ℂ–ORBIT REFLEXIVE OPERATORS

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Abstract. We introduce the notion of ℂ-orbit reflexivity and study its properties. An operator on a finite-dimensional space is ℂ-orbit reflexive if and only if the two largest blocks in its Jordan form corresponding to nonzero eigenvalues with the largest modulus differ in size by at most one. Most of the proofs of our results in infinite dimensions are obtained from purely algebraic results we obtain from linear-algebraic analogs of ℂ-orbit reflexivity.

1. Introduction

The term *reflexive* for an algebra of operators was coined by P. R. Halmos [17], but the first theorem about reflexivity was proved earlier by D. Sarason [29]. Since that time there have been many papers written on the topic, and various notions of reflexivity (e.g., algebraic reflexivity [12], approximate reflexivity [11], orbit reflexivity [16]) for linear subspaces, convex sets and other sets of operators, have been studied extensively, e.g., [1], [3], [4], [6], [7], [9], [22], [23], [28], [30]. In [13] a very general notion of reflexivity that contained many of these notions as special cases. Orbit reflexivity was introduced and studied in [16], but it wasn't until over twenty years later that an operator was constructed on a Hilbert space that is not orbit reflexive [10] (see also [27] and [8]). John von Neumann's classic double commutant theorem [33] can be viewed as the statement that every von Neumann algebra is reflexive. In fact, many view reflexive algebras as nonselfadjoint analogues of von Neumann algebras. Nest algebras, i.e., reflexive algebras whose lattice of invariant subspaces is a chain, have received a great deal of attention (see [5]). W. Arveson [2] relates reflexivity to spectral synthesis in commutative harmonic analysis and remarks that it is appropriate to consider reflexivity questions as "noncommutative harmonic analysis". Reflexivity appears in other guises, often in the form of a "local" property, e.g., local derivations, local automorphisms, local multiplications [14], [15], [18], [19], [21], [23], [31].

In this paper we introduce a new notion of reflexivity for operators, ℂ-*orbit reflexivity*, and we also define a linear-algebraic analogue. This notion is related to the notion of orbit reflexivity [16]. We first prove a number of results in the purely algebraic case, and we use these to prove several results for operators on a normed space or a Hilbert space. We also give an easy proof that every subnormal operator is orbit reflexive. In finite-dimensions a characterization of reflexivity for a single matrix was given [7] in

terms of the Jordan form, i.e., for each eigenvalue the largest two Jordan blocks differ in size by at most 1. Every matrix is orbit reflexive [16]. However, $\mathbb{C}$-orbit reflexivity has a characterization in terms of the Jordan form that is similar to, but quite different from, the one for reflexivity, i.e., among the nonzero eigenvalues with maximum modulus the largest two Jordan blocks differ in size by at most 1.

Suppose X is a normed space and $\mathscr{A}$ is an algebra of (bounded linear) operators on X. A (closed linear) subspace M of X is $\mathscr{A}$-*invariant* if $A(M) \subseteq M$ for every $A \in \mathscr{A}$. We let $\operatorname{Lat}\mathscr{A}$ denote the set of all invariant subspaces for $\mathscr{A}$, and we let $\operatorname{AlgLat}\mathscr{A}$ denote the algebra of all operators that leave invariant every $\mathscr{A}$-invariant subspace. The algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A} = \operatorname{AlgLat}\mathscr{A}$. If the algebra $\mathscr{A}$ contains the identity operator 1, then $S \in \operatorname{AlgLat}\mathscr{A}$ if and only if, for every $x \in X$, Sx is in the closure of $\mathscr{A}x$. This characterization works equally well for linear subspace $\mathscr{S}$ of $B(X)$ (the set of all operators on X), i.e., we define $\operatorname{ref}\mathscr{S}$ to be the set of all operators A such that, for every $x \in X$, we have Ax is in the closure of $\mathscr{S}x$, and we say that $\mathscr{S}$ is reflexive if $\mathscr{S} = \operatorname{ref}\mathscr{S}$. If we let T be a single operator and let $\mathscr{S} = \operatorname{Orb}(T) = \{T^n : n \geqslant 0\}$, we apply the same process to obtain the notion of orbit reflexivity. (Note that in this case $\mathscr{S}$ is not a linear space.) We define $\operatorname{OrbRef}(T)$ to be the set of all operators A such that, for every vector x, we have Ax is in the closure of $\operatorname{Orb}(T,x) = \operatorname{Orb}(T)x$. We say that T is *orbit reflexive* if $\operatorname{OrbRef}(T)$ is the closure of $\operatorname{Orb}(T)$ in the strong operator topology (SOT). For the next notion we allow powers and scalar multiples. We define, for the field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$

$$\mathbb{F}\text{-}\operatorname{Orb}(T) = \{\lambda T^n : n \in \mathbb{N}, \lambda \in \mathbb{F}\},$$

and

$$\mathbb{F}\text{-}\operatorname{Orb}(T,x) = \{\lambda T^n x : n \geqslant 0, \lambda \in \mathbb{F}\},$$

$$\mathbb{F}\text{-}\operatorname{OrbRef}(T) = \{S \in B(H) : Sx \in \mathbb{F}\text{-}\operatorname{Orb}(T,x)^- \text{ for every } x \in H\}.$$

And we say that T is $\mathbb{F}$-*orbit reflexive* if $\mathbb{F}\text{-}\operatorname{OrbRef}(T)$ is the strong operator closure of $\mathbb{F}\text{-}\operatorname{Orb}(T)$.

2. Algebraic Results

Throughout this section $\mathbb{F}$ will denote an arbitrary field, X will denote a vector space over $\mathbb{F}$, and $\mathscr{L}(X)$ will denote the algebra of all linear transformations on X. If $T \in \mathscr{L}(X)$, we define $\mathbb{F}\text{-}\operatorname{OrbRef}_0(T)$ to be the set of all $S \in \mathscr{L}(X)$ such that, for every $x \in X$, $Sx \in \operatorname{Orb}(T,x)$. We say that T is algebraically $\mathbb{F}$-*orbit reflexive* if $\mathbb{F}\text{-}\operatorname{OrbRef}_0(T) = \mathbb{F}\text{-}\operatorname{Orb}(T)$.

A transformation $T \in \mathscr{L}(X)$ is *locally nilpotent* if $X = \cup_{n \geqslant 1} \ker(T^n)$. More generally T is *locally algebraic* if, for each $x \in X$, there is a nonzero polynomial $p_x \in \mathbb{F}[t]$ such that $p_x(T)x = 0$. If $p_x(t)$ is chosen to be monic with minimal degree, we call p_x a *local polynomial* for T at x.

