wf$, $W:\mathrm{Dom}(A')\to\mathrm{Dom}(A')$ and $WA'\subset A'W$.

Hence $A' \in (T)''_{ub}$ is a closed restriction of $\varphi(T)$, and Proposition 2 implies that $\varphi(T) = A' \subset A$. Since we now have that $A \supset \varphi(T)$, a second application of Proposition 2 shows that $A = \varphi(T)$. $\square$

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