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# ON MINIMAL POTENTIALLY POWER–POSITIVE SIGN PATTERNS

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*Abstract.* An $n$-by-$n$ sign pattern $\mathscr{A}$ is said to be potentially power-positive if there exists some $A \in Q(\mathscr{A})$ such that $A$ is power-positive, i.e., $A^k > 0$ for some positive integer $k$. Catral, Hogben, Olesky and van den Driessche [Sign patterns that require or allow power-positivity, *Electron. J. Linear Algebra*, 19 (2010), 121-128] investigated the sign patterns that require or allow power-positivity. It has been shown that an $n$-by-$n$ sign pattern $\mathscr{A}$ is potentially power-positive if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is potentially eventually positive. But as the identification of sufficient and necessary conditions for potentially eventually positive sign patterns remains open, the characterization of potentially power-positive sign patterns is still open. In this paper, we introduce the minimal potentially power-positive sign patterns to classify the potentially power-positive sign patterns. Some properties of minimal potentially power-positive sign patterns are presented. It is shown that for an $n$-by-$n$ sign pattern $\mathscr{A}$ with at most $n+1$ negative entries, $\mathscr{A}$ is minimal potentially power-positive if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is minimal potentially eventually positive. Finally, we classify the minimal potentially power-positive sign patterns of order $n \leqslant 3$.

## 1. Introduction

In qualitative and combinatorial matrix theory, a methodology based on the signs of the entries of a matrix is often quite useful in study of some properties of matrices. A sign pattern is a matrix $\mathscr{A} = [a_{ij}]$ with entries in $\{+,-,0\}$. For a real matrix $B$, $\operatorname{sgn}(B)$ is the sign pattern whose entries are the signs of the corresponding entries in $B$. For an $n$-by-$n$ sign pattern $\mathscr{A}$, the qualitative class of $\mathscr{A}$, denoted by $Q(\mathscr{A})$, is defined as

$$Q(\mathscr{A}) = \{B = [b_{ij}] \in M_n(R) | \operatorname{sgn}(b_{ij}) = a_{ij}, \text{for all } i,\ j\}.$$

A subpattern of an $n$-by-$n$ sign pattern $\mathscr{A} = [a_{ij}]$ is an $n$-by-$n$ sign pattern $\mathscr{B} = [b_{ij}]$ such that $b_{ij} = 0$ whenever $a_{ij} = 0$; if, in addition, $\mathscr{B} \neq \mathscr{A}$, then $\mathscr{B}$ is a proper subpattern of $\mathscr{A}$. A permutation pattern is a square sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to $0$. A product of the form $P^T \mathscr{A} P$, where $P$ is a permutation pattern and $\mathscr{A}$ is a sign pattern matrix of the same order as $P$, is called a permutation similarity. Two sign patterns

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$\mathscr{A}$ and $\mathscr{B}$ are equivalent if $\mathscr{A} = P^T \mathscr{B} P$, or $\mathscr{A} = P^T \mathscr{B}^T P$, where $P$ is a permutation pattern and $P^T$ is the transpose of $P$. A pattern $\mathscr{A}$ is reducible if there is a permutation matrix $P$ such that

$$P^T \mathscr{A} P = \begin{pmatrix} \mathscr{A}_{11} & 0 \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix},$$

where $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible. For a sign pattern $\mathscr{A}$, we define the positive part of $\mathscr{A}$ to be $\mathscr{A}^+ = [a_{ij}^+]$ and the negative part of $\mathscr{A}$ to be $\mathscr{A}^- = [a_{ij}^-]$, where

$$a_{ij}^+ = \begin{cases} + & \text{if } a_{ij} = +, \\ 0 & \text{if } a_{ij} = 0 \text{ or } -, \end{cases} \quad \text{and} \quad a_{ij}^- = \begin{cases} - & \text{if } a_{ij} = -, \\ 0 & \text{if } a_{ij} = 0 \text{ or } +. \end{cases}$$

For a real matrix, the positive part and the negative part are defined similarly.

