CHERNOFF'S THEOREM FOR BACKWARD PROPAGATORS
AND APPLICATIONS TO DIFFUSIONS ON MANIFOLDS

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Abstract. The classical Chernoff's theorem is a statement about discrete-time approximations of semigroups, where the approximations are consturcted as products of time-dependent contraction operators strongly differentiable at zero. We generalize the version of Chernoff's theorem for semigroups proved in [3] (see also [4] and [5]), and obtain a theorem about discrete-time approximations of backward propagators.

1. Introduction

Let E be a Banach space, and let $\mathscr{L}(E)$ denote the space of all bounded operators $E \to E$. Let $U(s,t)$ be a backward propagator on E possessing the left generator A_t. For convenience, we give definitions of backward propagators and their left generators (see [1]). A two-parameter family of operators $\{U(s,t) \in \mathscr{L}(E) : 0 \leqslant S \leqslant s \leqslant t \leqslant T\}$ is called a backward propagator on E if

$$U(s,t) = U(s,\tau)U(\tau,t), \qquad (1)$$
$$U(s,s) = I$$

for all s, τ, t such that $S \leqslant s \leqslant \tau \leqslant t \leqslant T$. The operator A_t on E, $0 < t \leqslant T$, defined as

$$A_t x = \lim_{h \downarrow 0} \frac{U(t-h,t)x - x}{h},$$

with the domain $D(A_t)$ consisting of those $x \in E$ for which the above limit exists, is called the left generator of the backward propagator $U(s,t)$.

Analogously, we can introduce the concept of the right generator of a backward propagator (see [1]): the operator A_t^+ on E, $0 \leqslant t < T$, defined as

$$A_t^+ x = \lim_{h \downarrow 0} \frac{U(t,t+h)x - x}{h},$$

with the domain $D(A_t^+)$ consisting of those $x \in E$ for which the above limit exists, is called the right generator of the backward propagator $U(s,t)$.

Let $Q_{s,t}$, $0 \leqslant S \leqslant s \leqslant t \leqslant T$, be a two-parameter family of contraction operators on E, whose left derivatives at $s = t$ equal to A_t. The discrete-time approximations of the

backward propagator $U(s,t)$ are constructed as products of Q_{t_1,t_2}, $s \leqslant t_1 \leqslant t_2 \leqslant t$. Note that we could equivalently use right generators of the backward propagator and right derivatives of $Q_{s,t}$ at $t = s$. The theorem will also work in the situation with (forward) propagators and the two-parameter family of contractions $Q_{t,s}$ parametrized by times t and s such that $T \geqslant t \geqslant s \geqslant S \geqslant 0$. We prove our main result for backward propagators because in the application to diffusions on manifolds (Section 3) backward propagators will be associated to transition probability functions. Specifically, we consider the situation when the backward propagator is represented by a transition probability function of a time-inhomogeneous diffusion on a compact Riemannian manifold, the contraction operators are integral operators with probabilistic kernels, the left generators are second-order differential operators on the manifold, and the discrete-time approximations are distributions of diffusion processes in the surrounding Euclidean space. We then obtain the approximation of the distribution on the manifold by distributions in the Euclidean space.

Compared to the situation considered in [7], the stochastic processes under consideration are non-homogeneous. In particular, the coefficients of the second-order differential operator representing the generator of the manifold-valued diffusion are time-dependent. Therefore, in the current paper we consider a more general situation compared to [3], [4], [5], and [7] for both Chernoff's theorem and its applications to diffusions on manifolds.

2. Chernoff's theorem for backward propagators

THEOREM 1. (Chernoff's theorem for backward propagators) *Let $U(s,t)$, $0 \leqslant S \leqslant s \leqslant t \leqslant T$, be a backward propagator with the left generators A_t, and let Q_{t_1,t_2}, $S \leqslant t_1 \leqslant t_2 \leqslant T$, be a two-parameter family of contractions $E \to E$. We assume that the following assumptions are fulfilled:*

1) *The subset $\cap_{t \in [S,T]} D(A_t)$ is dense in E.*

2) *There exists a dense in E Banach space Y such that $Y \subset \cap_{t \in [S,T]} D(A_t)$ and $U(s,t)Y \subset Y$ for all $s,t \in [S,T]$, $s < t$, and, moreover, so that there exists a constant $\gamma > 0$ such that the norm in Y satisfies the inequality $\|x\|_Y \geqslant \gamma [\|x\|_E + \sup_{\tau \in [S,T]} \|A_\tau x\|_E]$.*

3) *For every $x \in Y$ and $t \in [S,T]$, the function $[S,t] \to Y$, $\tau \mapsto U(\tau,t)x$ is continuous.*

4) *For every $x \in Y$, the function $[S,T] \to E$, $t \mapsto A_t x$ is continuous.*

5) *For all $x \in Y$ there exists the uniform in t limit*

$$\lim_{h \downarrow 0} \frac{Q_{t-h,t}x - x}{h} = A_t x.$$

Then, for any subinterval $[s,t] \subset [S,T]$, for any sequence of partitions $\{s = t_0 < t_1 < \cdots < t_n = t\}$ such that $\max |t_j - t_{j-1}| \to 0$ as $n \to \infty$, and for all $x \in E$,

$$Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to U(s,t)x, \quad n \to \infty.$$

Proof. Fix an $x \in Y$. First consider the case $s > S$. Using relation (1), we obtain:

$$Q_{t_0,t_1} Q_{t_1,t_2} \ldots Q_{t_{n-1},t_n} - U(s,t)$$
$$= \sum_{j=1}^{n} Q_{t_0,t_1} \ldots Q_{t_{j-2},t_{j-1}} (Q_{t_{j-1},t_j} - U(t_{j-1},t_j))U(t_j,t). \quad (2)$$

