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A REVIEW OF A RIESZ BASIS PROPERTY FOR INDEFINITE STURM–LIOUVILLE PROBLEMS

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Abstract. For an indefinite weight function r on $[-1,1]$ with $xr(x)>0$ we consider connections between a Riesz basis property of the indefinite Sturm-Liouville eigenvalue problem

$$-y''=\lambda ry, \qquad y(-1)=y(1)=0$$

and various different conditions, for example HELP-type inequalities

$$\left(\int_0^1 |h'|^2\frac{1}{r}\,dx\right)^2 \leqslant k\left(\int_0^1 |h|^2 dx\right)\left(\int_0^1 \left|\left(\frac{h'}{r}\right)'\right|^2 dx\right)$$

for certain classes of functions h on $[0,1]$. We show that for so-called strongly odd dominated functions r (including odd r) these problems are equivalent. This allows us to apply known results from the theory of one problem to the others.

1. Introduction

We shall consider the Sturm-Liouville problem

$$-y''=\lambda ry \text{ a.e. on } (-1,1), \quad y(\pm 1)=0 \tag{1.1}$$

where $r\in L_1(-1,1)$ and

$$xr(x)>0. \tag{1.2}$$

The aim is to review some of the conditions on r known to be necessary or sufficient (or both) for the existence of a Riesz basis of eigenfunctions in the weighted space $H:=L_{2,|r|}(-1,1)$ with inner product

$$(y,z)=\int_{-1}^1 |r|y\bar{z}. \tag{1.3}$$

We refer to this as the Riesz basis property (RBP), and we shall explore the connections between it and the conditions mentioned above, and also make some extensions to more general conditions on r.

Recall that a Riesz basis is the image under a linear operator U, say, of an orthonormal basis, where U is (a) bounded and (b) boundedly invertible. At times it will also be useful to consider the related problem

$$-z''=\mu|r|z \text{ a.e. on } (-1,1), \quad z(\pm 1)=0. \tag{1.4}$$

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Even for this right definite problem, Riesz bases can play a rôle. Suppose for the moment that $r, \frac{1}{r} \in L_\infty(-1,1)$ (so H is equivalent to $L_2(-1,1)$ in this case) and write (1.4) in the form

$$Lz = \mu\,|R|\,z, \tag{1.5}$$

where $-L$ is the Dirichlet Laplacian and R is multiplication by r, both operators acting in $L_2(-1,1)$. Since $|R|^{-1}L$ is self-adjoint with compact inverse in H, (1.5) admits an orthonormal basis of eigenfunctions $z_1, z_2, \ldots$ in H. Thus, for $U = |R|^{1/2}$, Uz_n are orthonormal in $L_2(-1,1)$. Since U satisfies (a) and (b) above, z_n form a Riesz basis of eigenfunctions for (1.4) in the unweighted space $L_2(-1,1)$.

We remark that the above U is also (c) positive definite, and conditions like (a), (b) and (c) will play a role for problems with indefinite r below. Expansions using eigenfunction bases for indefinite weight Sturm-Liouville problems were already investigated by Hilbert over a century ago in a space denoted below by H_1. The impetus for considering the larger space H instead (at least in indefinite cases) seems to have come initially from two-way processes in particle physics and probability, related to "half-range" expansions.

While such processes have a long history, there were several mathematical investigations in the 1960s and 1970s. For example, Case [11], Hangelbroek [23] and others discussed integrodifferential equations with an indefinite time derivative from transport theory, and we cite Baouendi and Grisvard [5] and Pagani [29] for existence and uniqueness of forward-backward parabolic equations. The last two papers correspond (after separation of variables) to equations including (1.1) with $r(x) = x$ (where there are many applications), and

$$r(x) = r_\alpha(x) := \operatorname{sgn}(x)\,|x|^\alpha, \tag{1.6}$$

respectively. Several works on (1.1) and related equations were published in the period 1975–1985, particularly by Beals, Kaper and their collaborators. We cite [3] with several applications for (an abstract version of) $r = r_\alpha$, [26] for a treatment of (1.1) with $r(x) = x$ via properties of Airy functions, and [4] for cases with $r = fr_\alpha$ where $f \in C^1$. In these works the equivalence of two norms on H_1 played a fundamental rôle, and we return to this aspect later.

Our review in Section 2 takes the study of conditions for the RBP from the position it reached with [4] up to the present. We split the conditions into five groups, 2.1 being devoted to conditions which are equivalent to the RBP for abstract problems. Explicitly these began (to our knowledge) with Pyatkov's work [32] using interpolation spaces, but there were several earlier conditions implicit in [12]. Subsequently Ćurgus and Najman [14] and Volkmer [36] gave several further conditions independently. In 2.2 we examine generalised Beals conditions on the weight function r which are sufficient for the RBP, in [4, 13] using conditions generalising (1.6) on both sides of the turning point $x = 0$, and in [18, 19, 36] where a one-sided condition suffices. A necessary condition, which we denote by V, was given in [36] when $r \in L_\infty$, and is examined in 2.3 along with further conditions based on V from [2, 9]. In 2.4 we discuss some versions of the so-called HELP inequality. This has a long history but the works most relevant to our topic are by Evans and Everitt [15, 16] and Bennewitz [6]. In [36]

Volkmer showed that a particular version of the HELP inequality was necessary for the RBP if r was L_∞ and odd. Finally in 2.5 we discuss Parfenov's developments [30] of a sufficient condition from [32], leading to some conditions which are necessary as well, at least for odd r.

In Section 3, we shall review and extend some connections between the groups of conditions in 2.1–2.5. At the same time, we also relax the assumptions on r from L_∞ to L_1 and from oddness to a strongly odd dominated (SOD) condition introduced in [9]. The net result is that we obtain equivalence of conditions from all the groups except 2.2. For the latter we give two new conditions, one necessary and one sufficient for the RBP, and we obtain equivalence if r satisfies an extra limit condition at the turning point. These equivalences are summarised in 3.5. We conclude in Section 4 with a discussion of our results together with some examples illustrating the theory and its limitations.

Before proceeding, we wish to point out that several works in the area (including some cited above and below) deal with more general problems than (1.1) (e.g., the differential expression could contain extra coefficients [4], be of higher order [13] or in higher dimensions [14, 32], or contain measures [19, 31]; the boundary conditions could be more general [7, 8], there could be several turning points [13, 33], etc.) A theory for these settings equivalent to the one presented in Section 3 would require additional research, and some of the above references already give counterexamples limiting the possibilities. We have focussed instead on some specific properties of r for (1.1), which allows comparison of much of the literature.

NOTATION. As above, H is the weighted Hilbert space $L_{2,|r|}(-1,1)$ with inner product from (1.3), and we denote the corresponding norm by $\|\cdot\|$. Equation (1.4) can be written in the form $Az = \mu z$ where A is a (self-adjoint) operator in H. Here $Az = -\frac{1}{|r|}z''$ on a domain consisting of those z such that $z(\pm 1) = 0$ with z, z' absolutely continuous (AC) on $[-1,1]$ and $\frac{1}{r}z'' \in H$. Similarly, (1.1) can be written $JAy = \lambda y$ where J is the (self-adjoint) operator of multiplication by sgn r in H. It is well known that A is positive and we can define (as in [14]) a scale of Hilbert spaces $H_s(-2 \leqslant s \leqslant 2)$ in the following way. For $s \geqslant 0$ we take $D(A^{s/2})$ with norm given by $\|y\|_s := \|A^{s/2}y\|$, and for $s < 0$ we take the completion of H with respect to $\|.\|_s$ as above. In all cases we denote the corresponding inner product by $(.,.)_s$. Note that $H_0 = H$, while H_1 and H_2 are the form and operator domains of A.