THEOREM 1. *Every locally nilpotent linear transformation on a vector space X over field $\mathbb{F}$ is algebraically $\mathbb{F}$-orbit reflexive. Moreover, if $S \in \mathbb{F}\text{-}\operatorname{OrbRef}_0(T)$, $f \in X$, $\beta \in \mathbb{F}$, and $Sf = \beta T^k f \neq 0$, then $S = \beta T^k$.*

Proof. Suppose first that X is finite-dimensional and that $J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_k}$ is the Jordan form for T with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$. Suppose $S \in \mathbb{F}\text{-OrbRef}_0(T)$. Suppose e is in the domain of J_{n_1} and $J_{n_1}^{n_1-1}e \neq 0$. We first assume that $Se \neq 0$. Then, since $\{e, Te, \ldots, T^{n_1-1}e\}$ is linearly independent, there is a unique m, $0 \leqslant m < n_1$ and a unique $\lambda \in \mathbb{F}$ such that $Se = \lambda T^m e$. Suppose g is in the domain of $J_{n_2} \oplus \cdots \oplus J_{n_k}$. Then there is an $\alpha \in \mathbb{F}$ and a j, $1 \leqslant j \leqslant n_1$ such that $Se + Sg = S(e+g) = \alpha T^j (e+g) = \alpha T^j e + \alpha T^j g$, and by projecting onto the domain of J_{n_1}, we have $Se = \alpha T^j e$ which implies $j = m$ and $\alpha = \lambda$. Thus $Sg = \lambda T^m g$ for every g in the domain of $J_{n_2} \oplus \cdots \oplus J_{n_k}$. A similar argument, considering the coefficient of $T^m e$ of $S(e+g)$, shows that $Sg = \lambda T^m g$ if g is any member of the linearly independent set $\{Te, T^2 e, \ldots, T^{n_1-1}e\}$. Hence $S = \lambda T^m$. Repeating the same argument when $Se = 0$, gives $S = 0$. Hence $S \in \mathbb{F}\text{-Orb}(T)$.

We now move to the general case. Suppose T is locally nilpotent and $S \in \mathbb{F}\text{-}$OrbRef$_0(T)$. If $S = 0$, we are done. Suppose $f \in X$ and $Sf \neq 0$. Choose $n \geqslant 1$ so that $T^n f = 0$ and $T^{n-1}f \neq 0$. It follows that there is a $\beta \neq 0$ in $\mathbb{F}$ and a k, $0 \leqslant k < n$ such that $Sf = \beta T^k f \neq 0$. Suppose $h \in X$. Since T is locally algebraic, it follows that $sp(\{T^m f : m \geqslant 0\} \cup \{T^m h : m \geqslant 0\}) = M$ is a finite-dimensional invariant subspace for T, and, hence, for S. It follows from our finite-dimensional case that there is an $m \geqslant 0$ and a $\gamma \in \mathbb{F}$ such that $Sx = \gamma T^m x$ for every $x \in M$. In particular, $0 \neq Sf = \gamma T^m f$, so $\gamma \neq 0$ and $m < n$. We know that $\{f, Tf, \ldots, T^{n-1}f\}$ is linearly independent, so we have $m = k$ and $\gamma = \beta$. Thus $Sh = \beta T^k h$. Since $h \in X$ was arbitrary, we have $S = \beta T^k \in \mathbb{F}\text{-Orb}(T)$. $\square$

For infinite fields the next theorem reduces the problem of algebraic $\mathbb{F}$-orbit reflexivity to the case of locally algebraic transformations. A key ingredient in the proof is an algebraic reflexivity result from [12] that says if $\mathbb{F}$ is infinite and $T \in \mathscr{L}(X)$ is not locally algebraic, then, whenever $S \in \mathscr{L}(X)$ and for every $x \in X$ there is a polynomial p_x such that $Sx = p_x(T)x$, we must have $S = p(T)$ for some polynomial p.

THEOREM 2. *Suppose X is a vector space over an infinite field $\mathbb{F}$, and suppose $T \in \mathscr{L}(X)$ is not locally algebraic. Then T is algebraically $\mathbb{F}$-orbit reflexive.*

Proof. Suppose $S \in \mathbb{F}\text{-OrbRef}_0(T)$. Then $Sx \in \mathbb{F}\text{-Orb}(T)$ for every $x \in X$. It follows from [12] that T is algebraically reflexive, so we know there is a polynomial $p \in \mathbb{F}[t]$ such that $S = p(T)$. Since T is not locally algebraic, there is a vector $e \in X$ such that for every nonzero polynomial $q \in \mathbb{F}[t]$, we have $q(T)e \neq 0$. Since $S \in \mathbb{F}\text{-}$OrbRef$_0(T)$, we know that there is an $n \in \mathbb{N}$ and a $\lambda \in \mathbb{F}$ such that $Se = \lambda T^n e$. Hence $p(t) = \lambda t^n$, and thus $S \in \mathbb{F}\text{-Orb}(T)$. $\square$

The following lemma dashes all hope, at least for some fields, that in finite dimensions every transformation is algebraically $\mathbb{F}$-orbit reflexive.

LEMMA 1. *Suppose $\mathbb{F}$ is a field and T is the linear transformation on $\mathbb{F}^2$ defined by the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The following are equivalent:*

1. T is algebraically $\mathbb{F}$-orbit reflexive

2. $\mathbb{F}$ is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

3. Whenever X is a vector space over $\mathbb{F}$ and A, S are linear transformations on X, $v \in X$ such that there is an $\beta \in \mathbb{F}$ and there are integers $k \geqslant 0, m \geqslant 2$ such that

    (a) $(A-1)v \neq 0$,
    (b) $(A-1)^m v = 0$,
    (c) $S \in \mathbb{F}\text{-OrbRef}_0(A)$
    (d) $Sv = \beta A^k v$,

    then we must have $S(A-1)v = \beta A^k (A-1)v$.

Proof. $(1) \implies (2)$. Suppose (2) is not true, and let $S = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Suppose $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}^2$. If $y = 0$, then $S\begin{pmatrix} x \\ y \end{pmatrix} = 0 = 0T\begin{pmatrix} x \\ y \end{pmatrix}$. Assume $y \neq 0$, then there is a positive integer m such that

$$ym = 1 \bmod p$$

and let n be a positive integer such that

$$n = (y-x)m \bmod p.$$

Then we have

$$T^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ny \\ y \end{pmatrix} = \begin{pmatrix} x+(y-x)my \\ y \end{pmatrix} = S\begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence $S \in \mathbb{F}\text{-OrbRef}_0(T)$, but $ST \neq TS$, so (1) is not true.