We now introduce some graph theoretical concepts (see, for example, [2, 4, 9, 11]), since graph theoretical methods are often useful in the study of sign patterns.

The signed digraph of an $n$-by-$n$ sign pattern $\mathscr{A} = [a_{ij}]$, denoted by $\Gamma(\mathscr{A})$, is the digraph with a vertex set $\{1, 2, \cdots, n\}$ where $(i, j)$ is an arc (bearing $a_{ij}$ as its sign) if and only if $a_{ij} \neq 0$. A (directed) simple cycle of length $k$ is a sequence of $k$ arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct.

A digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is $1$. It is well known that a digraph $D = (V, E)$ is primitive if and only if there exists a natural number $k$ such that for all $V_i \in V$, $V_j \in V$, there is a walk of length $k$ from $V_i$ to $V_j$. A sign pattern $\mathscr{A}$ is primitive if its signed digraph $\Gamma(\mathscr{A})$ is primitive.

We say that a sign pattern $\mathscr{A}$ requires a property $P$ referring to real matrices if every real matrix $A \in Q(\mathscr{A})$ has the property $P$ and that $\mathscr{A}$ allows $P$ or $\mathscr{A}$ is potentially $P$ if there exists at least one $A \in Q(\mathscr{A})$ that has the property $P$.

In the last few years, there has been an increasing interest in requiring or allowing problems of sign patterns; see, e.g., [1, 5-8, 10]. In [6], Ellison, Hogben and Tsatsomeros studied the sign patterns that require eventual positivity or require eventual nonnegativity. Sign patterns that allow eventual positivity have been studied in [1]. Several necessary or sufficient conditions for an $n$-by-$n$ sign pattern to be potentially eventually positive have been established. A characterization of potentially power-positive sign patterns was given in [5] by Catral, Hogben, Olesky and van den Driessche. It has been shown that the sign pattern $\mathscr{A}$ is potentially power-positive if and only if $\mathscr{A}$ or $-\mathscr{A}$ is potentially eventually positive. However, since the identification of sufficient and necessary conditions for an $n$-by-$n$ sign pattern to allow eventual positivity remains open, the characterization of potentially power-positive sign patterns is also open.

In this paper, we consider the minimal potentially power-positive sign patterns. This paper is organized as follows: Definitions and notations are given in the end of Section 1. In Section 2, some properties of minimal potentially power-positive sign patterns are presented. For an $n$-by-$n$ sign pattern $\mathscr{A}$ with at most $n+1$ negative entries, it can be shown that $\mathscr{A}$ is minimal potentially power-positive if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is minimal potentially eventually positive. Furthermore, for $n \leqslant 3$, it can be shown that the condition that $\mathscr{A}$ has at most $n+1$ negative entries can be removed. In Section 3, we classify the minimal potentially power-positive sign patterns of order $n \leqslant 3$. Conclusions and some open questions are given in Section 4.

In order to state our results clearly, we need the following definitions.

DEFINITION 1.1. [6, 12] An $n$-by-$n$ real matrix $A$ is said to be eventually positive if there exists a positive integer $k_0$ such that $A^k > 0$ for all $k \geqslant k_0$.

DEFINITION 1.2. [3, 5] An $n$-by-$n$ real matrix $A$ is said to be power-positive if there exists a positive integer $k$ such that $A^k > 0$.

DEFINITION 1.3. [1] An $n$-by-$n$ sign pattern $\mathscr{A}$ is said to be potentially eventually positive (PEP) if there exists some $A \in Q(\mathscr{A})$ that is eventually positive.

DEFINITION 1.4. [5] An $n$-by-$n$ sign pattern $\mathscr{A}$ is said to be potentially power-positive (PPP) if there exists some $A \in Q(\mathscr{A})$ that is power-positive.

Next, we consider the minimal PEP and PPP sign patterns.