Finally, if $\mathbf{C}$ denotes a condition on $r:(-1,1) \to \mathbb{R}$ then $\mathbf{C}+$ (resp. $\mathbf{C}-$) denotes the same condition with r restricted to $(0,1)$ (resp. (-1,0)).

2. Groups of known conditions for the Riesz basis property

In this section we give an overview of different groups of conditions which are known to be necessary and/or sufficient for the RBP (at least under some additional assumptions on r).

2.1. Necessary and sufficient conditions on an abstract level

We shall treat some of the conditions in this subsection somewhat informally, since (being equivalent to the RBP) they will not be needed explicitly in what follows. They were mostly formulated for abstract equations, with applications to (1.1) that were in some ways more general (e.g., in higher dimensions), but sometimes with restrictions to $r \in L_\infty$, $\frac{1}{r} \in L_\infty$, or both. Our aim is to give the reader a flavour, without full details and definitions, of some of the different areas that have been visited in the search for equivalent statements.

The first condition that we know to have been explicitly formulated as equivalent to the RBP was by Pyatkov [32] as follows:

(i) $[H_1, H_{-1}]_{1/2} = H$.

Here H, H_s come from Section 1 and $[X,Y]_t$ denotes (complex) interpolation between the spaces X and Y. Interpolation is discussed in books and [32] refers to [35] for this topic. We note here only that there are other (real) methods of interpolation which are related, and, for example, H_s can be obtained as $[H_1, H]_{1-s}$ or via the real methods as $(H, H_1)_{s,2}$.

Later Ćurgus and Najman [14] gave several conditions equivalent to the RBP, including versions of (i) with H_1, H_{-1} replaced by H_s, H_{-s} for $0 < s \leqslant 2$. Another of their conditions can be expressed in the form

(ii) infinity is a regular critical point of JA,

where J and A are as in Section 1. Regularity of critical points is also discussed in books and [14] refers to [1, 10, 28] for this topic. We note here, however, that (ii) concerns the spectral measure E (also called spectral function) of JA in the Krein space $K = (H, [.,.])$, with indefinite inner product $[y,z] = \int_{-1}^{1} ry\bar{z}$, the topology being that of H. For a given eigenvalue λ of (1.1), $E(\{\lambda\})$ is a K-orthogonal projector taking the sign of λ. Since the eigenvalues accumulate at $\pm\infty$, E changes sign at ∞, which is therefore a critical point. The values of E (on, say, two-sided neighbourhoods of ∞) are not in general H-orthogonal projectors, however, and (ii) corresponds to the case when these projectors remain bounded.

We remark that Ćurgus already gave a list of conditions (not including the RBP) equivalent to (ii) in [12]. In view of [14], then, Ćurgus's list implicitly contains the RBP, and also predates [32]. Another condition from [12] is

(iii) JA is similar to a self-adjoint operator in H.

Indeed, since (1.1) leads to a discrete spectrum, the similarity transformation can be taken as the operator U in the definition of Riesz basis in Section 1, but (iii) (and the other conditions above) also apply in more general cases (and then Riesz bases must be interpreted in terms of spectral measures – cf. [13, Remark 4.3]). We note that (iii) has also been studied in its own right – see [27] for a recent contribution.

Ćurgus [12] gave another condition involving (a), (b) and (c) used for U in Section 1, but in the Krein space K instead of H. Of course the meaning of (a) and (b) is unchanged since the H and K topologies coincide, but (c) must be taken in K. The statement of the condition is

(iv) there exists an operator W on K satisfying (a), (b) and (c) from Section 1, and with H_2 as an invariant subspace.

There are also various modifications in [12], [14] and [36] involving different spaces. For example, W could be bounded in H_s and/or have H_s as an invariant subspace for $0 < s \leqslant 2$.

The final necessary and sufficient condition that we shall consider is as follows. We follow the construction in [36], noting that equivalent versions can be found in [12] and [14]. Let T denote the absolute value of the self-adjoint operator $(JA)^{-1}$ in H_1. With the norm on H_1 given by $\|y\|_T := \|T^{1/2}y\|_1$ consider the condition

(v) the norms $\|.\|_T$ and $\|.\|$ are equivalent on H_1, i.e., for some $c > 0$,

$$c^{-1}\|y\| \leqslant \|y\|_T \leqslant c\|y\| \text{ for all } y \in H_1. \tag{2.1}$$

We remark that this formed a key tool in the earlier approaches of Beals, e.g., [3, 4] and Kaper et al, e.g., [26] to establish the RBP under specific conditions on r mentioned in Section 1. For $r \in L_\infty$, Volkmer [36] showed that the separate inequalities in (2.1) were each equivalent to the RBP.

2.2. Beals-type sufficient conditions

Our starting point is the following modification of (1.6):

(B) there are $\alpha > -\frac{1}{2}$, $\varepsilon > 0$, $f \in C^1[-\varepsilon, \varepsilon]$ with $f(0) \neq 0$ such that

$$r(x) = \operatorname{sgn}(x)\, |x|^\alpha f(x) \text{ for all } x \in (-\varepsilon, \varepsilon). \tag{2.2}$$

This was employed by Beals [4] as a sufficient condition for the existence of (full- and half-range) bases for (1.1). There have been various generalisations. For example Beals [4] already mentioned the possibility of using two functions $f_\pm$ replacing f for $\pm x > 0$, and in [13] Ćurgus and Langer extended this to allow the possibility of two powers $\alpha_\pm > -1$ as well. They also showed that these conditions produced Riesz bases. The later investigations [18] and [36] showed independently that such conditions sufficed from one side $x > 0$ or $x < 0$, leading to so-called one-sided Beals conditions **(B$\pm$)**.

Various authors have also examined the proof of [4, Lemma 1], to find weaker implicit conditions. These are sometimes called generalized Beals conditions, for example

(GB+) for some $\varepsilon, \mu > 0$ there is $g \in C^1[0, \varepsilon]$ such that

$$g(x) = \frac{r(x)}{r(\mu x)} \quad \text{for all} \quad x \in (0, \varepsilon) \quad \text{and} \quad g(0) \neq \mu.$$

This condition was shown to be sufficient for the RBP in [19, Theorem 3.7]. We remark that there was a similar two-sided condition mentioned in [13, Remark 3.3], a related condition in [36, Corollary 2.7], and a more general (but less direct) one-sided condition in [19, Theorem 3.3]. Of course, there is a condition **(GB-)** with similar properties.

Further results on Beals-type conditions will be given in Section 3, but the following summarises some of the main points from above.