$(2) \implies (1)$. Suppose (2) is true. Then we can choose $w \in \mathbb{F}$ so that $w \notin \mathbb{Z}1$. Suppose $S \in \mathbb{F}\text{-OrbRef}_0(T)$. Then $S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$ for some $a \in \mathbb{F}$. Hence $S = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $b, c \in \mathbb{F}$. Suppose $c = 0$. Then

$$\begin{pmatrix} b \\ 0 \end{pmatrix} = S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha T^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha n \\ \alpha \end{pmatrix},$$

so $\alpha = b = 0$. Now,

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = S\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha T^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha(1+n) \\ \alpha \end{pmatrix},$$

which implies $\alpha = 0 = a$. Thus $c = 0$ implies $S = 0 \in \mathbb{F}\text{-Orb}(T)$. Hence we can assume $c \neq 0$.

We now want to show $c = a$. Assume, via contradiction, that $c - a \neq 0$. Then, for some $\alpha \in \mathbb{F}$, and some integer $n \geqslant 0$,

$$\begin{pmatrix} a\frac{cw-b}{a-c} + b \\ c \end{pmatrix} = S\begin{pmatrix} \frac{cw-b}{a-c} \\ 1 \end{pmatrix} = \alpha T^n \begin{pmatrix} \frac{cw-b}{a-c} \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} \frac{cw-b}{a-c} + n \\ 1 \end{pmatrix},$$

which implies that $\alpha = c \neq 0$ and

$$n = \frac{a}{c}\frac{cw-b}{a-c} + \frac{b}{c} - \frac{cw-b}{a-c} = w,$$

which contradicts the choice of w. Thus $a = c$, so $S = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $a \neq 0$.

Finally we see that there is an $\alpha \in \mathbb{F}$ and an integer $n \geqslant 0$ such that

$$\begin{pmatrix} b \\ a \end{pmatrix} = S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha T^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha n \\ \alpha \end{pmatrix},$$

which implies $b = na$; whence, $S = aT^n$. Hence (1) is true.

(3) $\Longrightarrow$ (1). Apply (3) to the vector $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $A = T$.

(2) $\Longrightarrow$ (3). Suppose (2) is true. We can assume that m is the smallest positive integer for which $(A-1)^m v = 0$. It follows that $\{(A-1)^s v : 0 \leqslant s < m\}$ is a linearly independent set whose linear span Y is an invariant subspace for A. Similarly, the linear span M of $\{(A-1)^s v : s \geqslant 2\}$ is also an invariant subspace for A. Since $S \in \mathbb{F}$-OrbRef$_0(A)$, we also have $S(Y) \subseteq Y$ and $S(M) \subseteq M$. Hence

$$\hat{S}(x+M) = Sx + M \text{ and } \hat{A}(x+M) = Ax + M$$

define linear transformations $\hat{S}$ and $\hat{A}$ on Y/M. It is easy to see that $\hat{S} \in \mathbb{F}$-OrbRef$_0(\hat{A})$ and the matrix for $\hat{A}$ with respect to the basis $\{(A-1)u + M, u + M\}$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, by (1), we know that there is a $\gamma \in \mathbb{F}$ and an integer $t \geqslant 0$ such that

$$\hat{S} = \gamma \hat{A}^t.$$

Thus,

$$\hat{S}(u+M) = \gamma(u+M) + \gamma t((A-1)u + M).$$

But $Su = \beta A^k u = \beta u + \beta k(A-1)u + h$ with $h \in M$. Therefore,

$$\hat{S}(u+M) = \beta(u+M) + \beta k((A-1)u + M),$$

which implies $\gamma = \beta$ and $\gamma t = \beta k$. On the other hand,

$$\hat{S}((A-1)u + M) = \gamma(A-1)u + M$$

and, for some α and some n,

$$S(A-1)u = \alpha A^n(A-1)u = \alpha(A-1)u + h$$

with $h \in M$, which implies $\alpha = \gamma$. Hence if $\beta = 0$, we have $Su = 0 = S(A-1)u$. If $\beta \neq 0$, then $t = k$ and $\beta = \gamma = \alpha$. However, there exist η and q such that

$$S(u + (A-1)u) = \eta A^q (A-1)u = \eta u + \eta(q+1)(A-1)u + g$$

with $g \in M$, which implies

$$\hat{S}(u + (A-1)u + M) = \eta(u+M) + \eta(q+1)((A-1)u+M).$$

Comparing this with

$$\hat{S}(u + (A-1)u + M) = \gamma \hat{A}^t(u + (A-1)u + M) = \beta(u+M) + \beta(k+1)((A-1)u+M),$$

we see that $\eta = \beta$ and $q = k$. Hence,

$$S(A-1)u = S(u + (A-1)u) - Su = \beta A^k(u + (A-1)u) - \beta A^k u = \beta A^k(A-1)u. \qquad \square$$

EXAMPLE 1. Let $\omega_n = e^{\frac{2\pi i}{n}}$ for $n \geqslant 1$. Let Y be a vector space over $\mathbb{C}$ with a basis $\{e_1, e_2, \ldots\}$. Define linear transformations $A, B : Y \to Y$ by

$$Ae_1 = 0, Ae_{n+1} = e_n \text{ for } n \geqslant 1,$$

and

$$Be_n = \omega_n e_n \text{ for } n \geqslant 1.$$

Let $T = A \oplus B$ acting on $X = Y \oplus Y$. Let $S = 0 \oplus 1$ acting on X. Suppose $x \in X$. Then there is a positive integer n and scalars $a_1, b_1, \ldots, a_n, b_n \in \mathbb{C}$ such that

$$x = \sum_{k=1}^{n} a_k e_k \oplus \sum_{k=1}^{n} b_k e_k.$$

Then $Sx = T^{n!}x$. However, there is no integer N and scalar α such that $S = \alpha T^N$, since $\alpha T^N(e_{N+1} \oplus e_{N+1}) \neq S(e_{N+1} \oplus e_{N+1})$. Hence T is neither algebraically orbit reflexive nor algebraically $\mathbb{C}$-orbit reflexive.

The preceding example makes us look at the *strict topology* on $\mathscr{L}(X)$, where a basic neighborhood of a transformation T is given by a finite subset E of X, defined by

$$U(T,E) = \{A \in \mathscr{L}(X) : Ax = Tx \text{ for all } x \in E\}.$$

It is easy to show that if S and T are as in the preceding example, then $T^n \to S$ in the strict topology. It is also easy to see that $\mathbb{F}$-OrbRef$_0(T)$ and OrbRef$_0(T)$ are closed in the strict topology. It is natural to define a linear transformation T on a vector space X over a field $\mathbb{F}$ to be *strictly algebraically $\mathbb{F}$-orbit reflexive* if $\mathbb{F}$-OrbRef$_0(T)$ is the strict closure of $\mathbb{C}$-Orb(T).