Let $\delta_n = \max_j(t_j - t_{j-1})$, $j \geqslant 1$, be the mesh of the partition $\{s = t_0 < t_1 < \cdots < t_n = t\}$. Relation (2) implies:

$$\|(Q_{t_0,t_1} Q_{t_1,t_2} \ldots Q_{t_{n-1},t_n} - U(s,t))x\|_E$$
$$\leqslant \sum_{j=1}^{n} \Delta t_j \left\| \left(\frac{Q_{t_{j-1},t_j} - I}{t_j - t_{j-1}} - \frac{U(t_{j-1},t_j) - I}{t_j - t_{j-1}} \right) U(t_j,t) x \right\|_E$$
$$\leqslant (t-s) \sup \left\{ \left\| \left(\frac{Q_{\tau-h,\tau} - I}{h} - \frac{U(\tau-h,\tau) - I}{h} \right) U(\tau,t) x \right\|_E : \tau \in (s,t], h \in (0,\delta_n) \right\}$$
$$\leqslant (t-s) \sup \left\{ \left\| \left(\frac{Q_{\tau-h,\tau} - I}{h} - A_\tau \right) U(\tau,t) x \right\|_E : \tau \in (s,t], h \in (0,\delta_n) \right\} \quad (3)$$
$$+ (t-s) \sup \left\{ \left\| \left(\frac{U(\tau-h,\tau) - I}{h} - A_\tau \right) U(\tau,t) x \right\|_E : \tau \in (s,t], h \in (0,\delta_n) \right\}. \quad (4)$$

Note that for every $x \in Y$,

$$\left(\frac{U(\tau-h,\tau) - I}{h} - A_\tau \right) x \quad (5)$$

converges to zero uniformly in $\tau \in [s,t]$. Indeed, by Assumption 4, one can find a $\theta \in (0,1)$ such that $\frac{U(\tau-h,\tau)x-x}{h} = A_{\tau-\theta h}x$. Since the function $[s,t] \to E$, $\tau \mapsto A_\tau x$ is continuous by assumption, it is also uniformly continuous which implies the uniform convergence in (5). Let $B_{\tau-h,\tau}$ denote one of the operators $\frac{Q_{\tau-h,\tau}-I}{h} - A_\tau$ or $\frac{U(\tau-h,\tau)-I}{h} - A_\tau$. We know that for every $x \in Y$, $B_{\tau-h,\tau}x$ converges to zero uniformly in $\tau \in [s,t]$. We would like to prove that $B_{\tau-h,\tau}U(\tau,t)x$ also converges to zero uniformly in $\tau \in [s,t]$. By the continuity of the map $[s,t] \to Y$, $\tau \mapsto U(\tau,t)y$, the set $\{U(\tau,t)x, \tau \in [s,t]\}$ is a compact in Y. Let us consider now $B_{\tau-h,\tau}$ as an operator from Y to the Banach space $\mathscr{E}$ of continuous functions $[s,t] \to E$ with the norm $\sup_{\tau \in [s,t]} \|f_\tau\|_E$. By the Banach-Steinhaus theorem the norms $\|B_{\tau-h,\tau}\|_{\mathscr{L}(Y,\mathscr{E})}$ are bounded. This implies the uniform in $\tau \in [s,t]$ convergence to zero of $B_{\tau-h,\tau}U(\tau,t)x$, and therefore the convergence to zero of terms (3) and (4).

Thus, we have proved that $Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to U(s,t)x$ as $n \to \infty$ for each $x \in Y$ where Y is dense in E. Since the operators $Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n}$ are contractions, the convergence $Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to U(s,t)x$ holds for all $x \in E$ by the Banach-Steinhaus theorem. We proved the theorem for the case $s > S$.

Let us consider the case $s = S$. Fix an $x \in Y$. Let $s_N > s$ be a decreasing sequence of real numbers such that $\lim_{N \to \infty} s_N = s$. Consider a partition $\mathscr{P}_N = \{s_N < t_1 < \cdots < t_n = t\}$ of $[s_N, t]$. We have:

$$
\begin{aligned}
&\|Q_{s,s_N} Q_{s_N,t_1} \ldots Q_{t_{n-1},t_n} x - U(s,t)x\|_{\mathscr{L}(E)} \\
&\leqslant \|Q_{s,s_N}(Q_{s_N,t_1} \ldots Q_{t_{n-1},t_n} x - U(s_N,t)x)\|_{\mathscr{L}(E)} + \|(Q_{s,s_N} - U(s,s_N))U(s_N,t)x\|_{\mathscr{L}(E)}.
\end{aligned}
\tag{6}
$$

Let us prove that as $N \to \infty$, $(Q_{s,s_N} - U(s,s_N))U(s_N,t)x \to 0$. We have:

$$
\begin{aligned}
(Q_{s,s_N} - U(s,s_N))U(s_N,t)x &= \big(Q_{s_N-(s_N-s),s_N} - I\big)U(s_N,t)x \\
&\quad - \big(U(s_N - (s_N - s), s_N) - I\big)U(s_N,t)x.
\end{aligned}
\tag{7}
$$

We have proved that for every $x \in Y$, $B_{\tau-h,\tau} U(\tau,t)x$ converges to zero uniformly in $\tau \in [s,t]$. This implies that the both summands in (7) converge to zero as $N \to \infty$. Further note that as the mesh of $\mathscr{P}_N$ tends to zero,

$$
Q_{s_N,t_1} \ldots Q_{t_{n-1},t_n} x - U(s_N,t)x \to 0,
\tag{8}
$$

since we can repeat the argument that leads to estimates (3) and (4). To make our argument precise, we define $U(\tau - h, \tau) = U(s, \tau)$ and $Q_{\tau-h,\tau} = Q(s, \tau)$ if $\tau - h < s$. Next, since Q_{s,s_N} is a contraction, we conclude that the first summand in (6) converges to zero as the mesh $|\mathscr{P}_N|$ goes to zero. Thus for any $x \in Y$, the left-hand side of (6) converges to zero. By the Banach-Steinhaus theorem it converges to zero for all $x \in E$. The theorem is proved. $\quad\square$

COROLLARY 1. (The case of commuting generators) *Let A_t be a stable (see* [2]) *family of pairwise commuting generators of strongly continuous semigroups, and let Q_{t_1,t_2}, $t_1, t_2 > 0$, be a two-parameter family of contraction operators $E \to E$, such that Assumptions 1–5 of Theorem 1 are fulfilled. Then, for any subinterval $[s,t] \subset [S,T]$, for any sequence of partitions $\{s = t_0 < t_1 < \cdots < t_n = t\}$ of $[s,t]$ such that $\max(t_{j+1} - t_j) \to 0$ as $n \to \infty$, and for all $x \in E$,*

$$
Q_{t_0,t_1} \ldots Q_{t_{n-1},t_n} x \to e^{\int_s^t A_r dr} x, \quad n \to \infty.
$$

For the proof of Corollary 1 we will need Proposition 1 below (see [2], p.489 for details).