THEOREM 2.1. *We have (B) $\Rightarrow$ (B+) $\Rightarrow$ (GB+) $\Rightarrow$ RBP.*

2.3. Volkmer-type necessary conditions

Most of the necessary conditions for the RBP so far are based on the inequality

$$\left(\int_{-1}^{1} \frac{1}{|r|}|h'|^2\right)^2 \leqslant k\left(\int_{-1}^{1} |h|^2\right)\left(\int_{-1}^{1} \left|\left(\frac{h'}{r}\right)'\right|^2\right) \tag{2.3}$$

introduced by Volkmer in [36, (4.3)]. Let D_V denote the set of AC functions h on $[-1,1]$, for which $\frac{h'}{r}$ is AC and $(\frac{h'}{r})'$ is L_2, and for which $(\frac{h'}{r})(-1) = (\frac{h'}{r})(1) = 0$. We write

(V) there is $k > 0$ such that (2.3) holds for all $h \in D_V$.

Volkmer [36] established the following

THEOREM 2.2. *If $r \in L_\infty(-1,1)$, then RBP $\Rightarrow$ (V).*

Various authors have given more direct necessary conditions, for example,

(AP+) there are no sequences $a_n,\ b_n$ satisfying $0 < a_n < b_n \leqslant 1$, and

$$\frac{a_n}{b_n} \to 0 \quad \text{and} \quad \frac{\int_0^{a_n} r}{\int_0^{b_n} r} \to 1 \text{ as } n \to \infty.$$

For odd r, Abasheeva and Pyatkov introduced an equivalent condition and established (V) $\Rightarrow$ (AP+) in [2, Corollary 1]. They also gave a more involved condition in between (V) and (AP+) in [2, Theorem 1]. Their proofs were based on applying (V) to sequences of specific functions h. In [9] a two-sided version **(AP)** was introduced with the above integrals replaced by $\int_{-a_n}^{a_n} |r|$ and $\int_{-b_n}^{b_n} |r|$, and also a condition **(AP-)** with upper integration limits zero. Of course these conditions all coincide when r is odd, but not in general.

For $0 < a < b < 1$, consider the functions

$$g_{a,b}(x) := \begin{cases} 0 & (x \in [-1,-b)) \\ \frac{b+x}{b-a} & (x \in [-b,-a]) \\ 1 & (x \in (-a,a)) \\ \frac{b-x}{b-a} & (x \in [a,b]) \\ 0 & (x \in (b,1]) \end{cases}, \quad h_{a,b}(x) := \int_{-1}^{x} g_{a,b}\, r \tag{2.4}$$

on $[-1,1]$. Then $h_{a,b} \in D_V$, and we can formulate the condition

($\mathbf{V}_{ab}$) there is $k>0$ so that (2.3) holds for all $h=h_{a,b}$ with $0<a<b<1$.
This condition (for sequences a_n, b_n, and another slightly weaker one), can be found in [9], where the implication $(V_{ab}) \Rightarrow$ (AP+) was established for odd $r \in L_\infty$. Actually, in [9] all the above conditions were shown to be equivalent under a weaker condition than oddness, which will be studied in Section 3. Combining the above, we obtain

THEOREM 2.3. *For odd $r \in L_\infty(-1,1)$, the conditions (V), (V_{ab}), (AP) and (AP$\pm$) are all equivalent.*

2.4. HELP-type conditions

In [36] Volkmer observed a connection between the RBP and one of the so-called HELP-type inequalities, which take the form

$$\left(\int_0^b p\,|y'|^2\right)^2 \leqslant k\left(\int_0^b w|y|^2\right)\left(\int_0^b \frac{1}{w}\,|(py')'|^2\right) \tag{2.5}$$

where $0<b\leqslant\infty$, $w>0$ and $\frac{1}{p}, w$ are locally L_1. (Actually potential terms in the first and third integrals are usually included, but they will not be relevant to our discussion). Such inequalities (which are to hold for some $k>0$ for all y from a specified domain) have a long history, but a brief version is as follows.

In 1932, Hardy and Littlewood [24] discussed the case $p=w=1$, $b=\infty$, leading to a so-called HL inequality. After it featured prominently in the well-known book [25] this was sometimes referred to as the HLP inequality, and Everitt's discussion [17] which allowed a C^1 positive coefficient p (and potential terms) led to the acronym HELP. After various authors had considered particular classes of weight function w, Evans and Everitt [15] studied cases with p, w as above, using a "strong limit point" condition to ensure vanishing of the boundary term at $b=\infty$ in the Dirichlet form for the associated equation

$$-(py')' = \lambda wy. \tag{2.6}$$

In [15, Section 12] they also introduced the regular case (which we distinguish by setting $b=1$, with $0<\frac{1}{p}, w \in L_1(0,1)$), and discussed the corresponding inequalities on various domains in [15, 16]. In the following we consider this regular case.

Let D_M denote the (maximal) domain consisting of those $y \in AC[0,1]$ for which py' is AC and the third integral in (2.5) is finite. With D_N as the set of $y \in D_M$ satisfying the right-hand Neumann condition $(py')(1)=0$, we write

($\mathbf{EE}_N$) for some $k>0$, (2.5) holds for all $y \in D_N$.
This condition was studied in [16] and was shown to be equivalent to a condition on the Titchmarsh-Weyl function

$$m(\lambda) = \frac{(p\psi_\lambda')(1)}{(p\chi_\lambda')(1)} \qquad (\lambda \in \mathbb{C}\setminus\mathbb{R})$$

where χ_λ, ψ_λ form a fundamental pair of solutions of (2.6) at $x=0$, so

$$\chi_\lambda(0)=1, \quad (p\chi_\lambda')(0)=0, \quad \psi_\lambda(0)=0, \quad (p\psi_\lambda')(0)=1.$$

One can then express the m function condition equivalent to (EE_N) in the following form:

($\mathbf{EE}_m$) for some $\theta_0 \in (0, \frac{\pi}{2})$,

$$(Im\lambda)Im[\lambda^2 m(\lambda)] \leqslant 0 \text{ for all } \lambda \in C(\theta_0) \tag{2.7}$$

where $C(\theta_0)$ is the (double) cone $\{se^{i\theta} : s \in \mathbb{R} \setminus \{0\}, \theta \in [\theta_0, \frac{\pi}{2}]\}$.
(Note that (2.7) is automatic for $\theta_0 = \frac{\pi}{2}$). Evans and Everitt also gave the best possible constant k for (EE_N) in terms of the minimal angle θ_0, together with a similar result for a right hand Dirichlet condition.

There is an obvious similarity between (2.3) and (2.5) in the case $p = \frac{1}{r}$, $w = 1$, but there is more involved than simply changing the interval of integration, and in general the relationship between the (V) and (EE_N) conditions is not clear. For example, (V) involves a turning point, and the number of boundary conditions imposed differs between D_N and D_V. Nevertheless Volkmer [36, Theorem 4.1] was able to use the two-sided condition (V) to establish the following connection between the RBP and the one-sided condition (EE_N).