THEOREM 3. *Suppose X is a finite-dimensional vector space over a field $\mathbb{F}$ not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Then every linear transformation on X whose minimal polynomial splits over $\mathbb{F}$ is algebraically $\mathbb{F}$-orbit reflexive.*

Proof. Since the minimal polynomial for T splits over $\mathbb{F}$, we can assume T has a Jordan canonical form. Moreover, if T is nilpotent, then, by Theorem 1, T is algebraically $\mathbb{F}$-orbit reflexive. Thus we can assume that T has at least one nonzero eigenvalue λ with largest Jordan block of size m, which we can assume is 1. We can write

$$T = (1 + J_m) \oplus \sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i}) \oplus \sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$$

with $\alpha_1, \ldots, \alpha_s$ nonzero and $n_1 \geqslant \cdots \geqslant n_t$, $m \geqslant m_1 \geqslant \cdots \geqslant m_s$. Suppose $S \in \mathbb{F}$-OrbRef$_0(T)$. First suppose there is a nonzero f in the domain of $\sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$ and a $\beta \in \mathbb{F}$ and an integer $k \geqslant 0$ such that $Sf = \beta T^k f$. Then, by Theorem 1, this uniquely defines k and β and uniquely defines $S = \beta T^k$ on the domain of the nilpotent transformation $\sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$. If x is in the domain of $(1 + J_m) \oplus \sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i})$, then there is an integer n and a scalar γ such that $S(x + f) = \gamma T^n(x + f)$. But $S(x + f) = Sx + Sf$ and $\gamma T^n(x + f) = \gamma T^n x + \gamma T^n f$. It follows that $Sx = \gamma T^n x$ and $Sf = \gamma T^n$, which implies $\gamma = \beta$ and $n = k$. Hence, $Sx = \beta T^k x$. Therefore $\ker(S - \beta T^k)$ contains the domains of both $(1 + J_m) \oplus \sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i})$ and $\sum_{1 \leqslant i \leqslant t}^{\oplus} J_{n_j}$, which implies $S = \beta T^k$.

We now consider the case in which $S = 0$ on the domain of $\sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$. Choose a vector g in the domain of J_{n_1} such that $J_{n_1}^{n_1 - 1} g \neq 0$. If $m = 1$, then $m_1 = \cdots = m_s = 1$ and $(1 + J_m) \oplus \sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i})$ is a diagonal matrix with eigenvectors $u, u_1, \ldots, u_s$ and there is a scalar η and an integer r such that

$$S(u + u_1 + \cdots + u_s + g) = \eta T^r(u + u_1 + \cdots + u_s + g),$$

which implies that $Su = \eta T^r u, Su_i = \eta T^r u_i \ (1 \leqslant i \leqslant s)$, and $0 = Sg = \eta T^r g$, which implies $\eta T^r = 0$ on the domain of $\sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$, and, hence, $S = \eta T^r$.

We are left with the case where $S = 0$ on the domain of $\sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$ and $m \geqslant 2$. Let $\{e_1, \ldots, e_m\}$ be the basis shifted by J_m. Then there is a scalar ρ and an integer $N \geqslant 0$ such that $Se_1 = \rho T^N e_1$. It follows from part (3) of Lemma 1 that $Se_j = \rho T^N e_j$ for $1 \leqslant j \leqslant m$. Suppose y is in the domain of $\sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i}) \oplus \sum_{1 \leqslant j \leqslant t}^{\oplus} J_{n_j}$, then there is a scalar α and an integer $d \geqslant 0$ such that

$$Se_1 + Sy = S(e_1 + y) = \alpha T^d(e_1 + y) = \alpha T^d e_1 + \alpha T^d y,$$

and it follows that

$$\rho T^N e_1 = \alpha T^d e_1 \text{ and } Sy = \alpha T^d y.$$

However, the representation of $\rho T^N e_1$ with respect to the basis $\{e_1, \ldots, e_m\}$ is $\rho e_1 + \rho N e_2 + \cdots$ and the expansion for $\alpha T^d e_1$ is $\alpha e_1 + \alpha d e_2 + \cdots$, so $\alpha = \rho$, and if $\alpha = \rho = 0$, then $Sy = 0 = \rho T^N y$, and if $\rho \neq 0$, then $d = N$ and $Sy = \rho T^N y$. Hence $S = \rho T^N$. $\square$

COROLLARY 1. *If X is a finite-dimensional vector space on an algebraically closed field $\mathbb{F}$, then every linear transformation on X is $\mathbb{F}$-algebraically reflexive.*

Recall from ring theory that if $\mathscr{R}$ is a principal ideal domain and M is an $\mathscr{R}$-module and $0 \neq r \in \mathscr{R}$ and $rM = \{0\}$, then M is a direct sum of cyclic $\mathscr{R}$-modules; Applying this fact to $\mathscr{R} = \mathbb{F}[t]$, we get that any algebraic linear transformation on a vector space is a direct sum of transformations on finite-dimensional subspaces, and therefore has a Jordan form when the minimal polynomial splits over $\mathbb{F}$. (See [20] for details.) This gives us the following corollary.

COROLLARY 2. *Suppose X is a vector space over a field $\mathbb{F}$ not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Then every algebraic linear transformation on X whose minimal polynomial splits over $\mathbb{F}$ is algebraically $\mathbb{F}$-orbit reflexive.*

The next corollary follows from the technique in the last paragraph of the proof of Theorem 3. Recall from the beginning of Section 2 that if T is locally algebraic and x is a vector, then the *local minimal polynomial* for T at x is the unique monic polynomial $p(t)$ of minimal degree for which $p(T)x = 0$.

COROLLARY 3. *Suppose X is a vector space over a field $\mathbb{F}$ that is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p, and suppose T is a locally algebraic linear transformation on X whose local minimal polynomial splits over $\mathbb{F}$. If there is a nonzero $\lambda \in \mathbb{F}$ such that $\ker(T - \lambda) \neq \ker(T - \lambda)^2$, then T is algebraically $\mathbb{F}$-orbit reflexive.*

The next corollary follows from the fact that if T is a locally algebraic linear transformation and E is any finite set of vectors, then there is a finite-dimensional invariant subspace M for T that contains E.

COROLLARY 4. *Suppose X is a vector space over a field $\mathbb{F}$ that is not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p, and suppose T is a locally algebraic linear transformation on X whose local minimal polynomials split over $\mathbb{F}$. Then T is strictly algebraically $\mathbb{F}$-orbit reflexive.*

THEOREM 4. *If $\mathbb{F}$ is an algebraically closed field, then every linear transformation on a vector space over $\mathbb{F}$ is strictly algebraically $\mathbb{F}$-orbit reflexive.*

Proof. Since $\mathbb{F}$ is algebraically closed, we know $\mathbb{F}$ is infinite and is therefore not isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime number p. Suppose X is a vector space and T is a linear transformation on X. If T is not locally algebraic, then T is algebraically $\mathbb{F}$-orbit reflexive. If T is locally algebraic, then, by Corollary 4, T is strictly algebraically $\mathbb{F}$-orbit reflexive. $\square$

3. $\mathbb{F}$-orbit reflexivity with $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$

PROPOSITION 1. *Every normal operator is $\mathbb{C}$-orbit reflexive.*

Proof. This is an immediate consequence of [16, Proposition 3]. $\square$

The next two results are consequences of Theorem 1.

