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NONNEGATIVE REFLEXIVE GENERALIZED INVERSES AND APPLICATIONS TO GROUP MONOTONICITY

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Abstract. A classical finite dimensional result of Berman and Plemmons says that a nonnegative matrix with a nonnegative reflexive generalized inverse has a nonnegative rank factorization. In this article, we propose a notion of nonnegative rank factorization that is applicable in the infinite dimensional setting over more general cones and prove an infinite dimensional generalization of Berman and Plemmons's result. As a consequence, a simpler proof of the finite dimensional result (on the existence of nonnegative rank factorizations) is obtained. Characterizations of nonnegativity of the group inverse (when it exists) in infinite dimensions are also presented.

1. Introduction

A non-singular matrix whose inverse has nonnegative entries is called inverse positive or monotone. Such matrices were first studied by Collatz [10] in connection with iterative methods to solve PDEs. A natural question arises as to what happens when A is singular or more generally, rectangular. In this context, nonnegativity of various generalized inverses were investigated and various other notions of monotonicity were proposed [3, 4]. Some of the commonly used notions are rectangular monotonicity (existence of a nonnegative left inverse), semi-monotonicity (nonnegativity of the Moore-Penrose inverse) and group monotonicity (nonnegativity of the group inverse, when it exists). Many of the above mentioned notions of monotonicity have been used profoundly in areas like linear economic models, numerical analysis, eigenvalue problems etc., to name a few. A good source of reference on this subject matter is the monograph by Berman and Plemmons [5]. Not much work has been done in the infinite dimensional setting, in relation to nonnegativity of various generalized inverses. For some recent work on this topic, refer [16, 21, 22, 23, 29, 30, 32]. Very little is known concerning nonnegativity of reflexive generalized inverses in infinite dimensions. A generalization of a finite dimensional result of Mangasarian was obtained by Kulkarni and Sivakumar (Theorem 2.16, [21]). Some interesting results were also obtained by Jayaraman and Sivakumar recently (Theorems 3.16, 3.26 and 3.27, [16]). Recently, Sivakumar applied nonnegative Moore-Penrose inverses to a special class of optimization problems in infinite dimensions [31].

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Perhaps the most prominent use of nonnegative matrices having nonnegative reflexive generalized inverses is in deriving a nonnegative rank factorization. For $A \in \mathbb{R}^{m\times n}$, the set of all $m \times n$ matrices over $\mathbb{R}$, a factorization $A = BC$ such that $B \in \mathbb{R}^{m\times r}, C \in \mathbb{R}^{r\times n}$ and $r = rank(A)$ is called a rank factorization of A. If B and C are entrywise nonnegative, then such a factorization is called a nonnegative rank factorization. Berman and Plemmons posed the problem of characterizing those nonnegative matrices that admit a nonnegative rank factorization [2]. This question was answered by Thomas [35], who characterized the existence of a nonnegative rank factorization using simplicial cones and went on to prove that every nonnegative matrix with rank 1 or 2 has a nonnegative rank factorization. Thomas also gave an example of a nonnegative matrix that does not admit any nonnegative rank factorization. The matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, identified by Thomas, has no nonnegative rank factorization. Later on, a large class of matrices was constructed by Jeter and Pye [19] such that no member of the class possessed a nonnegative rank factorization. In fact, the matrix A above falls into that class. They also obtained necessary and sufficient conditions for the existence of nonnegative rank factorizations in terms of polar cones [20]. It is a well known result due to Berman and Plemmons that if a nonnegative matrix A has a nonnegative reflexive generalized inverse then, A has a nonnegative rank factorization [3]. The proof given by Berman and Plemmons (Theorem 4, [3]) uses a particular representation of nonnegative idempotent matrices due to Flor [12]. A more detailed work on this topic was later on carried out by Tam, involving simpliciality of the cones $A(\mathbb{R}^n_+)$ and $R(A) \cap \mathbb{R}^m_+$ [33, 34]. Campbell and Poole also discussed methods of computing a nonnegative rank factorization [6]. The importance of nonnegative rank factorizations was brought out recently in a paper by Jayaraman and Sivakumar [17] in disproving a conjecture of Peris and Subiza concerning weak monotonicity of a nonnegative matrix A and the existence of a $\{1\}$-inverse that is nonnegative on the range space of A. (Refer Theorem 4 and the paragraph following it, [25].) Nonnegative rank factorization was also used by Jayaraman and Sivakumar in deriving certain structural results concerning weak monotone operators in infinite dimensional spaces [16]. Interestingly, Jain and Tynan [14] used the nonnegativity of $A^{\dagger}A$ and/or $AA^{\dagger}$ to obtain nonnegative rank factorizations of a nonnegative matrix A over the nonnegative orthants of $\mathbb{R}^n$. Interesting contributions were also made by Chen [9] in generalizing certain results of Richman and Schneider [27] on semigroups of nonnegative matrices.