THEOREM 2.4. *For odd $r \in L_\infty(-1,1)$, RBP $\Rightarrow$ (EE_N) with $p = \frac{1}{r}$, $w = 1$.*

In between the publication of [15] and [16], Bennewitz (for example, in [6]) studied (2.5) on the maximal domain D_M above. This was in fact the original formulation in [15, Section 12], and we shall label the corresponding condition

($\mathbf{EE}_M$) for some $k > 0$, (2.5) holds for all $y \in D_M$.
For simplicity we shall give a version of [6] only for the case $p = \frac{1}{r}$, $w = 1$ relevant to the RBP as noted above. In this case, Bennewitz's assumption [6, eq. (1.1)] is automatic and [6, eq. (1.2)] leads to conditions of the form

($\mathbf{B}_0+$) for some $t \in (0,1)$, $\limsup_{x \searrow 0} S_0(t,x) \neq 1$ where

$$S_0(t,x) := \frac{\int_0^{tx} r}{\int_0^x r}, \tag{2.8}$$

and a similar condition **($\mathbf{B}_1+$)** involving lim sup as $x \nearrow 1$ for integrals with upper limits 1. There is also a third condition **($\mathbf{B}_2+$)** that the boundary term $[(py')\bar{y}]_0^1$ in the Dirichlet form for the associated equation (2.6) should vanish for all solutions y of (2.6) with $\lambda = 0$. Bennewitz expressed this in terms of certain eigenvalue problems for (2.6), and his main result (for the special case we are considering) was

THEOREM 2.5. *If $p = \frac{1}{r}$, $w = 1$ then (EE_M) is equivalent to the combination of (B_0+), (B_1+) and (B_2+).*

2.5. Parfenov-type conditions

While the core of this subsection is from Parfenov's paper [30], the foundation for one of his conditions dates somewhat earlier. In [32], Pyatkov showed that boundedness of the operator J on H_s for some $s > 0$ (see Section 1 for notation) was sufficient for

(i) in 2.1, and hence for the RBP. Using real interpolation spaces, Parfenov was able to rework Pyatkov's condition into the following equivalent form expressed more directly in terms of r:

(P) there are $c,d>0$ such that, whenever $0<\eta\leqslant\varepsilon\leqslant 1$,

$$\min\Big(\int_0^\eta r, \int_{-\eta}^0 |r|\Big) \leqslant c\left(\frac{\eta}{\varepsilon}\right)^d \int_{-\varepsilon}^{\varepsilon} |r|. \tag{2.9}$$

Indeed [30, Corollary 4] contains the following

THEOREM 2.6. *If $r\in L_1(-1,1)$ then (P) $\Rightarrow$ RBP.*

Parfenov then turned to one-sided conditions, one being **(P+)**, i.e., (P) with (2.9) replaced by

$$\int_0^\eta r \leqslant c\left(\frac{\eta}{\varepsilon}\right)^d \int_0^\varepsilon r.$$

We shall also define the analogous condition **(P$-$)** for $-1\leqslant\varepsilon\leqslant\eta<0$. The remaining conditions below will be discussed via the following, for $c\in(0,1)$:

(P$_c$+) there is $t\in(0,1)$ such that for all $\varepsilon\in(0,1)$

$$\int_0^{t\varepsilon} r \leqslant c\int_0^\varepsilon r. \tag{2.10}$$

We define **(P$_c$)** via the inequality

$$\int_{-t\varepsilon}^{t\varepsilon} |r| \leqslant c\int_{-\varepsilon}^{\varepsilon} |r|,$$

and similarly for **(P$_c-$)**. Parfenov introduced (P$_{\frac{1}{2}}+$) in [30], and (P$_{\frac{1}{2}}$), (P$_{\frac{1}{2}}\pm$) were discussed in [9]. We can now state the following variant of [30, Theorem 6].

THEOREM 2.7. *For odd $r\in L_\infty(-1,1)$ and $c\in(0,1)$, the following are all equivalent to the RBP: (P), (P$\pm$), and (P$_c$) and (P$_c\pm$).*

We remark that Parfenov's list of equivalent conditions includes (AP+), and the argument proceeds via the implication RBP $\Rightarrow$ (AP+) which was given in Theorem 2.3 only for $r\in L_\infty$. On the other hand we shall improve this to $r\in L_1$ in Section 3. Also Parfenov's argument is for (P$_{\frac{1}{2}}+$) but it carries over directly to (P$_c+$), and Pytakov [34, Theorem 3.1] states (referring to Parfenov's thesis for proof) a result implicitly providing equivalence of various conditions including (P+), (P$_{\frac{1}{2}}+$) and (P$_c+$). For the remaining equivalences, we note that (P) and (P$\pm$) coincide for odd r, as do (P$_c$) and (P$_c\pm$).

3. Connections between the groups

In this section we change our focus in various ways. We shall review inter- rather than intra-group relations, provide some new relations including equivalences, and extend other relations from $r \in L_\infty$ to $r \in L_1$, and from odd r (as in [2, 30, 36]) to a weaker condition defined as follows. With

$$r^e(x) := \frac{1}{2}(r(x)+r(-x)), \quad r^o(x) := \frac{1}{2}(r(x)-r(-x)) \qquad (x \in [-1,1])$$

our standing assumption (1.2) is equivalent to

$$|r^e(x)| < r^o(x) \quad \text{a.e. on} \quad [0,1].$$

We call r *weakly odd-dominated* **(WOD)** if a function ρ exists satisfying $\rho(\varepsilon) < 1$ for all $\varepsilon \in (0,1]$ and

$$\int_0^x |r^e| \leqslant \rho(\varepsilon) \int_0^x r^o \quad \text{for all } x \in (0,\varepsilon)$$

and *strongly odd-dominated* **(SOD)** if ρ can also be chosen so that

$$\rho(\varepsilon) = o\left(\varepsilon^{1/2}\right) \quad \text{as} \quad \varepsilon \to 0.$$

Clearly, all odd weight functions are SOD (and hence also WOD). Also r is SOD if $r(x) = ax + o(|x|^{\frac{3}{2}})$ as $x \to 0$, assuming $a \neq 0$ – further examples will be given in Section 4. Odd domination conditions were discussed in [9] after an earlier condition, stronger than SOD but weaker than oddness, had been introduced in [22].

3.1. Conditions from 2.1 and 2.3

In [36], Volkmer established the implication RBP $\Rightarrow$ (V) of Theorem 2.2 for $r \in L_\infty$, yet there are various results stated in the literature for $r \in L_1$ but depending explicitly or implicitly on this implication. Since we have not seen an explicit proof for unbounded r, we shall provide one now for completeness.

THEOREM 3.1. *If $r \in L_1(-1,1)$ then RBP $\Rightarrow$ (V).*

Proof. We shall adapt the proof from [36], using an abstract form

$$Sy = \lambda Ty \tag{3.1}$$

of (1.1), where S and T are self-adjoint operators on a Hilbert space H_* with inner product $(.,.)_*$, S has a positive compact inverse and T is bounded. Assume the RBP in H_* for (3.1). Then [36, Theorem 2.2] gives the necessary condition

$$(|T|y,y)_* \leqslant c\|S^{-1}Ty\|_S\|y\|_S \text{ for all } y \in H_S. \tag{3.2}$$

Here $H_S = D(S^{\frac{1}{2}})$ with norm $\|y\|_S := \|S^{\frac{1}{2}}y\|_*$.