THEOREM 5. *Suppose T is a bounded linear operator on a normed space X over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ such that $\cup_{n=1}^{\infty} \ker(T^n)$ is dense in X. Then T is $\mathbb{F}$-orbit reflexive and $\mathbb{F}$-Orb(T) is SOT-closed. Moreover, if $S \in \mathbb{F}$-OrbRef(T), $x \in \cup_{n=1}^{\infty} \ker(T^n)$, $\beta \in \mathbb{F}$, $k \geqslant 0$, and $Sx = \beta T^k x \neq 0$, then $S = \beta T^k$.*

Proof. Suppose $S \in \mathbb{F}$-OrbRef(T) and let $M = \cup_{n=1}^{\infty} \ker(T^n)$. It is clear that $S(M) \subseteq M$ and $T(M) \subseteq M$ and $S|_M \in \mathbb{F}$-OrbRef$(T|M)$. But $T|M$ is locally nilpotent, and if $x \in M$ and $T^n x = 0$, then

$$\mathbb{F}\text{-Orb}(T)x = \cup_{k=0}^{n} \mathbb{F} T^k x$$

is norm closed. Hence, $\mathbb{F}$-OrbRef$(T|M) = \mathbb{F}$-OrbRef$_0(T|M)$, which, by Theorem 1 is $\mathbb{F}$-Orb(T). Hence there is a $\lambda \in \mathbb{F}$ and an $n \geqslant 0$ such that $S|M = \lambda T^n|M$. However, M is dense in X, so $S = \lambda T^n \in \mathbb{F}$-Orb$(T)$. $\square$

The preceding theorem implies a stronger version of itself.

COROLLARY 5. *Suppose X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $T \in B(X)$, and there is a decreasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of T-invariant closed linear subspaces such that*

1. *for every $\lambda \in \Lambda$, $\cup_{n=0}^{\infty} (T^n)^{-1}(X_\lambda)$ is dense in X, and*

2. *$\cap_{\lambda \in \Lambda} X_\lambda = \{0\}$.*

Then T is $\mathbb{F}$-orbit reflexive.

Proof. Suppose $S \in \mathbb{F}$-OrbRef(T) and $S \neq 0$. Choose $e \in X$ such that $Se \neq 0$. It follows from (2) that both (1) and (2) remain true if we consider only those X_λ that contain neither e nor Se. Since $T(X_\lambda) \subseteq X_\lambda$, $\hat{T}_\lambda(x + X_\lambda) = Tx + X_\lambda$ defines a bounded linear operator $\hat{T}_\lambda$ on X/X_λ. Condition (1) implies that $\cup_{n=1}^{\infty} \ker(\hat{T}_\lambda^n)$ is dense in X/X_λ; whence $\hat{T}_\lambda$ is $\mathbb{F}$-orbit reflexive. However, $S \in \mathbb{F}$-OrbRef(T) implies that $S(X_\lambda) \subseteq X_\lambda$, so $\hat{S}_\lambda(x + X_\lambda) = Sx + X_\lambda$ defines an operator on X/X_λ such that $\hat{S}_\lambda \in \mathbb{F}$-OrbRef$(\hat{T}_\lambda)$. Hence, by Theorem 5, since $\hat{S}_\lambda(e + X_\lambda) \neq 0$, there is a unique $\beta \in \mathbb{F}$ and a unique nonnegative integer n such that $\hat{S}_\lambda(e + X_\lambda) = \beta \hat{T}_\lambda^n(e + X_\lambda)$, and for this β and n, we have $\hat{S}_\lambda = \beta \hat{T}_\lambda^n$. Suppose $\eta \in \Lambda$. Since the X_λ's are decreasingly directed, there is a $\sigma \in \Lambda$ such that $X_\sigma \subseteq X_\lambda \cap X_\eta$. Applying the same arguments we used on X_λ, there is a unique $\alpha \in \mathbb{F}$ and a unique integer $m \geqslant 0$ such that $\hat{S}_\sigma(e + X_\sigma) = \alpha T_\sigma^m(e + X_\sigma)$. However, it follows that

$$Se - \alpha T^n e \in X_\sigma \subseteq X_\lambda,$$

which implies $\hat{S}_\lambda(e + X_\lambda) = \alpha \hat{T}_\lambda^n(e + X_\lambda)$, which implies that $\alpha = \beta$ and $m = n$. Thus $\hat{S}_\sigma = \beta \hat{T}_\sigma^n$, which in turn implies $\hat{S}_\eta = \beta \hat{T}_\eta^n$. Therefore $\hat{S}_\eta = \beta \hat{T}_\eta^n$ for every $\eta \in \Lambda$. Therefore, for every $\eta \in \Lambda$ and for every $x \in X$,

$$Sx - \beta T^n x \in X_\eta,$$

which, by (2), implies $S = \beta T^n$. $\square$

The following corollary applies to operators that have a strictly upper-triangular operator matrix with respect to some direct sum decomposition.

COROLLARY 6. *If a normed space X over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a direct sum of spaces $\{X_n : n \in \mathbb{N}\}$ such that $T(X_1) = \{0\}$, and for every $n > 1$,*

$$T(X_n) \subseteq \left(\sum_{k<n}^{\oplus} X_k \right)^-,$$

then T is $\mathbb{F}$-orbit reflexive and $\mathbb{F}$-Orb(T) is SOT-closed.

The preceding corollary has some familiar special cases.

COROLLARY 7. *If T is an operator-weighted shift or if T is a direct sum of nilpotent operators on a normed space X over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then T is $\mathbb{F}$-orbit reflexive.*

THEOREM 6. *Suppose X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $T \in B(X)$ and $\cap_{n=1}^{\infty} T^n(X)^- = \{0\}$. Then T is $\mathbb{F}$-orbit reflexive and $\mathbb{F}$-OrbRef$(T) = \mathbb{F}$-Orb(T). Moreover, if $S \in \mathbb{F}$-OrbRef(T), $f \in X$, and $0 \neq Sf = \beta T^k f$, then $S = \beta T^k$.*