A notion of nonnegative rank factorization in the infinite dimensional setting and also over more general cones was proposed recently by Jayaraman and Sivakumar [16] and was used to study properties of weak monotone operators that possessed such factorizations. In this article, we prove the existence of a nonnegative rank factorization for nonnegative operators $A \in \mathscr{B}(H_1, H_2)$ (nonnegative with respect to two self-dual cones P_1, P_2) and having a nonnegative reflexive generalized inverse. Our result thus generalizes the existing finite dimensional result due to Berman and Plemmons over the usual nonnegative orthant. We also give a simpler proof of the same. Finally, we apply our results on nonnegative rank factorization to characterize nonnegativity of the group inverse in infinite dimensions, when it exists.

The paper is organised as follows. The next section contains the basic definitions and notations used throughout the manuscript. The main results are presented in Section 3. A notion of nonnegative rank factorization that is applicable in the infinite dimensional setting and also over more general cones is introduced in Definition 3.3. The importance of self-dual cones in the definition of nonnegative rank factorization is brought out next. An important result (Theorem 3.5) due to Sivakumar concerning obtuseness of the image cone $A(P)$ is stated next. A generalization of Sivakumar's result seems elusive if we replace the assumption on nonnegativity of $A^{\dagger}A$ by that of XA for some reflexive generalized inverse X of A. However, when X is nonnegative (with respect to two self-dual cones), obtuseness of $A(P)$ can be deduced. We prove this result (Theorem 3.11) after proving a preliminary result (Lemma 3.9). As a consequence of Theorem 3.11, a generalization of Berman and Plemmons's result on the existence of a nonnegative rank factorization is presented (Theorem 3.12). Characterizations of existence and nonnegativity of the group inverse in terms of rank and nonnegative rank factorizations, respectively, are presented in the end (Theorems 3.20 and 3.21), with a view to generalize the finite dimensional results of Berman and Plemmons.

2. Notations, Definitions and Preliminaries

Let H_1 and H_2 be real Hilbert spaces. We shall denote the set of all bounded operators between H_1 and H_2, by $\mathscr{B}(H_1,H_2)$. For $A \in \mathscr{B}(H_1,H_2)$ the range and null spaces will be denoted by $R(A)$ and $N(A)$, respectively.

A subset P of a vector space V is called a cone if $\alpha P \subseteq P, \forall\, \alpha \geqslant 0$ and $P+P=P$. P is said to be pointed if $P \cap -P = \{0\}$ and generating if $V = P-P$.

DEFINITION 2.1. Let V_1 and V_2 be vector spaces with cones P_1 and P_2, respectively. A linear map $A : V_1 \to V_2$ is said to be nonnegative, if $A(P_1) \subseteq P_2$.

We shall henceforth use the phrase A *is* (P_1,P_2)*-nonnegative*, to denote nonnegativity of A with respect to two cones P_1 and P_2. For a cone P in a Hilbert space H, the dual cone of P, denoted by P^*, is defined by

$$P^* := \{x \in H \,:\, \langle x,y\rangle \geqslant 0 \;\forall\, y \in P\}.$$

A cone P is said to be acute if $\langle x,y\rangle \geqslant 0$ for all $x,y \in P$ and obtuse if $P^* \cap \{\overline{span(P)}\}$ is acute [8]. In particular, for the image cone $A(P)$, obtuseness is defined to be acuteness of $(A(P))^* \cap R(A)$ [8]. According to Novikoff [24], a cone P in a Hilbert space is said to be acute if $P \subseteq P^*$ and obtuse if $P^* \subseteq P$. Therefore for the image cone $A(P)$, obtuseness should be replaced by the condition $(A(P))^* \cap R(A) \subseteq A(P)$. Notice that when A is invertible, the above inclusion becomes $(A(P))^* \subseteq A(P)$ and coincides with Novikoff's definition. This is not clear when the operator is singular. The inclusion $(A(P))^* \cap R(A) \subseteq A(P)$ always implies acuteness of $(A(P))^* \cap R(A)$ whereas, the reverse implication requires nonnegativity of $A^{\dagger}A$. This was pointed out by Sivakumar (Lemma 3.11, [32]). A cone is self-dual if it is both acute and obtuse. The nonnegative orthant $\mathbb{R}^n_+$ of the Euclidean space $\mathbb{R}^n$ and the standard cone in ℓ^2, denoted by ℓ^2_+