For $r \in L_\infty$, Volkmer used (3.2) with $H_* = L_2(-1,1)$ and $S = L, T = R$ from Section 1 to derive

$$\left(\int_{-1}^{1} |r||y|^2\right)^2 \leqslant k\left(\int_{-1}^{1} |z'|^2\right)\left(\int_{-1}^{1} |y'|^2\right) \tag{3.3}$$

for all $y \in H_S = W_{2,0}^1(-1,1)$, where $Sz = Ty$, so z satisfies

$$-z'' = ry, \tag{3.4}$$

$$z(\pm 1) = 0. \tag{3.5}$$

For $r \in L_1$, we claim that (3.2), with $H_* = H$, $S = A$ (which does have a positive compact inverse, and moreover $H_S = H_1$) and $T = J$ again yields (3.3). Indeed the first integral is obvious since $y \in H_1 \subset H$, and for the second we integrate by parts to give

$$\|u\|_1 = \left(\int_{-1}^{1} |u'|^2\right)^{\frac{1}{2}} := \nu(u) \text{ for all } u \in H_2, \tag{3.6}$$

in particular for $u = z$. For the third integral, we note that H_1 is the completion of H_2 in the H_1 norm, so for any $y \in H_1$ there is a sequence $z_n \in H_2$ converging to y in H_1. By (3.6) with $u = z_n$, the z_n also converge in the ν norm, to a limit v, say. Moreover A is uniformly positive (see above), so the norms in (3.6) dominate the H norm. Thus the z_n also converge in H, and we can write $u = y = v$ in (3.6) to establish our claim. (Note that the third integral above can also be treated by the methods of, e.g., [19, Chapter 2]).

Volkmer's argument extending (3.3) to any z satisfying (3.4) (without boundary conditions) can now be repeated, and (2.3) then follows from the substitution $y = -\frac{h'}{r}, z' = h$. Thus condition (V) continues to hold even for $r \in L_1(-1,1)$. $\square$

Note that in Theorems 2.3, 2.4 and 2.7 the only step depending on $r \in L_\infty$ was Theorem 2.2. Therefore, Theorem 3.1 implies

COROLLARY 3.2. *Theorems* 2.3, 2.4 *and* 2.7 *remain valid for all* $r \in L_1(-1,1)$.

3.2. Beals- and HELP-type conditions from 2.2 and 2.4

Our aim now is to connect (B_0+) of 2.4 with new conditions related to (GB+) and (B+) of 2.2. At first sight such a connection may seem surprising since (B_0+) is only one of three conditions in Theorem 2.5 and moreover (EE_M) has as far as we know not previously been connected to the RBP. Moreover Beals-type conditions have usually been viewed only as sufficient for the RBP, although a necessary aspect was explored in [21] under a smoothness condition on r and we shall return to this later.

We shall need a variant of de l'Hôpital's rule, which is usually proved via Rolle's theorem and requires a certain function to have a vanishing derivative at an extremum. In our case this derivative need not exist, so we shall present a different proof of (a generalisation of) this famous rule. In what follows we shall use "ess" to mean modulo

(Lebesgue) null sets. For example, if $f \in L_1(0,1)$ and $x \in (0,1)$, then ess $\inf_{(0,x)} f$ is the largest number η so that one can redefine f to value η on a null set in $(0,x)$, giving a new function $\tilde{f}$ with $\inf_{(0,x)} \tilde{f} = \eta$. Similarly, ess $\lim_{x\searrow 0} f = \eta$ means $\lim_{x\searrow 0} \tilde{f} = \eta$ for an appropriate $\tilde{f}$ (redefined as above).

LEMMA 3.3. *For some $\eta > 0$ let $f, g \in L_1(0,\eta)$ with g of one sign a.e. Then with $F(x) := \int_0^x f$, $G(x) := \int_0^x g$ we have*

$$\lim_{x\searrow 0} \operatorname{ess\,inf}_{(0,x)} \frac{f}{g} \leqslant \liminf_{x\searrow 0} \frac{F(x)}{G(x)} \leqslant \limsup_{x\searrow 0} \frac{F(x)}{G(x)} \leqslant \lim_{x\searrow 0} \operatorname{ess\,sup}_{(0,x)} \frac{f}{g}.$$

Proof. Suppose without loss that $g > 0$ and note that

$$\operatorname{ess\,inf}_{(0,x)} \frac{f}{g} \text{ is nonincreasing in } x. \tag{3.7}$$

Thus the left hand limit exists, perhaps infinite. We write ℓ for this limit, and assume it to be finite, as in our application below (the proof in the infinite case is a simple adaptation).

Given $\varepsilon > 0$ there is $\delta > 0$ so that ess $\inf_{(0,x)} \frac{f}{g} > \ell - \varepsilon$ for all $x \in (0,\delta)$, so $F(x) > (\ell - \varepsilon)G(x)$ for all $x \in (0,\delta)$. Obviously $G(x) > 0$ for such x, so the first of our required inequalities follows. The third is similar, and the second is trivial. □

We note that a standard version of de l'Hôpital's rule follows if F and G are differentiable and the outer two limits above are equal. This version was used under related circumstances in [21].

For $\mu \in (0,1)$ we now write

$$\sigma_\mu(x) := \frac{\mu r(\mu x)}{r(x)} \tag{3.8}$$

and we introduce two generalised Beals-type conditions

(GB$_i$+) $\lim_{x\searrow 0}$ ess $\inf_{(0,x)} \sigma_\mu < 1$ for some $\mu \in (0,1)$

and

(GB$_s$+) $\lim_{x\searrow 0}$ ess $\sup_{(0,x)} \sigma_\mu < 1$ for some $\mu \in (0,1)$.

Note that these limits always exist, by (3.7) and its ess sup analogue.

THEOREM 3.4. *If $r \in L_1(-1,1)$ then (GB$_s$+) $\Rightarrow$ (B$_0$+) $\Rightarrow$ (GB$_i$+).*

Proof. Assume (GB$_s$+) and apply Lemma 3.3 with $f(x) := \mu r(\mu x)$ and $g(x) := r(x)$. Then

$$\limsup_{\varepsilon\searrow 0} \frac{\int_0^{\varepsilon\mu} r}{\int_0^{\varepsilon} r} = \limsup_{\varepsilon\searrow 0} \frac{\int_0^{\varepsilon} \mu r(\mu x)\,dx}{\int_0^{\varepsilon} r(x)\,dx} \leqslant \lim_{\varepsilon\searrow 0} \text{ ess } \sup_{(0,\varepsilon)} \sigma_\mu < 1.$$

This yields (B_0+). Now, assuming this condition, we apply Lemma 3.3 again to give

$$\lim_{\varepsilon\searrow 0} \text{ ess inf}_{(0,\varepsilon)}\sigma_\mu \leqslant \liminf_{\varepsilon\searrow 0} \frac{\int_0^\varepsilon \mu r(\mu x)\,dx}{\int_0^\varepsilon r(x)\,dx} \leqslant \limsup_{\varepsilon\searrow 0} \frac{\int_0^{\varepsilon\mu} r}{\int_0^\varepsilon r}$$

which is less than 1 because (B_0+) implies $0 \leqslant S_0(t,x) < 1$ in (2.8), since $r > 0$ a.e. on $(0,1)$ and $t \in (0,1)$. Thus (GB_i+) must hold. □

Under an additional limit condition we can do better:

COROLLARY 3.5. *Assume that* $r \in L_1(-1,1)$ *and that* ess $\lim_{x\searrow 0}\sigma_\mu(x)$ *exists for some* $\mu \in (0,1)$. *Then* (GB_s+)*,* (B_0+) *and* (GB_i+) *are all equivalent.*

Proof. By assumption, $\lim_{x\searrow 0}$ ess $\inf_{(0,x)}\sigma_\mu = \lim_{x\searrow 0}$ ess $\sup_{(0,x)}\sigma_\mu$, so (GB_s+) and (GB_i+) are equivalent. The result then follows from Theorem 3.4. □

Conditions of this type on r are clearly weaker than (GB+). One may also give more direct conditions on r, as follows.