Proof. We will first show that T is algebraically $\mathbb{F}$-orbit reflexive. If M is a finite-dimensional invariant subspace for T and $T|M$ is not nilpotent, then there is a nonzero T-invariant subspace N of M such that $\ker(T|N) = 0$. Thus $T(N) = N \neq 0$, which violates $\cap_{n=1}^{\infty} T^n(X)^- = \{0\}$. Thus, either T is not locally algebraic or T is locally nilpotent. In these cases it follows either from Theorem 2 or Theorem 1 that T is indeed algebraically $\mathbb{F}$-orbit reflexive. Furthermore, the hypothesis on T implies, for each $x \in X$, that

$$\cap_{N=1}^{\infty} \left\{ \lambda T^k x : \lambda \in \mathbb{F}, k \geqslant N \right\}^{-SOT} = \{0\},$$

so $\mathbb{F}$-Orb(T) is closed in X. Thus $\mathbb{F}$-OrbRef$(T) = \mathbb{F}$-OrbRef$_0(T) = \mathbb{F}$-Orb(T). For the last statement suppose $f \in X$, $\alpha, \beta \in \mathbb{F}$, and $k, n \geqslant 0$ are integers, and

$$0 \neq Sf = \alpha T^n f = \beta T^k f.$$

Clearly if $n = k$, then $\alpha = \beta$. Suppose $k < n$. Then $M = sp\{f, Tf, \ldots, T^{n-1}f\}$ is a nonzero finite-dimensional invariant subspace for T with $\dim M \leqslant n$. Since $T^n f \neq 0$, we know $T|M$ is not locally nilpotent, which, as remarked earlier, contradicts $\cap_{n=1}^{\infty} T^n(X)^- = \{0\}$. $\square$

This theorem also implies a stronger version of itself.

COROLLARY 8. *Suppose X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $T \in B(X)$, and there is an increasingly directed family $\{X_\lambda : \lambda \in \Lambda\}$ of T-invariant linear subspaces such that*

1. for every $\lambda \in \Lambda$, $\cap_{n=1}^{\infty} \overline{T^n(X_\lambda)} = \{0\}$, and

2. $\cup_{\lambda \in \Lambda} X_\lambda$ is dense in X.

Then T is $\mathbb{F}$-orbit reflexive, and $\mathbb{F}$-OrbRef$(T) = \mathbb{F}$-Orb(T). Moreover, if $S \in \mathbb{F}$-OrbRef(T), $f \in X$, and $0 \neq Sf = \beta T^k f$, then $S = \beta T^k$.

Proof. Suppose $0 \neq S \in \mathbb{F}$-OrbRef(T). It follows from (2) that there is a $\lambda_0 \in \Lambda$ and an $f \in X_{\lambda_0}$ such that $0 \neq Sf$. However, we must have $S(X_{\lambda_0}) \subseteq X_{\lambda_0}$, and $S|X_{\lambda_0} \in \mathbb{F}$-OrbRef$(T|X_{\lambda_0}) = \mathbb{F}$-Orb$(T)$ (by (1) and the preceding theorem). Thus there is a unique scalar β and an integer $k \geqslant 0$ such that

$$S|X_{\lambda_0} = \beta T^k | X_{\lambda_0}.$$

The same β and k must work for any X_λ that contains X_{λ_0}. It follows from the fact that the family is increasingly directed and (2) that $S = \beta T^k$.   □

I. Kaplansky [20] (see also [22], [22] , [26]) proved that a (bounded linear) operator on a Banach space is locally algebraic if and only if it is algebraic. This immediately gives us the following result from Corollary 2.

PROPOSITION 2. *Suppose X is a Banach space over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $T \in B(X)$ is not algebraic. Then T is algebraically $\mathbb{F}$-orbit reflexive.*

If T is an operator on a Banach space, then $r(T)$ denotes the spectral radius of T, i.e.,

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

THEOREM 7. *Suppose $T \in \mathscr{M}_d(\mathbb{C})$ and T is not nilpotent. The following are equivalent.*

1. *T is $\mathbb{C}$-orbit reflexive*

2. *Among all the Jordan blocks with eigenvalues having modulus equal to $r(T) > 0$, the two largest blocks differ in size by at most 1.*

Proof. We begin with some basic computations. Suppose J_m is an $m \times m$ nilpotent Jordan block, i.e., there is an orthonormal basis $\{e_0, \ldots, e_{m-1}\}$ for the domain of J_m such that $J_m e_{m-1} = 0$ and $J_m e_k = e_{k+1}$ for $0 \leqslant k < m-1$. Note that J_m is lower triangular with respect to the basis $\{e_0, \ldots, e_{m-1}\}$. Then, for $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$ and $n \geqslant 0$, we have from the binomial theorem that

$$\|(\alpha + J_m)^n\| = \left\| \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} J_m^k \right\| = \left\| \sum_{k=0}^{\min(n,m-1)} \binom{n}{k} \alpha^{n-k} J_m^k \right\| \leqslant mn^{m-1} |\alpha|^{n-m+1},$$

so if $|\alpha| = 1$ and $N > m-1$ or $|\alpha| < 1$, then

$$\lim_{n \to \infty} \frac{1}{\binom{n}{N}} \|(\alpha + J_m)^n\| = 0.$$

Moreover, for $0 \leqslant t < m - 1$, we have

$$(\alpha + J_m)^n e_t = \sum_{k=0}^{m-1} \binom{n}{k} \alpha^{n-k} J_m^k J_m^t e_0 = \sum_{k=0}^{m-t-1} \binom{n}{k} \alpha^{n-k} e_{t+k},$$

and, when $m \geqslant 2$, we have

$$\langle (\alpha + J_m)^n e_0, e_0 \rangle = \alpha^n \text{ and } \langle (\alpha + J_m)^n e_0, e_1 \rangle = n\alpha^{n-1}$$

In particular, if $|\alpha| = 1$ and $0 \leqslant i < m$ we have

$$\lim_{n \to \infty} \frac{1}{\binom{n}{m-i-1} \alpha^{n-m+i+1}} (\alpha + J_m)^n e_i = e_{m-1}.$$

We can assume T is already equal to its Jordan canonical form. By replacing T with $\frac{1}{\lambda} T$, where λ is an eigenvalue of T with modulus $r(T) > 0$ and having the largest, say $m \times m$, Jordan block among such eigenvalues, we can assume that this largest block has eigenvalue 1. We write T as

$$T = (1 + J_m) \oplus \sum_{1 \leqslant i \leqslant s}^{\oplus} (\alpha_i + J_{m_i}) \oplus A$$

with each $|\alpha_i| = 1$, and $m \geqslant m_1 \geqslant \cdots \geqslant m_s$, and the modulus of every eigenvalue of A less than 1. It follows that $A^n \to 0$ as $n \to \infty$. Note that we allow the possibility that $s = 0$ or A is not present.