and defined by $\ell^2_+ := \{x = (x_i) \in \ell^2 : x_i \geqslant 0 \ \forall\, i\}$ are examples of pointed, self-dual (closed) generating cones. $\mathbb{R}^n_+$ has non-empty interior, while ℓ^2_+ has empty interior.

DEFINITION 2.2. Given an $m \times n$ real /complex matrix A, its Moore-Penrose inverse, denoted by $A^\dagger$, is the unique $n \times m$ matrix such that $AXA = A, XAX = X, (AX)^* = AX, (XA)^* = XA$.

For $A \in \mathscr{B}(H_1, H_2)$ consider the following four equations for $X \in \mathscr{B}(H_2, H_1)$:

$$AXA = A \tag{2.1}$$

$$XAX = X \tag{2.2}$$

$$(AX)^* = AX \tag{2.3}$$

$$(XA)^* = XA \tag{2.4}$$

It is well known that a solution exists and is unique. Such an X is denoted by $A^\dagger$ and is called the Moore-Penrose inverse of A. Also, $A^\dagger$ is bounded iff $R(A)$ is closed [7].

For a non-empty subset λ of $\{1,2,3,4\}$, $X \in \mathscr{B}(H_2, H_1)$ is called a λ-inverse of A, if X satisfies those equations corresponding to each element of λ. For example, $X \in \mathscr{B}(H_2, H_1)$ is called a $\{1,2\}$-inverse of $A \in \mathscr{B}(H_1, H_2)$, if X satisfies the equations $AXA = A$ and $XAX = X$. Again, such a transformation exists as a bounded operator if and only if $R(A)$ is closed [7]. For a non-empty subset λ of $\{1,2,3,4\}$, we say that A is λ-monotone if A has a nonnegative λ-inverse.

ASSUMPTIONS. We shall confine ourselves to bounded operators with closed range throughout this article.

3. Main results

The notion of rank factorization can be posed over the infinite dimensional setting as well. Although there is no notion of rank in the infinite dimensional case, we retain the term *rank* to compare with the existing finite dimensional notion. The definition is given below.

DEFINITION 3.1. $A \in \mathscr{B}(H_1, H_2)$ is said to have a rank factorization if there exists a Hilbert space H, operators $B \in \mathscr{B}(H, H_2)$ and $C \in \mathscr{B}(H_1, H)$ such that $A = BC$, with $N(B) = \{0\}$, $R(C) = H$.

Note that if $A = BC$ is a rank factorization, then $R(B)$ and $R(C)$ are closed by definition. It is well known that any finite matrix over $\mathbb{R}$ or $\mathbb{C}$ always has a rank factorization [1]. We prove below that any bounded linear map with closed range has a rank factorization.

LEMMA 3.2. (Lemma 3.3, [16]) *Every $A \in \mathscr{B}(H_1, H_2)$ with closed range has a rank factorization $A = BC$.*

Proof. Since we are assuming that $R(A)$ is closed, $R(A)$ is itself a Hilbert space. Let us denote this space by H. Let $C: H_1 \to H$ be defined by $Cx = Ax$ and let $B: H \hookrightarrow H_2$ be the inclusion operator. Then B is injective, C is surjective and $A = BC$. $\square$

It should be remarked that any rank factorization is *isomorphic* to the one constructed above [7]. (Also refer Theorem 2, [26].) We now define the notion of a nonnegative rank factorization over general cones in the infinite dimensional setting.

DEFINITION 3.3. $A \in \mathscr{B}(H_1, H_2)$ is said to have a nonnegative rank factorization $A = BC$, if in addition to being a rank factorization, P_1, P_2 and P are self-dual cones in H_1, H_2 and H, respectively, such that $B(P) \subseteq P_2$ and $C(P_1) \subseteq P$.