COROLLARY 3.6. *Assume that* $r \in L_1(-1,1)$ *and that* $r(x) = ax^\alpha + o(x^\alpha)$ *as* $x \searrow 0$*, for some real* $a\,(>0)$ *and* $\alpha\,(>-1)$. *Then* (GB_s+)*,* (B_0+) *and* (GB_i+) *are all equivalent.*

Proof. By assumption, $\sigma_\mu(x) = \mu^{\alpha+1} + o(1)$ as $x \searrow 0$ for some (in fact all) $\mu \in (0,1)$ so the result follows from Corollary 3.5. □

Again, the above conditions on r are clearly weaker than (B+). Note that all the conditions above refer to $x > 0$. There is of course a corresponding theory involving analogous conditions **($\mathbf{B}_0-$)** (defined as for (B_0+) but with $x < 0$) and **($\mathbf{GB}_s-$)**, **($\mathbf{GB}_i-$)** (with $(0,x)$ above replaced by $(x,0)$) and $x < 0$).

3.3. Volkmer- and HELP-type conditions from 2.3 and 2.4

We start with the following

PROPOSITION 3.7. *If* $r \in L_1(-1,1)$ *is odd then (V)* $\Rightarrow$ *(*EE_N*) for* $p = \frac{1}{r}$, $w = 1$.

Proof. Let $y \in D_N$ – see condition (EE_N) in 2.4 – and extend y to an even function h on $[-1,1]$. Then $h \in D_V$ and

$$\int_{-1}^1 \frac{1}{|r|}|h'|^2 = 2\int_0^1 \frac{1}{r}|h'|^2,$$
$$\int_{-1}^1 |h|^2 = 2\int_0^1 |h|^2,$$
$$\int_{-1}^1 |\left(\frac{h'}{r}\right)'|^2 = 2\int_0^1 |\left(\frac{h'}{r}\right)'|^2.$$

If (V) holds then (2.3) must be satisfied for h, and hence, from the above equations, (2.5) is satisfied for y, and so (EE_N) must hold. $\square$

We remark that the above result is implicit in [36], but we are unaware of any results in the converse direction. Our aim now is to establish such a result, moreover with oddness of r weakened to SOD. To this end we first obtain the following estimates:

LEMMA 3.8. *If $r \in L_1$ is SOD then there are constants $c_1, c_2 > 0$ such that*

$$\int_0^1 |g_{a,b}|^2\, |r^e| \leqslant c_1 \int_0^1 |g_{a,b}|^2\, r, \tag{3.9}$$

$$\int_{-1}^0 |g_{a,b}|^2\, |r| \leqslant c_2 \int_0^1 |g_{a,b}|^2\, r \tag{3.10}$$

for all $0 < a < b < 1$, where $g_{a,b}$ is defined in (2.4).

Proof. It follows from [9, Lemma 4.1] that there is a constant $c_1 > 0$ such that

$$\int_0^x |r^e| \leqslant c_1 \int_0^x r$$

for all $x \in (0,1)$. Therefore the function $\varphi(x) := \int_0^x (c_1\, r - |r^e|)$ is nonnegative on $[0,1]$. Then, since $\varphi(0) = g_{a,b}(1) = 0$ and $\left(|g_{a,b}|^2\right)' \leqslant 0$ on $(0,1)$, integration by parts gives

$$\int_0^1 |g_{a,b}|^2\, (c_1\, r - |r^e|) = -\int_0^1 \left(|g_{a,b}|^2\right)'\, \varphi \geqslant 0,$$

and this is (3.9). Then we obtain

$$\begin{aligned}\int_{-1}^0 |g_{a,b}|^2\, |r| &= -\int_{-1}^0 |g_{a,b}|^2\, (r^o + r^e) = \int_0^1 |g_{a,b}|^2\, (r^o - r^e)\\ &= \int_0^1 |g_{a,b}|^2\, (r - 2r^e) \leqslant (1 + 2c_1) \int_0^1 |g_{a,b}|^2\, r,\end{aligned}$$

which is (3.10). $\square$

Let us define condition **($\mathbf{EE}_{ab}$)** as for (EE_N) in 2.4 but with D_N replaced by the set of all restrictions of the functions $h_{a,b}$ in (2.4) to $[0,1]$ with $0 < a < b < 1$ – compare the definition of (V_{ab}) in 2.3. Then we have the following

PROPOSITION 3.9. *If $r \in L_1(-1,1)$ is SOD and $p = \frac{1}{r}$, $w = 1$ then*

$$(EE_N) \Rightarrow (EE_{ab}) \Rightarrow (V_{ab}) \Rightarrow (V).$$

Proof. (EE_N) $\Rightarrow$ (E_{ab}) is obvious so assume the latter. Note that by Lemma 3.8

$$\int_{-1}^1 |h_{a,b}'|^2 \frac{1}{|r|} = \int_{-1}^1 |g_{a,b}|^2 |r| \leqslant (1 + c_2) \int_0^1 |g_{a,b}|^2 r$$

for all $0 < a < b < 1$. Therefore, with $c := (1+c_2)^2$, (E_{ab}) implies

$$\begin{aligned}\left(\int_{-1}^{1} |h'_{a,b}|^2 \frac{1}{|r|}\right)^2 &\leqslant c\left(\int_{0}^{1} |h'_{a,b}|^2 \frac{1}{r}\right)^2 \\ &\leqslant ck\left(\int_{0}^{1} |h_{a,b}|^2\right)\left(\int_{0}^{1} \left|\left(\frac{h'_{a,b}}{r}\right)'\right|^2\right) \\ &\leqslant ck\left(\int_{-1}^{1} |h_{a,b}|^2\right)\left(\int_{-1}^{1} \left|\left(\frac{h'_{a,b}}{r}\right)'\right|^2\right)\end{aligned}$$

with a fixed constant $k > 0$. This establishes (V_{ab}), and its equivalence with (V) follows from [9, Theorem 4.3] and Theorem 3.1 (cf. Corollary 3.2). $\square$

3.4. Bennewitz- and Parfenov-type conditions from 2.4 and 2.5

In this subsection we shall relate the conditions of 2.5 to each other and to further conditions from Section 2. Most of these relations are either known or can be derived from the previous results of the present section.