PROPOSITION 1. *Let $\{A_t\}$ be a stable family of pairwise commuting generators of strongly continuous semigroups. Let us assume that there exists a space $Y \subset \cap_{t \in [S,T]} D(A_t)$ which is dense in E, and let for all $y \in Y$, the mapping $[S,T] \to E, t \mapsto A_t y$ be continuous. Then, $(\int_s^t A_r dr, Y)$ is closable and its closure (which we still denote by $\int_s^t A_r dr$) is a generator. Moreover, the backward propagator with the left generator A_t takes the form:*

$$
U(s,t) = e^{\int_s^t A_r dr}.
$$

Proof of Corollary 1. Proposition 1 and Theorem 1 imply Corollary 1. $\quad\square$

3. Application to diffusions on manifolds

Let M be a d-dimensional compact Riemannian manifold isometrically embedded into a Euclidean space $\mathbb{R}^m$. Further let $\sigma(t)$ be a nongenerate matrix in $\mathbb{R}^m$. We assume that the map $[S,T] \to GL(m)$, $t \mapsto \sigma(t)$ is continuously differentiable, where $GL(m)$ denotes the space or real nongenerate matrices $m \times m$. Consider the transition density function

$$p(s,x,t,y) = \frac{\det \sigma(t)}{(2\pi(t-s))^{\frac{m}{2}}} \exp\left(-\frac{|\sigma(t)y - \sigma(s)x|^2_{\mathbb{R}^m}}{2(t-s)}\right). \tag{9}$$

One can easily verify that the non-homogeneous Markov process associated to (9) is $x + \sigma(t)^{-1}W_t$, where W_t is an $\mathbb{R}^m$-valued Brownian motion.

3.1. A short time asymptotic of a Gaussian-type integral operator

In this paragraph we obtain a short time asymptotic for the intergral of the form $\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz)$, where λ_M is the volume measure on M. Unlike the short time asymptotic of the same integral obtained in [3], we compute the coefficient at t precisely. In [3], the authors do not obtain the precise expression for this coefficient.

Let scal_M denote the scalar curvature, and Δ_M denote the Laplace-Beltrami operator on M.

PROPOSITION 2. *Let $g \in C^2(M)$. Then, there exist a constant K, a time t_0, and a function $R : [0,t_0] \times M \to \mathbb{R}$ satisfying $|R(t,y)| < Kt^{\frac{1}{2}}$ for all $y \in M$ and for all $t \in [0,t_0]$ such that*

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz) = g(y) - \frac{t}{2}\Delta_M g(y)$$

$$- g(y)\left(\frac{1}{6}\mathrm{scal}(y) + \frac{1}{16}\Delta_M\Delta_M |\cdot - y|^2\big|_y\right)t + tR(t,y) \quad (10)$$

for all $y \in M$ and for all $t \in [0,t_0]$.

Proof. Let ι be the isometrical embedding of M into $\mathbb{R}^m$. It is well known that $|\iota(z) - \iota(y)|^2 = d(y,z)^2 + \varphi(y,z)$, where d is a geodesic distance in M, and $\varphi(y,z) = O(d(y,z)^4)$. Let $U_y \subset M$ be a neighborhood of y, U be a neighborhood of zero. Let $\psi_y : U \to U_y$ be the diffeomorphism providing the normal coordinates in U_y, $f_y(x) = \varphi(y, \psi_y(x))$, $h_y(x) = \sqrt{\det g_{ij}(x)}\, g(\psi_y(x))$ where g_{ij} is the metric tensor. We have:

$$\int_{U_y} e^{-\frac{|z-y|^2}{2t}} g(z)\lambda_M(dz) = \int_{U_y} e^{-\frac{d(y,z)^2 + \varphi(y,z)}{2t}} g(z)\lambda_M(dz) = \int_U e^{-\frac{|x|^2 + f_y(x)}{2t}} h_y(x)dx.$$

By results of [3], there exist a function $\tilde{R}(t,\cdot)$ and a constant $\tilde{K}$ such that

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_U e^{-\frac{|x|^2 + f_y(x)}{2t}} h_y(x)dx = h_y(0) + \frac{t}{2}\Delta h_y(0) - \frac{t}{16}h_y(0)\Delta\Delta f_y(0) + t\tilde{R}(t,x), \quad (11)$$

and $|\tilde{R}(t,\cdot)| < \tilde{K}t^{1/2}$. By arguments of [3], the neighborhood $U \subset \mathbb{R}^d$ and the constant $\tilde{K}$ can be chosen the same for all $y \in M$. Note that $h_y(0) = g(y)$. Next, it was obtained in [3] that $\Delta h_y(0) = -\Delta_M u(y) - \frac{1}{3}u(y)\operatorname{scal}(y)$. Let us compute $\Delta\Delta f_y(0)$. Note that $\Delta\Delta d(y,\psi_y(x))^2 = \Delta\Delta|x|^2 = 0$. Hence,

$$\Delta\Delta f_y(0) = \Delta\Delta\big(|\iota \circ \psi_y(x) - \iota(y)|^2\big)|_{x=0} = \Delta_M\Delta_M \left. |\cdot - y|^2\right|_y.$$

Substitute the expressions for $\Delta h_y(0)$ and $\Delta\Delta f_y(0)$ into (11). Next, we need to estimate the integral $\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_{M\setminus U_y} g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz)$. Neighborhoods U_y can be choosen of the form $U_y = \{z \in M : |z-y| < \varepsilon_y\}$ where ε_y can be choosen bounded away from zero (see [3]), say, by ε. Let $t_0 > 0$ be small enough so that

$$\frac{1}{(2\pi t)^{\frac{d}{2}}}\int_{M\setminus U_y} g(z)e^{-\frac{|z-y|^2}{2t}}\lambda_M(dz) \leqslant \frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{\varepsilon^2}{2t}}\int_M |g(z)|\lambda_M(dz) < t^{3/2} \qquad (12)$$

for $t < t_0$. Estimate (12) and the choice of the function $\tilde{R}$ imply (10) with $R(t,y)$ satisfying $|R(t,y)| < Kt^{1/2}$, where the constant K does not depend on y. $\quad\square$