$(1) \implies (2)$. Assume $m_1 \leqslant m - 2$, i.e., the second largest Jordan block for the eigenvalues with modulus $r(T)$ differs from m by more than 1. In this case $m \geqslant 2$. Let $\{e_0, \ldots, e_{m-1}\}$ be the orthonormal basis above. Define a linear transformation S in terms of the inner product $\langle, \rangle$ on $\mathbb{C}^n$ by

$$Sx = [\langle x, e_0 \rangle + \langle x, e_1 \rangle] e_{m-1}.$$

Note that

$$(T - 1) Se_0 = J_m e_{m-1} = 0,$$

but

$$S(T - 1) e_0 = Se_1 = e_{m-1} \neq 0.$$

Hence $ST \neq TS$, so S is not in the SOT-closure of $\mathbb{C}$-Orb(T). However, we will show that $S \in \mathbb{C}$-OrbRef(T). If x is a vector and $Sx = 0$, then $Sx = 0 \cdot T^1 x \in \mathbb{C}$-Orb$(T, x)$. If $Sx = \beta e_m \neq 0$, then either $\langle x, e_0 \rangle \neq 0$ or both $\langle x, e_0 \rangle = 0$ and $\langle x, e_1 \rangle \neq 0$. In case $\langle x, e_0 \rangle \neq 0$, we have

$$Sx = \lim_{n \to \infty} \frac{\beta}{\langle x, e_0 \rangle} \frac{1}{\binom{n}{m-1}} T^n x \in \mathbb{C}\text{-Orb}(T, x)^-.$$

In case $\langle x, e_0 \rangle = 0$ and $\langle x, e_1 \rangle \neq 0$, we have

$$Sx = \lim_{n \to \infty} \frac{\beta}{\langle x, e_1 \rangle} \frac{1}{\binom{n}{m-2}} T^n x \in \mathbb{C}\text{-Orb}(T, x)^-.$$

Both of the above two formulas hold because

$$\lim_{n\to\infty} \frac{1}{\binom{n}{m-2}} \|(\alpha_i + J_{m_i})^n\| = 0$$

for $1 \leqslant i \leqslant s$ since either $m_i \leqslant m - 2$ and $A^n \to 0$.

$(2) \implies (1)$. Let $\{f_0, \ldots, f_{m_1-1}\}$ be the orthonormal basis shifted by J_{m_1}. Suppose $S \in \mathbb{C}\text{-OrbRef}(T)$. Relative to the direct sum decomposition for T above, we can write

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_r \oplus B.$$

In order to prove $S \in \mathbb{C}\text{-Orb}(T)^{-SOT}$, we consider each of the following cases.

Case 1. $m = 1$. Then $T = 1 \oplus U \oplus A$ with U unitary. Write $S = \lambda \oplus D \oplus B$.

Subcase 1.1. $\lambda = 0$. Suppose x is in the domain of $U \oplus A$. Then there is a sequence $\{c_n\}$ in $\mathbb{C}$ and a sequence $\{k_n\}$ of integers such that

$$Se_0 \oplus Sx = S(e_0 \oplus x) = \lim_{n\to\infty} c_n T^{k_n}(e_0 \oplus x) =$$

$$\lim_{n\to\infty} c_n e_0 \oplus \lim_{n\to\infty} c_n (U \oplus A)^{k_n} x.$$

Thus $c_n \to 0$, and, since $\left\{ \left\| (U \oplus A)^{k_n} \right\| \right\}$ is bounded, $Sx = 0$. Hence, $S = 0$.

Subcase 1.2. $B = 0$, so $S = \lambda \oplus D \oplus 0$. It is well-known that if $\alpha \in \mathbb{C}$ and $|\alpha| = 1$, then there is a sequence $\{k_n\} \to \infty$ such that $\alpha^{k_n} \to 1$. Thus there is a sequence $\{k_n\} \to \infty$ such that $U^{k_n} \to 1$. Thus $T^{k_n+1} \to 1 \oplus U \oplus 0$. Since $1 \oplus U$ is normal, $\lambda \oplus D \in \mathbb{C}\text{-Orb}(1 \oplus U)^{-SOT}$. Hence $S \in \mathbb{C}\text{-Orb}(T)^{-SOT}$.

Subcase 1.3. $B \neq 0$, $\lambda \neq 0$. Then, for every $x \in \text{Dom}A$, there are sequences $\{c_n\}$ and $\{k_n\}$ such that

$$\lambda e_0 \oplus 0 \oplus Bx = S(e_0 \oplus 0 \oplus x) = \lim c_n T^{k_n}(e_0 \oplus 0 \oplus x),$$

which implies, $c_n \to \lambda$. By choosing a subsequence, we can assume $k_n \to \infty$ or $k_n \to k < \infty$. If $k_n \to \infty$, then $Bx = 0$. If $k_n \to k$, then $Bx = \lambda A^k x$. Hence

$$\text{Dom}(B) = \ker B \cup \bigcup_{n=0}^{\infty} \ker(B - \lambda A^n),$$

which implies, by the Baire Category theorem, that $B \in \{0, \lambda, \lambda A, \lambda A^2, \ldots\}$. Since $B \neq 0$, there is a $k \geqslant 0$ such that $B = \lambda A^k$. Since $1 - A^t$ is invertible for $t \geqslant 1$, and $A^k = \frac{1}{\lambda} B \neq 0$, the integer k is unique. Applying the same technique with e_0 replaced with an eigenvector for U, we get $D = \lambda U^k$. Thus, $S = \lambda T^k$.

Case 2. $m \geqslant 2$. Then $m_1 \geqslant m - 1 \geqslant 1$. Write $S = S_0 \oplus S_1 \oplus \cdots \oplus S_s \oplus B$.

Subcase 2.1. $B \neq 0$. As in Subcase 1.3, $S = \lambda T^k$.

Subcase 2.2. $Se_0 = 0$. Suppose x is orthogonal to the domain of S_0. Then

$$S(e_0 \oplus x) = \lim_{n\to\infty} c_n T^{k_n} e_0 \oplus c_n T^{k_n} e_0 x$$

implies that

$$\lim_{n \to \infty} c_n \binom{k_n}{m-1} = 0,$$

which implies

$$\lim_{n \to \infty} c_n \left\| ((1+J_m) \oplus (\alpha_1 + J_{m_1}) \oplus \cdots \oplus (\alpha_r + J_{m_r}) \oplus A)^{k_n} \right\| \to 0.$$

This, in turn, implies $Sx = 0$. Thus $S_1 \oplus \cdots \oplus S_r \oplus B = 0$. Applying the same idea with e_0 replaced with f_0 and x replaced with any of $e_1, \ldots, e_{m-1}$, we conclude that $S_0 = 0$. Hence $S = 0$.