In [30] Parfenov gave equivalence of (P+), ($\mathrm{P}_{\frac{1}{2}}+$), (AP+) and the RBP for odd r. This was partially extended to equivalence of (P+) and the RBP under a condition between oddness and SOD of $r \in L_\infty(-1,1)$ in [22]. Then these results were extended to SOD $r \in L_\infty(-1,1)$, and also to include (V), in the main result of [9]. Actually, one can extend this to $r \in L_1(-1,1)$:

PROPOSITION 3.10. *If $r \in L_1(-1,1)$ is SOD then the conditions of Theorems* 2.3 *and* 2.7 *are all equivalent.*

Proof. All the steps in the proof of [9, Theorem 4.3] are valid for $r \in L_1$ except RBP $\Rightarrow$ (V), and for this we can use Theorem 3.1 instead. $\square$

In [21] and [30], certain connections and differences, respectively, were observed between the conditions of 2.2 and 2.5, and they will be discussed below. As we saw in 3.2, there is a connection between Beals-type conditions and (B_0+), and we now show that (B_0+) and (P_c+), originally formulated within the different frameworks of 2.4 and 2.5, are also intimately connected.

PROPOSITION 3.11. *For any $r \in L_1(-1,1)$ and $c \in (0,1)$, the conditions (P_c+) and (B_0+) are equivalent.*

Proof. Referring to (2.8), we see that (P_c+) with $c \in (0,1)$ implies $S_0(t,x) \leqslant c < 1$ for all $x \in (0,1)$, so (B_0+) holds.

For the converse, we assume (B_0+) and we note that $\neq 1$ can be replaced by < 1 in this condition for some $d \in (0,1)$ since $r > 0$ a.e. on $(0,1)$ and $t \in (0,1)$. If (P_d+) fails for all $d \in (0,1)$, then there is a sequence $\varepsilon_n \in (0,1)$ so that $S_0(t,\varepsilon_n) \to 1$. Without loss of generality we can assume $\varepsilon_n \to \varepsilon \in [0,1]$. If $\varepsilon > 0$ then $S_0(t,\varepsilon) = 1$

so $\int_{t\varepsilon}^{\varepsilon} r = 0$, contradicting (1.2). If $\varepsilon = 0$ then we contradict (B_0+). Thus (P_d+) does in fact hold for some $d \in (0,1)$.

Finally, define $\tilde{r}$ as the odd extension of $r|_{[0,1]}$, i.e.,

$$\tilde{r}(x) = r(x) \text{ if } x > 0 \text{ and } \tilde{r}(x) = -r(-x) \text{ if } x < 0. \tag{3.11}$$

Then (P_b+) for r and (P_b) for $\tilde{r}$ coincide for any $b \in (0,1)$. Moreover (P_c) and (P_d) are equivalent for the odd function $\tilde{r}$ by Theorem 2.7 and Corollary 3.2, so, from the previous paragraph, (P_c+) also holds. □

Of course there is a corresponding result involving (P_c-) and the analogous condition (B_0-) for $x < 0$ (see Corollary 3.6 et seq). In view of Proposition 3.10, then, we have the following

COROLLARY 3.12. *If $r \in L_1(-1,1)$ is SOD and $c \in (0,1)$, then the RBP is equivalent to each of the conditions ($B_0\pm$) and ($P_c\pm$).*

3.5. Equivalent statements

We are now ready to list as equivalent most of the statements in Section 2 for SOD $r \in L_1(-1,1)$.

THEOREM 3.13. *If $r \in L_1(-1,1)$ is SOD, $p = \frac{1}{r}$, $w = 1$ and $c \in (0,1)$, then the RBP is equivalent to each of the following conditions:*

(V), (V_{ab}), (AP) and (AP$\pm$) from 2.3,

(EE_N) and (EE_m) from 2.4 and (EE_{ab}) from Proposition 3.9,

($B_0\pm$) from 2.4 and 3.2, and

(P), (P$\pm$), and (P_c) and ($P_c\pm$) from 2.5.

Proof. In view of Propositions 3.9 and 3.10 and Corollary 3.12, it is enough to prove (P_c+) $\Rightarrow$ (EE_N) since (EE_N) $\Rightarrow$ (V) follows from Proposition 3.9.

To this end consider again the odd extension $\tilde{r}$ of (3.11). Then (P_c+) for r implies the same condition for $\tilde{r}$. Consequently, Proposition 3.10 gives (V) for the odd function $\tilde{r}$. By Proposition 3.7 this implies (EE_N) for $\tilde{r}$, and hence (EE_N) for r. □

In order to include Beals-type conditions, we appeal to the results of 3.2.

COROLLARY 3.14. *If $r \in L_1(-1,1)$ is SOD, then*

(i) *(GB_s+) or (GB_s-) $\Rightarrow$ RBP $\Rightarrow$ (GB_i+) and (GB_i-)*

(ii) *($GB_s\pm$) and ($GB_i\pm$) are each equivalent to the conditions of Theorem* 3.13 *if, in addition, $\sigma_\mu(x)$ of* (3.8) *has an essential limit as $x \to 0\pm$ for some $\mu \in (0,1)$ (for example, if $r(x) = a\,\mathrm{sgn}(x)|x|^\alpha + o(|x|^\alpha)$ as $x \to 0\pm$, for some real a (> 0) and α (> -1)).*

This follows from Theorems 3.4 and 3.13 and Corollaries 3.5 and 3.6, and improves previous results in the literature in various ways. For example, [21] contains equivalence of (GB+) and the RBP under some extra limit conditions on $r(x)$ and $r'(x)$ as $x \to 0$.

We conclude this section by noting that not only are some of the conditions of 2.4 helpful in the study of the RBP, but the converse is also true. For example, we have

COROLLARY 3.15. *If $r \in L_1(-1,1)$ is SOD, then (EE$_M$) with $p = \frac{1}{r}$, $w = 1$ is equivalent to the combination of (B_1+), (B_2+) and any one of the conditions in Theorem* 3.13.

In particular, (EE$_M$) is equivalent to (EE$_N$) (which takes the form of (EE$_M$) but for a restricted class of functions) together with the extra conditions (B$_1+$) and (B$_2+$).

Moreover, counterexamples to the RBP must also fail the other conditions of Theorem 3.13. For example, let $a_n := \frac{1}{(2n)!}$, $b_n := \frac{1}{(2n-1)!}$ for each $n \in \mathbb{N}$. For $x \in (0,1]$ put

$$r(x) := \begin{cases} x & \text{if } x \in [a_n, b_n],\, n \in \mathbb{N}, \\ 1 & \text{otherwise} \end{cases}$$

and take the odd extension $r(x) := -r(-x)$ for $x \in [-1,0)$. Obviously, $r \in L_\infty(-1,1)$ is odd, hence SOD, and by [2, Example 1] the RBP fails. (A similar example was given in [20]). Thus we have the following

COROLLARY 3.16. *For the preceding function r, condition (EE$_N$) is not valid, i.e., for all $k > 0$ there is a function $h \in D_N$ such that*

$$\left(\int_0^1 |h'|^2 \frac{1}{r}\, dx\right)^2 > k \left(\int_0^1 |h|^2 dx\right) \left(\int_0^1 \left|\left(\frac{h'}{r}\right)'\right|^2 dx\right).$$

4. Discussion

In the preceding sections we have reviewed some of the more recent literature on a Riesz basis property of (1.1). We have also provided a platform for the equivalence of most of these conditions, some of the implications involved being new, at least to our knowledge. This platform includes the SOD condition from the start of Section 3, and we shall next discuss some aspects of this condition in its own right.