REMARK 3.4. There are two primary reasons for imposing self-duality of the cones in Definition 3.3. First, the existing notion in finite dimensions should follow as a particular case, which it does. Second, if A has a nonnegative rank factorization $A = BC$, then we would like A^* to have the nonnegative rank factorization $A^* = C^*B^*$. It is a known fact that if P_1 and P_2 are self-dual cones in Hilbert spaces H_1 and H_2, respectively, then $A \in \mathscr{B}(H_1, H_2)$ is (P_1, P_2)-nonnegative if and only if $A^* \in \mathscr{B}(H_2, H_1)$ is (P_2, P_1)-nonnegative.

As stated earlier, a well known result due to Berman and Plemmons says that if a nonnegative matrix A is $\{1,2\}$-monotone, then it has a nonnegative rank factorization. The following theorem characterizes obtuseness of the image cone using nonnegativity of the Moore-Penrose inverse.

THEOREM 3.5. (Theorem 3.17, [32]) *Let $A \in \mathscr{B}(H_1, H_2)$ have closed range. Further, assume that $A^\dagger A(P) \subseteq P$, where P is a closed cone in H_1. Then, the following are equivalent :*

(1) $(A^\dagger)^*(P^*)$ *is acute.*

(2) $(A^*A)^\dagger(P^*) \subseteq P + N(A)$.

(3) $A(P)$ *is obtuse.*

(4) *For every $x \in H_2$, there exists $y \in H_2$ such that $y \pm x \in A(P)$ and $||y|| \leqslant ||x||$.*

It is not hard to prove that if $A \in \mathscr{B}(H_1, H_2)$ is nonnegative with respect to self-dual cones P_1 and P_2, then $A(P_1)$ is an acute cone in $R(A)$. Therefore, it follows from Theorem 3.5 that, in the presence of self-dual cones P_1 and P_2, nonnegativity of $A^\dagger$ guarantees self-duality of the image cone $A(P_1)$ in $R(A)$. As far as nonnegative rank factorizations are concerned, this is not a desirable result. Even over the nonnegative orthants nonnegativity of a reflexive generalized inverse is enough, as was observed by Berman and Plemmons. It is not clear if the above theorem can be generalized by replacing nonnegativity of $A^\dagger A$ by that of XA for some reflexive generalized inverse X of A. Nevertheless, obtuseness of the cone $A(P)$ can be deduced from nonnegativity of X, as we prove below. The following results will be used.

LEMMA 3.6. *Let $A \in \mathscr{B}(H_1,H_2)$ and $b \in H_2$ be such that the equation $Ax = b$ is consistent. Then, the general solution is given by $x = Tb + z$ for some $\{1\}$-inverse $T \in \mathscr{B}(H_2,H_1)$ and $z \in N(A)$.*

THEOREM 3.7. (Proposition 8, [11]) *Let $A : H_1 \to H_2$ be a linear operator. Then the following statements are equivalent:*

(1) *T is a $\{1,2\}$-inverse of A.*
(2) *AT is idempotent and $R(T) \oplus N(A) = H_1$.*
(3) *TA is idempotent and $R(A) \oplus N(T) = H_2$.*
(4) *AT is idempotent, $R(AT) = R(A)$ and $N(AT) = N(T)$.*
(5) *TA is idempotent, $R(TA) = R(T)$, and $N(TA) = N(A)$.*

LEMMA 3.8. (Lemma 3.1, [32]) $u \in (A(P))^* \Rightarrow A^*u \in P^*$.

LEMMA 3.9. *Let P_1 and P_2 be self-dual cones in Hilbert spaces, respectively. Let $X \in \mathscr{B}(H_2,H_1)$ be a reflexive generalized inverse of $A \in \mathscr{B}(H_1,H_2)$ such that $X(P_2) \subseteq P_1$. Then $P_1 \cap R(A^*) \subseteq A^*A(P_1) + N(A)$.*

Proof. Let $y = A^*x \in P_1, v := X^*y, u := Xv$. Since $X(P_2) \subseteq P_1$ and the cones are self-dual, we see that $X^*(P_1) \subseteq P_2$. Therefore, $v \in P_2, u \in P_1$. From $Xv = u$, we see that $v = Au + p, p \in N(X)$; that is, $p = v - Au \in R(A) \cap N(X)$ and so, $p = 0$ (by Theorem 3.7). Thus $X^*y = Au$. From this, we get $y = A^*Au + q, q \in N(A)$ (again, by Theorem 3.7). Therefore, $P_1 \cap R(A^*) \subseteq A^*A(P_1) + N(A)$. $\square$

The following result on obtuseness of the image cone $A(P)$ was obtained by Sivakumar.