Subcase 2.3. $Se_0 = \lambda T^k e_0 \neq 0$. Suppose x is orthogonal to the domain of S_0 and

$$S(e_0 + x) = \lim_{n \to \infty} c_n T^{k_n}(e_0 + x).$$

We then get

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} \left\langle c_n T^{k_n} e_0, e_0 \right\rangle = \langle Se_0, e_0 \rangle = \lambda,$$

and

$$\lim_{n \to \infty} k_n = \lim_{n \to \infty} \frac{1}{c_n} \left\langle c_n T^{k_n} e_0, e_1 \right\rangle = \left\langle \frac{1}{\lambda} Se_0, e_1 \right\rangle = k.$$

Hence $Sx = \lambda T^k x$ for every x orthogonal to the domain of S_0. In particular

$$S_1 = \lambda (\alpha_1 + J_{m_1})^k.$$

If $m_1 \geqslant 2$, we can make the same argument with e_0 replaced with f_0 to get that $S_0 = \lambda (1 + J_m)^k$, implying $S = \lambda T^k$. If $m_1 = 1$, then $m = 2$, and we need only show that $Se_1 = \lambda T^k e_1$; but we know that $Te_1 = e_1$. Suppose $Se_1 = \lambda \beta e_1$ and $\beta \neq 1$. Choose $y = \frac{1}{2(\beta-1)}$; we can write

$$\lambda e_0 + (\lambda k + \lambda \beta y) e_1 = S(e_0 + y e_1) = \lim_{n \to \infty} d_n T^{j_n}(e_0 + y e_1) = \lim_{n \to \infty} d_n e_0 + (d_n j_n + y) e_1,$$

which implies that

$$d_n \to \lambda,$$

and

$$j_n - k \to (\beta - 1) y = \frac{1}{2},$$

which is impossible. Thus $Se_1 = \lambda e_1$, and $S = \lambda T^k$.

Subcase 2.4. $Se_0 = \lim c_n T^{k_n} e_0 \neq 0$ with $n_k \to \infty$. Using the ideas in the proof of Subcase 2.3, we get $B = 0$, and $S_j = 0$ when $m_j < m$, and $S_j | ran(J_{m_j}) = 0$ when $m_j = m$. If $m_1 = m$, we can apply the same reasoning to $e_i \oplus f_0$ for $1 \leqslant i < m$ to get $Se_i = 0$, and then applying S to a sum of $e_0 \oplus f_0 \oplus h_2 \oplus \cdots \oplus h_t \oplus 0$, where $m_j = m$ and $h_j \in \ker J_{m_j}^*$ for $1 \leqslant j \leqslant t$, we conclude that there are sequences $\{d_n\}$ and $\{j_n\}$ such that $S = \lim_{n \to \infty} d_n T^{j_n}$ and $\lim_{n \to \infty} j_n = \infty$. If $m_1 = m - 1$, then $Sf_0 = 0$, and we can still look at $S(e_i \oplus f_0)$ for $1 \leqslant i < m$ to get $Se_i = 0$. In this case we get $S = \lim_{n \to \infty} c_n T^{k_n}$.

Hence in all of the possible cases, $S \in \mathbb{C}\text{-}\mathrm{Orb}(T)^{-SOT}$. Thus T is $\mathbb{C}$-orbit reflexive. $\square$

4. Orbit Reflexivity

We conclude with a few results on orbit reflexivity, most of which appeared in [25]. A key ingredient in the results of this section comes from [16, Theorem 5 (1)], which uses a simple Baire category argument to show, for an operator T on a Banach space, that if $\mathrm{Orb}(T,x)$ is closed for every x in a nonempty open set, then T is orbit reflexive.

LEMMA 2. *Suppose X is a normed space, $T \in B(X)$, λ is an eigenvalue of the adjoint $T^{\#}$ of T with unit eigenvector $\alpha \in X^{\#}$ and $|\lambda| > 1$. Then T is orbit reflexive.*

Proof. Suppose $f \in X$ and $\langle f, \alpha \rangle = \alpha(f) \neq 0$. Then

$$\|T^n f\| \geqslant |\langle T^n f, \alpha \rangle| = |\langle f, (T^n)^* \alpha \rangle| = |\lambda|^n |\langle f, \alpha \rangle| \to \infty$$

as $n \to \infty$. Since $\{f \in X : \langle f, \alpha \rangle \neq 0\}$ is an open set, it follows from [16, Theorem 5 (1)] that T is orbit reflexive. $\square$

COROLLARY 9. *Suppose X is a functional Hilbert space on a set E, and $f : E \to \mathbb{C}$ is a multiplier of X such that $\|M_f\| = \sup\{|f(t)| : t \in E\}$. Then M_f is orbit-reflexive.*

Proof. Suppose $t \in E$ and let $e_t \in X^{\#}$ be the evaluation functional at t. Then, for every $h \in X$, we have

$$\left(M_f^{\#}(e_t) \right) h = e_t(M_f h) = e_t(fh) = f(t) e_t(h).$$

Thus $M_f^{\#} e_t = f(t) e_t$ for every $t \in E$. If $|f(t)| > 1$ for some $t \in E$, then it follows from Lemma 2 that M_f is orbit reflexive. Otherwise, $\|M_f\| \leqslant 1$, which, by [16], implies M_f is orbit reflexive. $\square$

In [25] the third author used the preceding corollary and a result of J. E. Thomson [32] concerning bounded point evaluations for cyclic subnormal operators to show that every cyclic subnormal operator is a multiplication on a functional Hilbert space, implying that every cyclic subnormal operator is orbit reflexive. Here we give a more elementary proof that every subnormal operator is orbit reflexive.

THEOREM 8. *Suppose H is a Hilbert space and $T \in B(H)$ is subnormal. Then T is orbit reflexive.*

Proof. Suppose $f \in H$ and $\|f\| = 1$. Let $E_f = \{p(T)f : p \in \mathbb{C}[t]\}^-$ be the cyclic invariant subspace for T generated by f. We know (see [4]) that there is a probability measure μ whose support is $\sigma\left(T|_{E_f}\right)$ and a unitary operator U from E_f onto the closure $P^2(\mu)$ of the set of polynomials in $L^2(\mu)$ such that $Uf = 1$ and $UT|_{E_f} U^*$

is the multiplication operator M_z on $P^2(\mu)$. Since the norm of a subnormal operator equals it spectral radius and

$$\|T^n f\|^2 = \int_{\sigma\left(T|_{E_f}\right)} |z^{2n}| \, d\mu$$

for each $n \geqslant 1$, we have that

$$\{T^n f\} \text{ is bounded} \Leftrightarrow \left\|T|_{E_f}\right\| \leqslant 1 \Leftrightarrow \sup_{n \geqslant 1} \|T^n f\| \leqslant \|f\|,$$

and

$$\left\|T|_{E_f}\right\| > 1 \Leftrightarrow \lim_{n \to \infty} \|T^n f\| = \infty.$$

It follows that $\{f \in H : \{T^n f\} \text{ is bounded}\}$ is closed and the set

$$U = \left\{ f \in H : \lim_{n \to \infty} \|T^n f\| = \infty \right\}$$

is open. If $U = \varnothing$, then $\|T\| \leqslant 1$, which implies T is orbit reflexive [16]. On the other hand if $U \neq \varnothing$, then it follows from [16, Theorem 5 (1)] that T is orbit reflexive. $\square$

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