COROLLARY 2. *Let $g \in C^2(M)$. Then, there exist a constant K, a time t_0, and a function $\bar{R}: [0,t_0] \times M \to \mathbb{R}$ satisfying $|\bar{R}(t,x)| < Kt^{\frac{1}{2}}$ for all $x \in M$ and for all $t \in [0,t_0]$ such that for all $x \in M$, and for all $t \in [0,t_0]$,*

$$\frac{\int_M g(y)e^{-\frac{|y-x|^2}{2t}}\lambda_M(dy)}{\int_M e^{-\frac{|y-x|^2}{2t}}\lambda_M(dy)} = g(x) - \frac{t}{2}\Delta_M g(x) + t\bar{R}(t,x).$$

Proof. The statement of the corollary easily follows from Proposition 1 applied to the functions $g(y)$ and $g(y) \equiv 1$ respectively. $\quad\square$

3.2. A special case of Chernoff's theorem

Consider the integral operator $C(M) \to C(M)$:

$$(Q_{\tau-h,\tau}f)(x) = \frac{\int_M p(\tau-h,x,\tau,y)f(y)\lambda_M(dy)}{\int_M p(\tau-h,x,\tau,y)\lambda_M(dy)}. \qquad (13)$$

After introducing the notation

$$p^M(\tau-h,x,\tau,y) = \frac{\mathbb{I}_M(y)\,p(\tau-h,x,\tau,y)}{\int_M p(\tau-h,x,\tau,y)\lambda_M(dy)}$$

we can write (13) in the form:

$$(Q_{\tau-h,\tau}f)(x) = \int_M p^M(\tau-h,x,\tau,y)f(y)\lambda_M(dy). \qquad (14)$$

Consider the operator product:

$$
(Q_{t_0,t_1} Q_{t_1,t_2} \cdots Q_{t_{n-1},t_n} f)(x)
$$
$$
= \int_M p^M(t_0,x,t_1,x_1) \lambda_M(dx_1) \int_M p^M(t_1,x_1,t_2,x_2) \lambda_M(dx_2)
$$
$$
\cdots \int_M p^M(t_{n-1},x_{n-1},t_n,x_n) f(x_n) \lambda_M(dx_n). \quad (15)
$$

THEOREM 2. *Let the operator* $Q_{\tau-h,\tau}: C(M) \to C(M)$ *be defined by (14). Then, as the mesh of $\mathscr{P}$ tends to zero, the operator product defined by (15) converges at every point $f \in C(M)$ with respect to the norm of $C(M)$ to the backward propagator $U(s,t)$ whose left generator is given by*

$$
(A_t f)(x) = -\frac{1}{2} \Delta_{M_t} f_t(\sigma(t)x) + (\nabla_{M_t} f_t(\sigma(t)x), \sigma'(t)x)_{\mathbb{R}^m}, \quad (16)
$$

where $M_t = \sigma(t)M$, $f_t = f \circ \sigma(t)^{-1}$, $x \in M$.

Proof. Let us first show that the operators A_t generate a non-homogeneous diffusion on M. Note that M_t is also isometrically embedded into $\mathbb{R}^m$. The isometric embedding ι_t defines a metric tensor $\tilde{g}_{ij}(t,x) = \sum_\alpha \frac{\partial \iota_t^\alpha}{\partial x^i} \frac{\partial \iota_t^\alpha}{\partial x^j}(x)$ on M_t, and the Levi-Civita connection $\tilde{\Gamma}^i_{jk}(t,\cdot)$ of the metric $\tilde{g}_{ij}(t,\cdot)$. Let $f \in C^2(M)$ and $\tilde{x} = \sigma(t)x \in M_t$. Further let $\{\tilde{x}_i\}$ be local coordinates in a neighborhood U of $\tilde{x}$. We have:

$$
(A_t f)(x) = \tilde{g}^{ij}(t,\tilde{x}) \frac{\partial^2 f_t}{\partial \tilde{x}^i \partial \tilde{x}^j}(\tilde{x}) - \tilde{g}^{ij}(t,\tilde{x}) \tilde{\Gamma}^k_{ij}(t,\tilde{x}) \frac{\partial f_t}{\partial \tilde{x}^k}(\tilde{x}) + \frac{\partial f_t}{\partial \tilde{x}^i} \widetilde{(\sigma'(t)x)}^i
$$

where $\widetilde{(\sigma'(t)x)}^i$ are the coordinates, with respect to the basis $\frac{\partial}{\partial \tilde{x}^i}$, of the projection of the vector $\sigma'(t)x$ onto the tangent space $T_{\tilde{x}}(M_t)$. The matrix $\sigma(t)^{-1}$ can be regarded as a change of coordinates in U. Let $\{x_i\} = \sigma(t)^{-1}\{\tilde{x}_i\}$ be the new coordinates in U due to this change. Further let $g^{ij}(t,\cdot)$ and $\Gamma^k_{ij}(t,\cdot)$ denote the metric tensor and the Levi-Civita connection written in the coordinates $\{x_i\}$. Note that $\{x_i\}$ are also local coordinates in the neighborhood $\sigma(t)^{-1}U \subset M$ of the point $x \in M$. Also, g^{ij} and Γ^k_{ij} can be ragarded as the metric tensor and the Levi-Civita connection on M. Taking into account this, we obtain the following connection between g^{ij} and $\tilde{g}^{ij}$, Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$:

$$
\tilde{g}^{ij}(t,\tilde{x}) = g^{pq}(t,x) \sigma^i_p(t) \sigma^j_q(t),
$$
$$
\tilde{\Gamma}^k_{ij}(t,\tilde{x}) = \sigma^k_l(t) (\sigma^{-1})^p_i(t) (\sigma^{-1})^q_j(t) \Gamma^l_{pq}(t,x).
$$

Moreover, $\frac{\partial^2 f_t}{\partial \tilde{x}^i \partial \tilde{x}^j}(\tilde{x}) = \frac{\partial^2 f}{\partial x^k \partial x^l}(x)(\sigma^{-1})^k_i(\sigma^{-1})^l_j$ and $\frac{\partial f_t}{\partial \tilde{x}^k}(\tilde{x}) = \frac{\partial f}{\partial x^m}(x)(\sigma^{-1})^m_k$. This implies that

$$
(A_t f)(x) = g^{pq}(t,x) \frac{\partial^2 f}{\partial x^p \partial x^q}(x) - g^{pq}(t,x) \Gamma^k_{pq}(t,x) \frac{\partial f}{\partial x^k}(x) + \frac{\partial f}{\partial x^p} (\sigma'(t)x)^p \quad (17)
$$

where $(\sigma'(t)x)^p$ are the coordinates, with respect to the basis $\frac{\partial}{\partial x^p}$, of the projection of the vector $\sigma'(t)x$ onto the tangent space $T_x(M)$. The existence of a unique diffusion on $(M, g_{ij}(t,\cdot))$ generated by the time-inhomogeneous differential operator on the right-hand side of (17) is known (see, for example, [6]). Therefore the operator A_t defined by (16) generates a diffusion on M.