LEMMA 3.10. (Lemma 3.11, [32]) *Let $A^\dagger A(P) \subseteq P$. Then $(A(P))^* \cap R(A)$ is acute if and only if $(A(P))^* \cap R(A) \subseteq A(P)$.*

In the proof of Theorem 3.10, deriving acuteness of $(A(P))^* \cap R(A)$ from the inclusion $(A(P))^* \cap R(A) \subseteq A(P)$ does not require nonnegativity of $A^\dagger A$. The crux of the following theorem is that $(A(P))^* \cap R(A) \subseteq A(P)$ can actually be deduced from nonnegativity of X.

THEOREM 3.11. *Let $A \in \mathscr{B}(H_1,H_2)$ have closed range. Let P_1, P_2 be self-dual cones in H_1, H_2, respectively. Further, assume that $X(P_2) \subseteq P_1$, where X is a reflexive generalized inverse of A. Then $A(P_1)$ is obtuse.*

Proof. Let $y \in A(P_1)^* \cap R(A)$. Then, $y = Ax$ and $A^*y \in P_1$. Thus $A^*y \in P_1 \cap R(A^*)$. By Lemma 3.9, $A^*y = A^*Au + z, u \in P_1, z \in N(A)$. This implies, $y = (A^\dagger)^*A^*Au + w = Au + w, w \in N(A^*)$ (by Lemma 3.6). Since $y \in R(A)$, we have that $w = 0$. Therefore, $y \in A(P_1)$ and hence, $(A(P_1))^* \cap R(A) \subseteq A(P_1)$. It now follows from the sufficiency part of Theorem 3.10 that the cone $A(P_1)$ is obtuse. $\square$

As a consequence, we obtain the following generalization of Berman and Plemmons's result.

THEOREM 3.12. *Let $A \in \mathscr{B}(H_1,H_2)$ have closed range where, H_1 and H_2 are equipped wih self-dual cones P_1 and P_2, respectively. Assume that $A(P_1) \subseteq P_2$ and that A has a reflexive generalized inverse $X \in \mathscr{B}(H_2,H_1)$ such that $X(P_2) \subseteq P_1$. Then, A has a nonnegative rank factorization.*

Proof. Nonnegativity of A and self-duality of the cones imply that $A(P_1)$ is acute. Theorem 3.11 guarantees obtuseness of $A(P_1)$. The factorization constructed in Lemma 3.2 then yields a nonnegative rank factorization of A. $\square$

Theorem 3.12 not only generalizes Berman and Plemmons's result to infinite dimensions (and also over more general cones), it also gives a simple proof of the existing finite dimensional version. The finite dimensional version over the nonnegative orthants assumes that the Hilbert space H is $\mathbb{R}^r$ and the underlying cone is $\mathbb{R}^r_+$, whereas the above proof allows for more general cones in the Hilbert space $R(A)$. The following example illustrates our results.

EXAMPLE 3.13. Let A be the bounded operator on ℓ^2 given by $Ax = (2(x_1 + x_2), 2(x_1 + x_2), x_3, x_4, x_5, 0, \dots)$. A has as a reflexive generalized inverse the operator $T \in \mathscr{B}(\ell^2)$ given by $Tx = (x_1/2, 0, x_3, x_4, x_5, 0, \dots)$. Notice that both A and T are (ℓ^2_+, ℓ^2_+)-nonnegative. Therefore $A(\ell^2_+)$ is self-dual (by Theorem 3.12) and consequently, the canonical factorization as constructed in Lemma 3.2 gives a nonnegative rank factorization of A. Observe that $A = B^*B$ where, $B : \ell^2 \longrightarrow \mathbb{R}^6$ is defined by $Bx = (x_1 + x_2, x_3, x_1 + x_2, x_5, x_4, 0)^t$. B also has a nonnegative rank factorization $B = UV$ over the usual cones in $\mathbb{R}^n$ and ℓ^2 where, $U : \mathbb{R}^4 \longrightarrow \mathbb{R}^6$ is given by $Ux = (x_1, x_2, x_1, x_4, x_3, 0)^t$ and $V : \ell^2 \longrightarrow \mathbb{R}^4$ is given by $Vx = (x_1 + x_2, x_3, x_4, x_5)^t$. Then, B^* has a nonnegative rank factorization given by $B^* = V^*U^*$. Observe that the matrix U^*U is a 4×4 non-singular matrix with nonnegative entries. Therefore, A has a nonnegative rank factorization given by $A = V^*(U^*U)V$.