We start with the following

LEMMA 4.1. *If r is SOD then so is R, where*

$$R(x) := \begin{cases} r(x) & x \in (0,1) \\ r^o(x) & x \in (-1,0) \end{cases}.$$

Proof. Note that R satisfies (1.2), i.e., $R(x) = r(x) > 0$ and $R(-x) = r^o(-x) < 0$ for $x \in (0,1)$. We consider the even and odd parts of R for $x > 0$:

$$\begin{aligned} R^e(x) &= \frac{1}{2}\left(r(x) + r^o(-x)\right) = \frac{1}{2} r^e(x), \\ R^o(x) &= \frac{1}{2}\left(r(x) - r^o(-x)\right) = r^o(x) + \frac{1}{2} r^e(x). \end{aligned}$$

Since r is SOD, we have for all $x \in (0,\varepsilon)$

$$\int_0^x |R^e| \leqslant \frac{\rho(\varepsilon)}{2} \int_0^x r^o, \quad \int_0^x R^o \geqslant \left(1 - \frac{\rho(\varepsilon)}{2}\right) \int_0^x r^o$$

for some $\rho < 1$ satisfying $\rho(\varepsilon) = o\left(\varepsilon^{1/2}\right)$. Thus

$$\int_0^x |R^e| \leqslant \rho(\varepsilon) \int_0^x R^o$$

which implies that R is SOD. $\square$

We shall use this result in two ways, first to construct SOD examples via one-sided perturbations of odd functions. Indeed if r_* is odd and s is even on $(-1,1)$, and

$$\int_0^x |s| \leqslant \rho(\varepsilon) \int_0^x r_* \tag{4.1}$$

for $x \in (0,\varepsilon)$ with ρ as above, then $r_* + s$ is SOD on $(-1,1)$ by definition. Thus, by Lemma 4.1, the function R given by $R = r_*$ on $(-1,0)$ and $R = r_* + s$ on $(0,1)$ must also be SOD. It is enough here if s is defined only on $(0,1)$, and one can also have an independent perturbation of r_* on $(-1,0)$.

EXAMPLE 4.2. Let $r_* = ar_\alpha$ with $a > 0$ and $\alpha > -1$ – see (1.6) – and $s(x) = bx^\beta$ for $x \in (0,1)$, $|b| \leqslant a$ and $\beta > \alpha + \frac{1}{2}$. Then (4.1) holds, so the function R above is SOD.

Similarly, any two-sided perturbation of the form $ar_\alpha(x) + o(|x|^{\alpha+\frac{1}{2}})$ is SOD. Note that Beals-type functions as in (2.2) are included, being of the form $ar_\alpha(x) + O(|x|^{\alpha+1})$.

In view of the restriction $\alpha > -\frac{1}{2}$ in (2.2), such perturbing terms s are bounded, but it is easy to make unbounded versions, even if r_* is bounded, i.e., if $\alpha \geqslant 0$.

EXAMPLE 4.3. Let $I_n = (2^{-n}, 2^{1-n})$, and as before $r_* = ar_\alpha$ with $a \neq 0$. Then $\int_{I_n} r_* = b2^{-n(\alpha+1)}$ for some $b \neq 0$. For each n, choose a subinterval J_n of I_n with length $2^{-n\gamma}$ where $\gamma > \alpha + 2$, and let $s = 2^{n/2}$ on J_n and $s = 0$ elsewhere on I_n. Then s is unbounded (but integrable) on $(0,1)$ and (4.1) holds, so R as above is SOD.

Our second use of Lemma 4.1 is to connect the above perturbation approach with the RBP. This also sheds light on the additional "information" allowing a one-sided condition like (EE_N) to predict two-sided properties like the RBP for a non odd weight r.

THEOREM 4.4. *Assume $r \in L_1(-1,1)$ is SOD. Then the RBP holds for r if and only if this is true for the odd part r^o of r. In other words, the RBP holds for an odd weight function if and only if it holds for all SOD perturbations.*

Proof. Theorem 3.13 and Lemma 4.1 allow us to conclude the following chain of equivalences:

$$\begin{aligned}
&\text{RBP for } r\\
\Leftrightarrow\ &(\mathrm{P}_c+) \text{ for } r \Leftrightarrow (\mathrm{P}_c+) \text{ for } R \Leftrightarrow (\mathrm{P}_c-) \text{ for } R \Leftrightarrow (\mathrm{P}_c-) \text{ for } r^o\\
\Leftrightarrow\ &\text{RBP for } r^o \qquad \square
\end{aligned}$$

As an example, we have

EXAMPLE 4.5. Let r_* and s be as in Examples 4.2 and 4.3 respectively, and write s_* for the even extension of s over $(-1,1)$. Then the RBP holds for $r_* + s_*$.

Of course, many weight functions r admit the RBP without being SOD. The following is a more explicit version of Parfenov's final example in [30, Section 5].

EXAMPLE 4.6. Fix $\alpha > -1$ and let $d_k = (k+1)^{-(\alpha+2)}$, $x_k = 1/k!$ for $k \in \mathbb{N}$. Evidently $d_k > d_{k+1}$ and it is not difficult to show that

$$d_k < \frac{R_{k+1}}{(k+1)R_k} \text{ where } R_k := \int_{x_{k+1}}^{x_k} r_\alpha$$

and r_α is as above.

From the above properties of d_k we obtain

$$\frac{d_k R_k}{(1-d_{k+1})R_{k+1}} < \frac{1}{k} \to 0 \text{ as } k \to \infty. \tag{4.2}$$

Now define the function φ by

$$\varphi(s) := \begin{cases} d_{2k} & s \in (x_{2k+1}, x_{2k}], \\ 1-d_{2k-1} & s \in (x_{2k}, x_{2k-1}], \\ d_{2k-1} & s \in [-x_{2k-1}, -x_{2k}), \\ 1-d_{2k} & s \in [-x_{2k}, -x_{2k+1}). \end{cases}$$

Then $r := \varphi r_\alpha \in L_1(-1,1)$, and one can argue as in [30, Section 5] to show that r satisfies (P), but, because of (4.2), that it fails (AP+) with $a_k := x_{2k+1}$, $b_k := x_{2k}$. Therefore, by Theorem 3.13, the RBP holds for r, which cannot be SOD.

Moreover, if a given weight function r satisfies (GB+) then the RBP holds with no restriction (other than negativity and integrability) on $r(x)$ for $x < 0$. For example, we can combine $r_\alpha(x)$ for $x > 0$ with the weight function from Corollary 3.16 for $x < 0$. Then (GB+), and hence RBP and all right sided conditions from Theorem 3.13, are satisfied, whereas all left sided conditions, and hence SOD, fail to hold. Also several of the steps leading to Theorem 3.13 are valid under weaker assumptions than SOD, some even under WOD (see [9]). Thus relaxation of SOD can be added to the list in Section 1 of interesting areas of possible further research into the RBP.

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