Let us show that Assumptions 1–5 of Theorem 1 are fulfilled. Note that all the generators A_t have the same domain $C^2(M)$. Therefore, the space Y can be taken to be the common domain $C^2(M)$. The norm in Y is the following: $\|x\|_Y = \|x\|_{C(M)} + \sup_{\tau \in [S,T]} \|A_\tau x\|_{C(M)}$. Let $f \in C^2(M)$, and let $u(s,x)$ be the solution to the following final value problem on $C(M)$:

$$\begin{cases} \frac{\partial u}{\partial s}(s,x) = -A_s u(s,x) \\ \lim_{s \uparrow t} u(s,x) = f(x). \end{cases} \tag{18}$$

Further let $P(s,x,t,A)$ be the transition probability function of the diffusion generated by A_s. Then, the backward propagator $U(s,t)$ can be expressed via $P(s,x,t,A)$:

$$(U(s,t)f)(x) = \begin{cases} \int_M P(s,x,t,dy)\,f(y), \ s < t, \\ f(x), \ s = t. \end{cases}$$

Moreover, $u(s,x) = (U(s,t)f)(x)$ (see [1]). Clearly, $u(s,\cdot) \in C^2(M)$, and therefore Assumption 2 is fulfilled. Next, it is known that $u \in C^{2,1}(M \times [S,t])$, which implies that the map $[S,t] \to C^2(M)$, $s \mapsto u(s,\cdot)$ is continuous. Therefore, Assumption 3 of Theorem 1 is also fulfilled. Assumption 4 is implied by (17) and the compactness of M.

Let us show now that Assumption 5 is fulfilled. Let $y_t = \sigma(t)y$, $x_s = \sigma(s)x$. Then

$$p(s,x,t,y) = \det\sigma(t)\,q(t-s,x_s,y_t)$$

where $q(\tau,x,y)$ is the Gaussian density with respect to the Lebesgue measure on $\mathbb{R}^m$. It is easy to verify that

$$\int_M p(s,x,t,y)f(y)\lambda_M(dy) = \det\sigma(t)\int_{M_t} q(t-s,x_s,y_t)\,f_t(y_t)\,\lambda_{M_t}(dy_t). \tag{19}$$

Using this formula and canceling the multiplier $e^{\frac{-|x_t-x_{t-\delta}|^2_{\mathbb{R}^m}}{2\delta}}$ in the numerator and the denominator of the fraction below we obtain:

$$\frac{\int\limits_M p(t-\delta,x,t,y)f(y)\lambda_M(dy)}{\int\limits_M p(t-\delta,x,t,y)\lambda_M(dy)} = \frac{\int\limits_{M_t} q(\delta,x_t,y_t)e^{-(y_t-x_t,\sigma'(t-\theta\delta)x)_{\mathbb{R}^m}}f_t(y_t)\lambda_{M_t}(dy_t)}{\int\limits_{M_t} q(\delta,x_t,y_t)e^{-(y_t-x_t,\sigma'(t-\theta\delta)x)_{\mathbb{R}^m}}\lambda_{M_t}(dy_t)}. \tag{20}$$

Multiplying the numerator and the denominator by $\int_{M_t} q(\delta, x_t, y_t) \lambda_{M_t}(dy_t)$, and then applying Corollary 2, we continue (20):

$$\int_M p^M(t - \delta, x, t, y) f(y) \lambda_M(dy)$$
$$= f(x) + \delta \frac{-\frac{1}{2} \Delta_{M_t} f_t(x_t) + (\nabla_{M_t} f_t(x_t), \sigma'(t - \theta\delta)x)_{\mathbb{R}^m} - f(x)\tilde{R}(t, \delta) + \bar{R}(t, \delta)}{1 - \frac{\delta}{2} \Delta_{M_t} e^{-(y_t - x_t, \sigma'(t - \theta\delta)x)_{\mathbb{R}^m}} \big|_{y_t = x_t} + \delta \tilde{R}(t, \delta)}$$

where $\theta \in (0, 1)$ is the number satisfying $\sigma(t)x - \sigma(t - \delta)x = \delta \sigma'(t - \theta\delta)x$, and the functions $\bar{R}(t, \delta)$ and $\tilde{R}(t, \delta)$ are the higher-order terms that appear in the numerator and the denominator of (20) after applying Corollary 2. The term $(\nabla_{M_t} f_t(x_t), \sigma'(t - \theta\delta)x)_{\mathbb{R}^m}$ appears after computing

$$\nabla_{M_t} e^{-(y_t - x_t, \sigma'(t - \theta\delta)x)_{\mathbb{R}^m}} \Big|_{y_t = x_t} = -\text{Pr}_{T_{x_t}(M_t)} \sigma'(t - \theta\delta)x$$

where $\text{Pr}_{T_{x_t}(M_t)}$ denotes the projection onto the tangent space $T_{x_t}(M_t)$. Due to the continuity of the map $t \mapsto \sigma'(t)x$, $\sigma'(t - \theta\delta)x$ converges to $\sigma'(t)x$ uniformly in t as $\delta \to 0$. Also, as $\delta \to 0$, $\frac{\delta}{2} \Delta_{M_t} e^{-(y_t - x_t, \sigma'(t - \theta\delta)x)_{\mathbb{R}^m}} \big|_{y_t = x_t}$ converges to zero uniformly in t by boundedness of the second multiplier. Therefore, to show that Assumption 5 is fulfilled we have to prove that $\bar{R}(\delta, t)$ and $\tilde{R}(t, \delta)$ tend to zero uniformly in t as $\delta \to 0$. We prove it for the function $\bar{R}(t, \delta)$. In the proof of Proposition 2 we considered the neighborhoods $U_y = \{z \in M, |z - y| < \varepsilon_y\}$ where the normal coordinates can be introduced. Moreover ε_y is bounded away from zero by ε as $y \in M$ varies. Let $U_{y_t} = \sigma(t)U_y$, where $y_t = \sigma(t)y$, and $U_t = \sigma(t)U$. Clearly, the exponential map $\exp : U_t \to U_{y_t}$ is well-defined, and therefore we can introduce normal coordinates in U_{y_t}. Let $\varepsilon_{y_t} = \inf\{|z - y_t|, z \in U_{y_t}\}$. Due to the continuity of the map $t \mapsto \sigma(t)$, ε_{y_t} are bounded away from zero, say, by ε, as t runs over $[S, T]$ and y runs over M, i.e. when y_t runs over $\cup_{\tau \in [S,T]} M_\tau$. This and estimate (12) imply that

$$\frac{1}{(2\pi\tau)^{\frac{d}{2}}} \int_{M_t \setminus U_{y_t}} f_t(z) e^{-\frac{|z - y_t|^2}{2\tau}} \lambda_{M_t}(dz) < \tau^{3/2}.$$