Applications to group monotonicity

We now apply Theorem 3.12 to characterize group monotonicity in infinite dimensions. Recall that the group inverse of A is the unique operator X such that $AXA = A, XAX = X$ and $AX = XA$. Such an X, if it exists, is unique and is denoted by $A^\#$. $A^\# \in \mathscr{B}(H)$ if and only if $R(A)$ is closed. It is a well known result that in finite dimensions, $A^\#$ exists if and only if $N(A) = N(A^2)$; equivalently, $R(A) = R(A^2)$ [1]. However, in the infinite dimensional setting, $N(A) = N(A^2)$ does not imply $R(A) = R(A^2)$ (for example, the right shift operator on ℓ^2). It was proved by Robert that in infinite dimensional Hilbert spaces, $A^\#$ exists if and only if $N(A) = N(A^2)$ and $R(A) = R(A^2)$ (Theorem 4, [28]). It is also known that if a square matrix A has a rank factorization $A = BC$, then $A^\#$ exists if and only if CB is invertible, in which case $A^\# = B(CB)^{-2}C$. This is also true in infinite dimensions and we prove this below.

LEMMA 3.14. *Let $A \in \mathscr{B}(H)$ have a rank factorization $A = BC$. Then the group inverse $A^\#$ of A exixts if and only if CB is invertible.*

Proof. Suppose CB is invertible. It then follows easily that $X = B(CB)^{-2}C$ is a $\{1,2\}$-inverse of A that commutes with A. Thus $A^{\#}$ exists. Conversely, suppose $A^{\#}$ exists. Then $N(A)$ and $R(A)$ are complementary subspaces of H (Theorem 4, [28]). Suppose $CBx = 0$ Then, $Bx \in N(C) = N(A)$. Also, $Bx \in R(B) = R(A)$. It then follows that $x = 0$. Thus CB is injective. Note that since A has a rank factorization, there exists a Hilbert space H_1 such that $B \in \mathscr{B}(H_1, H)$, $C \in \mathscr{B}(H, H_1)$ with $A = BC$. Now let $y \in H_1$. Since C is surjective, there exists $x \in H$ such that $Cx = y$. Let $x = x_1 + x_2$ where, $x_1 \in R(A) = R(B), x_2 \in N(A) = N(C)$. Then, $Cx = Cx_1$. Since $x_1 \in R(B)$, there exists $z \in H_1$ such that $Cx_1 = CBz$ which implies $y = CBz$, proving the surjectivity of CB. Therefore CB is invertible. □

We now give the definition of an operator being group monotone.

DEFINITION 3.15. $A \in \mathscr{B}(H)$ is said to be group monotone if the group inverse $A^{\#}$ of A exists and is (P,P)-nonnegative, where P is a cone in H.

The following two results due to Berman and Plemmons characterize group monotonicity over the nonnegative orthants in terms of nonnegative rank factorizations.

THEOREM 3.16. (Theorem 2, [4]) *Let A be a nonnegative matrix. Then A is group monotone if and only if A has a nonnegative rank factorization $A = BC$, where CB is monomial. In this case every such factorization has this property.*

THEOREM 3.17. (Theorem 3, [4]) *Let A be a nonnegative matrix. Then $A = A^{\#}$ if and only if A has a nonnegative rank factorization $A = BC$ such that $CB = (CB)^{-1}$. In this case every such rank factorization has this property.*

Before proving generalizations of the above, we state the definition of a monomial operator.

DEFINITION 3.18. Let H_1, H_2 be Hilbert spaces with cones P_1, P_2, respectively. $A \in \mathscr{B}(H_1, H_2)$ is said to be monomial if A is invertible and in addition, A and A^{-1} are nonnegative with respect to (P_1, P_2) and (P_2, P_1), respectively.