DEFINITION 1.5. An $n$-by-$n$ sign pattern $\mathscr{A}$ is said to be minimal potentially eventually positive (MPEP) if $\mathscr{A}$ is PEP and no proper subpattern of $\mathscr{A}$ is PEP.

DEFINITION 1.6. An $n$-by-$n$ sign pattern $\mathscr{A}$ is said to be minimal potentially power-positive (MPPP) if $\mathscr{A}$ is PPP and no proper subpattern of $\mathscr{A}$ is PPP.

## 2. Some properties of MPPP sign patterns

We begin this section by quoting some fundamental results which were stated respectively in [1] and [5].

LEMMA 2.1. [1, Theorem 2.1] *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern. If the signed digraph $\Gamma(\mathscr{A}^+)$ of its positive part is primitive, then $\mathscr{A}$ is PEP.*

LEMMA 2.2. [5, Theorem 3.1] *The $n$-by-$n$ sign pattern $\mathscr{A}$ is PPP if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is PEP.*

For an $n$-by-$n$ sign pattern $\mathscr{A}$, identification of the sufficient and necessary condition for $\mathscr{A}$ to be PEP is open. So, Lemma 2.2 does not characterize the PPP sign patterns. The following lemma provides two sufficient conditions for an $n$-by-$n$ sign pattern $\mathscr{A}$ to be PPP.

LEMMA 2.3. [5, Theorem 3.2] *If $\Gamma(\mathscr{A}^+)$ or $\Gamma(\mathscr{A}^-)$ is primitive, then $\mathscr{A}$ is potentially power-positive.*

The following example illustrates that the sufficient condition in Lemma 2.3 is not a necessary condition for an $n$-by-$n$ sign pattern $\mathscr{A}$ to be PPP.

EXAMPLE 2.4. The 3-by-3 sign pattern

$$\mathfrak{A} = \begin{pmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{pmatrix}$$

is PPP, but neither $\Gamma(\mathfrak{A}^+)$ nor $\Gamma(\mathfrak{A}^-)$ is primitive. And the pattern

$$\mathfrak{B} = \begin{pmatrix} + & - & 0 \\ + & - & - \\ - & + & + \end{pmatrix}$$

is PPP with $\Gamma(\mathfrak{B}^-)$ primitive and $\Gamma(\mathfrak{B}^+)$ imprimitive.

*Proof.* $\mathfrak{A}$ is PEP by Example 2.2 in [1] and is PPP by Lemma 2.2. $\mathfrak{A}^+$ is reducible, so $\Gamma(\mathfrak{A}^+)$ is not primitive. Since the greatest common divisor of lengths of cycles in $\Gamma(\mathfrak{A}^-)$ is not 1, $\Gamma(\mathfrak{A}^-)$ is not primitive. $\mathfrak{A}$ is a subpattern of $\mathfrak{B}$, hence, it follows that $\mathfrak{B}$ is also PEP and PPP. But the greatest common divisor of lengths of cycles in $\Gamma(\mathfrak{B}^-)$ is 1 and $\Gamma(\mathfrak{B}^-)$ is strongly connected, so $\Gamma(\mathfrak{B}^-)$ is primitive. The fact that $\Gamma(\mathfrak{B}^+)$ is not primitive follows from that $\Gamma(\mathfrak{B}^+)$ is not strongly connected. $\square$

Next, we turn our attention to necessary conditions for MPPP sign patters.

THEOREM 2.5. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern with $n \geqslant 2$. If $\mathscr{A}$ is MPPP, then the following statements hold:*

*(1) $\mathscr{A}$ is irreducible;*

*(2) Each row of $\mathscr{A}$ has at lest one nonzero;*

*(3) Each column of $\mathscr{A}$ has at lest one nonzero;*

*(4) The number of nonzero entries of $\mathscr{A}$ is not less than $n+1$.*

*Proof.* Assume that $\mathscr{A}$ is reducible. Then there exists a permutation pattern $P$ such that $P^T\mathscr{A}P = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ with $A_{11}$ and $A_{22}$ being square. It follows that neither $P^TAP$ nor $-P^TAP$ is PEP. Thus $\mathscr{A}$ is not PPP, which is a contradiction.