Next, we have to analyze the higher-order term in every neighborhood $U_t = \exp^{-1} U_{y_t}$. We use the estimate of this term obtained in [3] (Lemma 2). All multipliers in the function estimating the higher-order term as well as the integral over $\mathbb{R}^m \setminus U_t$ are continuous in $t \in [S, T]$. This proves that $\bar{R}(t, \delta)$ is bounded by $K\delta^{\frac{1}{2}}$ where K does not depend on t. Now the statement of the theorem follows from Theorem 1. $\quad\square$

3.3. Surface measure generated by an M-valued non-homogeneous diffusion

Let us discuss now a probabilistic interpretation of Theorem 2. Let W_ξ, $\xi \in [\tau - h, \tau]$, be an $\mathbb{R}^m$-valued Brownian motion starting at 0, and let $\mathbb{W}_\sigma^x$ be the distribution of the process $x + \sigma(\xi)^{-1} W_\xi$. Further let $U_\varepsilon(M)$ denote the ε-neighborhood of M,

and $g : C([\tau - h, \tau], \mathbb{R}^m) \to \mathbb{R}$ be a Borel measurable function. The right-hand side of the equality

$$\int_{C([\tau-h,\tau],\mathbb{R}^m)} g(\omega) \, \mathbb{W}^x_{\varepsilon,\tau}(d\omega) = \frac{\int_{C([\tau-h,\tau],\mathbb{R}^m)} \mathbb{I}_{\{\omega:\omega(\tau)\in U_\varepsilon(M)\}} \, g(\omega) \, \mathbb{W}^x_\sigma(d\omega)}{\mathbb{W}^x_\sigma\{\omega : \omega(\tau) \in U_\varepsilon(M)\}}$$

defines a probability distribution $\mathbb{W}^x_{\varepsilon,\tau}$. Clearly, $\mathbb{W}^x_{\varepsilon,\tau}$ is supported on $C([\tau - h, \tau], \mathbb{R}^m)$. The diffusion associated with $\mathbb{W}^x_{\varepsilon,\tau}$ is a time-inhomogeneous Markov process that starts at $x \in M$ at time $\tau - h$, and is conditioned to come to the neighborhood $U_\varepsilon(M)$ at time τ. The transition probability function $P_{\tau-h,\tau}(x, \cdot)$ defined via $\mathbb{W}^x_{\varepsilon,\tau}$, i.e. $P_{\tau-h,\tau}(x,A) = \mathbb{W}^x_{\varepsilon,\tau}(\omega : \omega(\tau) \in A)$, possesses the density

$$p_\varepsilon(\tau - h, x, \tau, y) = \frac{\mathbb{I}_{U_\varepsilon(M)}(y) \, p(\tau - h, x, \tau, y)}{\int_{U_\varepsilon(M)} p(\tau - h, x, \tau, y) dy}.$$

As ε tends to zero, $p_\varepsilon(\tau - h, x, \tau, y) \, dy$ converges weakly relative to the family of bounded continuous functions to $p^M(\tau - h, x, \tau, y) \lambda_M(dy)$. The latter function defines a probability distribution on the algebra of cylindric subsets of $C([\tau - h, \tau], \mathbb{R}^m)$. The Markov process associated to this probability distribution starts at $x \in M$ at time $\tau - h$, and is conditioned to return to M at time τ. Consider a partition $\mathscr{P} = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of an interval $[s,t] \subset [S,T]$, and think of a Markov process $X^\mathscr{P}_t$ that starts at $x \in M$ at time s and is conditioned to return to M at all times $t_i \in \mathscr{P}$. Let $t_{i-1} \leqslant r < \tau \leqslant t_i$. If $\tau = t_i$ then the transition probability function $P^\mathscr{P}(r, z, \tau, \cdot)$ of $X^\mathscr{P}_t$, considered as a measure, is concentrated on M and $p^M(r, z, t_i, y)$ is its density with respect to the measure λ_M. Moreover, the latter holds also if $t_{i-1} < r$. If $\tau < t_i$ then $P^\mathscr{P}(r, z, \tau, \cdot)$ is a distribution on the enveloping space $\mathbb{R}^m$. The conditional probability argument implies that $P^\mathscr{P}(r, z, \tau, \cdot)$ has the density with respect to the Lebesgue measure on $\mathbb{R}^m$:

$$p^\mathscr{P}(r, z, \tau, y) = \frac{p(r, z, \tau, y) \int_M p(\tau, y, t_i, \bar{x}) \lambda_M(d\bar{x})}{\int_M p(r, z, t_i, \bar{x}) \lambda_M(d\bar{x})}. \tag{21}$$

Now let $t_{i-1} \leqslant r < t_i < t_{j-1} < \tau < t_j$. In this case the density of $P^\mathscr{P}(z, r, \tau, \cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^m$ is given by

$$p^\mathscr{P}(r, z, \tau, y) = \int_M p^M(r, z, t_i, x_i) \lambda_M(dx_i) \int_M p^M(t_i, x_i, t_{i+1}, x_{i+1}) \lambda_M(dx_{i+1})$$

$$\cdots \int_M p^M(t_{j-2}, x_{j-2}, t_{j-1}, x_{j-1}) \, p^\mathscr{P}(t_{j-1}, x_{j-1}, \tau, y) \lambda_M(dx_{j-1}). \tag{22}$$

COROLLARY 3. *As the mesh of $\mathscr{P}$ tends to zero, the finite-dimensional distributions of the process $X^\mathscr{P}_t$ converge weakly to the finite-dimensional distributions of the M-valued diffusion X_t genarated by A_t.*

REMARK 1. The disribution of the process X_t van be regarded as a surface measure on $C([S,T], M)$ while the distributions of the processes $X^\mathscr{P}_t$ can be regarded as

"volume" measures on $C([S,T],\mathbb{R}^m)$. These "volume" measures are concentrated on certain neigborhoods of $C([S,T],M)$ which can be associated with "ε-layers", Corrolary 3 states the convergence of the distributions of $X_t^{\mathscr{P}}$ ("volume" measures) to a surface measure on $C([S,T],M)$, where the latter is the distribution of the diffusion X_t generated by A_t.