We now present below a generalization of the Theorem 3.16 to infinite dimensions. Let us recall that an operator $A \in \mathscr{B}(H_1, H_2)$ is said to be (P_1, P_2) monotone if $Ax \in P_2 \Rightarrow x \in P_1$ and (P_1, P_2) weak monotone if $Ax \in P_2 \Rightarrow x \in P_1 + N(A)$. It can be easily proved that if A is $\{1,2\}$-monotone, then it is weak monotone (Lemma 3.6, [16]). The proof of the following theorem uses the following result due to Kulkarni and Sivakumar. The following terminology will be used in the theorems that follow. We say that a Hilbert space H equipped with a cone P has positive property with respect to P ($\mathscr{P}$ for short), if it has an orthonormal basis $\{u^{\alpha}\}$ such that $u^{\alpha} \in P, \forall \alpha \in J$, J being an index set. $\mathbb{R}^n$ with the usual cone $\mathbb{R}^n_+$ has property $\mathscr{P}$. The standard basis $\{e^1, ..., e^n\}$ of $\mathbb{R}^n$ satisfies $e^i \in \mathbb{R}^n_+ \ \forall i$. ℓ^2 with its usual cone $\ell^2_+ := \{x = (x_i) \in \ell^2 : x_i \geqslant 0 \ \forall i\}$ has property $\mathscr{P}$.

THEOREM 3.19. (Theorem 2.16, [21]) *Let H_1 and H_2 be Hilbert spaces with cones P_1 and P_2, respectively with P_1 generating and P_2 self-dual. Assume that H_1 has property $\mathscr{P}$. Let $A \in \mathscr{B}(H_1, H_2)$ with $N(A^*) + P_2$ closed. Then A is monotone if and only if there exists a nonnegative operator $X \in \mathscr{B}(H_2, H_1)$ such that $XA = I$.*

THEOREM 3.20. *If $A \in \mathscr{B}(H)$ is (P,P)-nonnegative (P is a self-dual generating cone in H) and has a nonnegative rank factorization $A = BC$ with CB monomial, then A is group monotone. Conversely, assume that H has property $\mathscr{P}$ with $N(A^*) + P$ closed and A being group monotone. Then, CB is monomial.*

Proof. We know that if $A^\#$ exists, it is given by $A^\# = B(CB)^{-2}C$. Since CB is monomial, $(CB)^{-1}$ is nonnegative. Thus, $A^\#$ is nonnegative.

Conversely, suppose A is group monotone. Then, A has a nonnegative $\{1,2\}$-inverse, namely, $A^\#$ itself. Therefore, A is weak monotone and also has a nonnegative rank factorization $A = BC$ (by Theorem 3.12). From this, it follows that B is also weak monotone and hence, monotone since it is injective. It now follows from Theorem 3.19 that B has a $(P, A(P))$-nonnegative left inverse, say X. Since $A^\#$ exists, CB is invertible. Also, CB is $(A(P), A(P))$-nonnegative. We also know that $A^\#$ is given by the formula $A^\# = B(CB)^{-2}C$. Now, nonnegativity of $A^\#$ implies that of $B(CB)^{-2}C$. Premultiplying by X and postmultiplying by B, we see that CB is monomial. $\square$

Similarly, the following result can be obtained.

THEOREM 3.21. *If $A \in \mathscr{B}(H)$ is (P,P)-nonnegative (P is a self-dual generating cone in H) and has a nonnegative rank factorization $A = BC$ with $CB = (CB)^{-1}$, then A is group monotone. Conversely, assume that H has property $\mathscr{P}$, that $N(A^*) + P$ is closed and that $A = A^\#$. Then, $CB = (CB)^{-1}$.*

Jain and Tynan [15] also studied nonnegative matrices A for which the group inverse existed and such that $AA^\#$ is nonnegative. Using this, they also obtained a characterization of nonnegative group inverses. It should be mentioned that the notion of nonnegativity of the group inverse of a matrix A over general polyhedral cones was studied by Wall and Haynsworth [37, 38], with a view to generalize the results of Berman and Plemmons. Their results were concerned with the existence of the group inverse $A^\#$ of a nonnegative matrix A such that $A^\# = A^k$ for some positive integer k. A generalization of Wall and Haynsworth's results does not seem amenable to be extended to the infinite dimensional setting, as the notion of polyhedral cones does not seem plausible.

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