Proof. We have to prove that for any partition $s < \tau_1 < \cdots < \tau_k < t$ and for any bounded continuous function $f : \mathbb{R}^k \to \mathbb{R}$,

$$\mathbb{E}\big[f(X_{\tau_1}^{\mathscr{P}}, \ldots, X_{\tau_k}^{\mathscr{P}})\big] \to \mathbb{E}\big[f(X_{\tau_1}, \ldots, X_{\tau_k})\big] \quad \text{as } |\mathscr{P}| \to 0.$$

First we consider only those partitions $\mathscr{P}$ that contain all the points τ_i, $1 \leqslant i \leqslant k$. Pick up two subsequent points τ_i and τ_{i+1}. Let $t_l \in \mathscr{P}$ and $t_m \in \mathscr{P}$ be such that $t_l = \tau_i$ and $t_m = \tau_{i+1}$. Then,

$$p^{\mathscr{P}}(\tau_i, z, \tau_{i+1}, y) = \int_M p^M(\tau_i, z, t_{l+1}, x_{l+1}) \lambda_M(dx_{l+1})$$

$$\ldots \int_M p^M(t_{m-2}, x_{m-2}, t_{m-1}, x_{m-1}) \, p^M(t_{m-1}, x_{m-1}, \tau_{i+1}, y) \lambda_M(dx_{m-1}). \quad (23)$$

Now let $f \in C(M^k)$, and let $P(s,x,t,A)$ be the transition probability function of the process X_t on M generated by A_t. We have:

$$\int_M p^{\mathscr{P}}(s, x, \tau_1, x_1) \lambda_M(dx_1) \ldots \int_M p^{\mathscr{P}}(\tau_{k-1}, x_{k-1}, \tau_k, x_k) f(x_1, \ldots, x_k) \lambda_M(dx_k)$$

$$- \int_M P(s, x, \tau_1, dx_1) \ldots \times \int_M P(\tau_{k-1}, x_{k-1}, \tau_k, dx_k) f(x_1, \ldots, x_k)$$

$$= \sum_{i=1}^{k-1} \int_M p^{\mathscr{P}}(s, x, \tau_1, x_1) \lambda_M(dx_1) \ldots \int_M p^{\mathscr{P}}(\tau_{i-1}, x_{i-1}, \tau_i, x_i) \lambda_M(dx_i)$$

$$\left[\int_M p^{\mathscr{P}}(\tau_i, x_i, \tau_{i+1}, x_{i+1}) \lambda_M(dx_{i+1}) - P(\tau_i, x_i, \tau_{i+1}, dx_{i+1}) \right]$$

$$\int_M P(\tau_{i+1}, x_{i+1}, \tau_{i+2}, dx_{i+2}) \ldots \int_M P(\tau_{k-1}, x_{k-1}, \tau_k, dx_k) f(x_1, \ldots, x_k) \lambda_M(dx_k). \quad (24)$$

Each term of this sum converges to zero as the mesh $|\mathscr{P}|$ goes to zero. Indeed, for every i, $1 \leqslant i < k$, for every function $g \in C(M^{i+1})$, the difference

$$\int_M \big(p^{\mathscr{P}}(\tau_i, \tilde{x}_i, \tau_{i+1}, x_{i+1}) \lambda_M(dx_{i+1}) - \int_M P(\tau_i, \tilde{x}_i, \tau_{i+1}, dx_{i+1})\big) g(x_1, \ldots, x_{i+1}) \quad (25)$$

converges to zero. This follows from Theorem 2. Indeed, the second term in (25) is the backward propagator with the left generator A_t, and the first term is the operator product (15). The convergence in (25) holds in $C(M^{i+1})$ by the argument of Theorem 2. The latter argument has to be applied to operators $C(M^{i+1}) \to C(M^{i+1})$ and with respect to the norm of $C(M^{i+1})$ instead of $C(M)$, as in Theorem 2, which, however, leaves the proof of Theorem 2 without changes.

Now let us assume that an infinite number of partitions $\mathscr{P}$ with the meshes decreasing to zero do not include some of the points τ_i. Then, instead of formula (23) for $p^{\mathscr{P}}$ we have to use formula (22). We would like to reduce this case to the previous one, i.e. when all the points τ_i are always among the partition points of $\mathscr{P}$. For this purpose we have to analyze the expression:

$$\int_{\mathbb{R}^m} p^{\mathscr{P}}(t_i, x_i, \tau, y)\, dy \int_M p^M(\tau, y, x_{i+1}, t_{i+1}) f(z, y, x_{i+1}) \lambda_M(dx_{i+1}), \qquad (26)$$

where the variable x_{i+1} comes from the subsequent integrals as the result of their replacement with the backward propagator. The variables z and y come from the original integrand function, τ is one of the points τ_i, $1 \leqslant i \leqslant k$, and the partition points t_i and t_{i+1} are choosen such that $t_i < \tau < t_{i+1}$. We would like to show that as $t_i, t_{i+1} \to \tau$, the difference between (26) and $\int_M p^M(t_i, x_i, t_{i+1}, x_{i+1}) f(z, x_i, x_{i+1}) \lambda_M(dx_{i+1})$ converges to zero. By the Banach-Steinhaus theorem, it suffices to prove this when f is continuously differentiable with respect to y. Applying formula (21) we observe that (26) equals to

$$\frac{\int_{\mathbb{R}^m} dy\, p(t_i, x_i, \tau, y) \int_M p(\tau, y, t_{i+1}, x_{i+1}) f(z, y, x_{i+1}) \lambda_M(dx_{i+1})}{\int_M p(t_i, x_i, t_{i+1}, \bar{x}_{i+1}) \lambda_M(d\bar{x}_{i+1})}. \qquad (27)$$

Applying formula (19), for the numerator of (27) we obtain:

$$\int_{\mathbb{R}^m} dy_\tau\, q(\tau - t_i, x_{t_i}, y_\tau) \int_{M_{t_{i+1}}} q(t_{i+1} - \tau, y_\tau, x_{t_{i+1}}) f_{\tau, t_{i+1}}(z, y_\tau, x_{t_{i+1}}) \lambda_{M_{t_{i+1}}}(dx_{t_{i+1}}) \qquad (28)$$

where $f_{\tau, t_{i+1}}(z, \cdot, \cdot) = f\big(z, \sigma(\tau)^{-1}(\cdot), \sigma(t_{i+1})^{-1}(\cdot)\big)$, $x_{t_i} = \sigma(t_i) x_i$, $y_\tau = \sigma(\tau) y$, $x_{t_{i+1}} = \sigma(t_{i+1}) x_{i+1}$. Also, in (28) we omitted the multiplier $\det \sigma(t_{i+1})$ which will be taken into consideration later again. Application of Taylor's formula to $f_{\tau, t_{i+1}}(z, \cdot, x_{t_i})$ at point x_{t_i} gives:

$$f_{\tau, t_{i+1}}(z, y_\tau, x_{t_{i+1}}) = f_{\tau, t_{i+1}}(z, x_{t_i}, x_{t_{i+1}}) + \partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})(y_\tau - x_{t_i}) \qquad (29)$$

where $p(x_{t_i}, y_\tau)$ is a point on the segment $[x_{t_i}, y_\tau]$ and ∂_2 means partial differentiation with respect to the second argument. If we substitute the first summand of (29) into (28) as an integrand, we obtain:

$$\int_{M_{t_{i+1}}} q(t_{i+1} - t_i, x_{t_i}, x_{t_{i+1}}) f_{\tau, t_{i+1}}(z, x_{t_i}, x_{t_{i+1}}) \lambda_{M_{t_{i+1}}}(dx_{t_{i+1}}).$$

Further, this substitution brings (27) to

$$\int_M p^M(t_i, x_i, t_{i+1}, x_{i+1}) f(z, \sigma(\tau)^{-1} \sigma(t_i) x_i, x_{i+1}) \lambda_M(dx_{i+1}).$$

The latter converges to $\int_M p^M(t_i, x_i, t_{i+1}, x_{i+1}) f(z, x_i, x_{i+1}) \lambda_M(dx_{i+1})$. Thus, we have to prove that

$$\frac{\det \sigma(t_{i+1})}{\int_M p(t_i, x_i, t_{i+1}, x_{i+1}) \lambda_M(dx_{i+1})} \int_{\mathbb{R}^m} dy_\tau\, q(\tau - t_i, x_{t_i}, y_\tau) \qquad (30)$$

$$\times \int_{M_{t_{i+1}}} q(t_{i+1} - \tau, y_\tau, x_{t_{i+1}})\, \partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})(y_\tau - x_{t_i}) \lambda_{M_{t_{i+1}}}(dx_{t_{i+1}})$$

converges to zero as $t_i, t_{i+1} \to \tau$. We change the order of integration in (30) and split the integral with respect to y_τ, taken over $\mathbb{R}^m$, into two: over the set $\{y_\tau : |y_\tau - x_{t_i}| < (\tau - t_i)^{\frac{1}{3}}\}$ and over its complement $\{y_\tau : |y_\tau - x_{t_i}| \geqslant (\tau - t_i)^{\frac{1}{3}}\}$. Estimation of the first term gives:

$$\left| \int_{\{|y_\tau - x_{t_i}| < (\tau - t_i)^{\frac{1}{3}}\}} dy_\tau \, q(\tau - t_i, x_{t_i}, y_\tau) \, q(t_{i+1} - \tau, y_\tau, x_{t_{i+1}}) \right.$$

$$\left. \times \partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})(y_\tau - x_{t_i}) \right| \tag{31}$$

$$\leqslant \sup_{x_i, x_{i+1} \in M, y \in U_\varepsilon(M), z \in K} |\partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})| \, q(t_{i+1} - t_i, x_{t_i}, x_{t_{i+1}}) (\tau - t_i)^{\frac{1}{3}}$$

where K is a compact, since without loss of generality we can consider that the totality of variables z belongs to a compact K. Indeed, the integrand $\partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})$ $(y_\tau - x_{t_i})$ is bounded by (29). The preceding integration w.r.t. each variable from the totality z is either taken over the manifold M or similar to the integration w.r.t. y in (26). In the latter case we can replace the integrals over $\mathbb{R}^m$ with integrals over compact neighborhoods of M because the integrals over the complements of these neighborhoods tend to zero as the mesh $|\mathscr{P}|$ goes to zero. Further, since y_τ is always in the $(\tau - t_i)^{\frac{1}{3}}$-neighborhood of M_{t_i}, then y is always in some ε-neighborhood of M. Therefore, the supremum in the last line of (31) is finite. Hence, the summand in (30) which corresponds to the integration over $\{y_\tau : |y_\tau - x_{t_i}| < (\tau - t_i)^{\frac{1}{3}}\}$ is bounded by

$$\sup |\partial_2 f_{\tau, t_{i+1}}| (\tau - t_i)^{\frac{1}{3}} \int_M p^M(t_i, x_i, t_{i+1}, x_{i+1}) \lambda_M(dx_{i+1})$$

which tends to zero as $t_i, t_{i+1} \to \tau$. The other summand, which corresponds to the integration over $\{y_\tau : |y_\tau - x_{t_i}| \geqslant (\tau - t_i)^{\frac{1}{3}}\}$, is bounded by

$$\frac{1}{(2\pi(\tau - t_i))^{\frac{m}{2}}} e^{-\frac{1}{2(\tau - t_i)^{\frac{1}{3}}}} \int_{M_{t_{i+1}}} \lambda_{M_{t_{i+1}}}(dx_{i+1})$$

$$\times \int_{\mathbb{R}^m} dy_\tau |\partial_2 f_{\tau, t_{i+1}}(z, p(x_{t_i}, y_\tau), x_{t_{i+1}})(y_\tau - x_{t_i})| q(t_{i+1} - \tau, y_\tau, x_{t_{i+1}})$$

which also tends to zero as $t_i, t_{i+1} \to \tau$ since the product of the partial derivative $\partial_2 f_{\tau, t_{i+1}}$ and $(y_\tau - x_{t_i})$ is bounded by formula (29). The multiplier in front of the integrals in (30) also converges to zero as $|\mathscr{P}| \to 0$, and therefore the convergence of (30) to zero as $t_i, t_{i+1} \to \tau$ is proved. $\square$

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