(2) and (3) follow readily from (1).

By Lemma 2.2 above and Corollary 4.5 in [1], The minimum number of nonzero entries of $\mathscr{A}$ is not less than $n+1$. $\square$

Next we consider some properties of MPPP sign patterns and some connections between the MPPP sign patterns and MPEP sign patterns. The following result is obvious and we omit its proof here.

PROPOSITION 2.6. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern. Then the following statements are equivalent:*

*(1) $\mathscr{A}$ is MPPP;*

*(2) $-\mathscr{A}$ is MPPP;*

*(3) $\mathscr{A}^T$ is MPPP;*

*(4) $P^T\mathscr{A}P$ is MPPP, where $P$ is an $n$-by-$n$ permutation pattern.*

THEOREM 2.7. *Let $\mathscr{A}$ be an $n$-by-$n$ $(n \geqslant 2)$ sign pattern with at most $n+1$ nonzero entries. Then $\mathscr{A}$ is PPP if and only if $\mathscr{A}$ is MPPP.*

*Proof.* It is sufficient to show the necessity. If $\mathscr{A}$ is PPP and not MPPP, then there exists a proper subpattern $\mathscr{B}$ such that $\mathscr{B}$ is PPP and has at most $n$ nonzero entries. So Theorem 2.5 is contradicted. Hence, $\mathscr{A}$ is MPPP. $\square$

EXAMPLE 2.8. The sign pattern

$$\mathfrak{A} = \begin{pmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{pmatrix}$$

is MPPP.

*Proof.* Example 2.4 shows that $\mathfrak{A}$ is PPP. Next we show that $\mathfrak{A}$ is the minimal PPP. Let $\mathbb{A}$ be a PPP subpattern of $\mathfrak{A}$ and let $\mathbb{A}_{ij}$ denote the $(i,j)$ entry of $\mathbb{A}$. We complete this proof by showing that $\mathbb{A}_{ij} \neq 0$ for all $i$ and $j$ such that the $(i,j)$ entry of $\mathfrak{A}$ is nonzero. If $\mathbb{A}_{11} = 0$, then neither $\mathbb{A}$ nor $-\mathbb{A}$ is PEP by Theorem 6.4 in [1]. So $\mathbb{A}$ is not PPP, which is a contradiction. Hence, $\mathbb{A}_{11} \neq 0$. $\mathbb{A}_{21} \neq 0$, $\mathbb{A}_{32} \neq 0$ and $\mathbb{A}_{33} \neq 0$ can be shown similarly. If $\mathbb{A}_{12} = 0$ or $\mathbb{A}_{23} = 0$, then $\mathbb{A}$ is reducible. By Theorem 2.5, $\mathbb{A}$ is not PPP. So, $\mathbb{A}_{21} \neq 0$ and $\mathbb{A}_{32} \neq 0$. If $\mathbb{A}_{31} = 0$, then $\mathbb{A}$ is a subpattern of the following sign pattern

$$\mathbb{A}^* = \begin{pmatrix} + & - & 0 \\ + & 0 & - \\ 0 & + & + \end{pmatrix}$$

We note that neither $\mathbb{A}^*$ nor $-\mathbb{A}^*$ is PEP by Theorem 5.2 in [1]. Hence, $\mathbb{A}^*$ is not PPP. It follows that no proper subpattern of $\mathfrak{A}$ is PPP and $\mathfrak{A}$ is MPPP. $\square$

Next we establish some connections between the MPPP sign patterns and MPEP sign patterns.

THEOREM 2.9. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern. If $\mathscr{A}$ is MPPP, then either $\mathscr{A}$ or $-\mathscr{A}$ is MPEP.*

*Proof.* If $\mathscr{A}$ is MPPP, then either $\mathscr{A}$ or $-\mathscr{A}$ is PEP by Lemma 2.2. We complete this proof by discussing the following two cases.

*Case* 1. *If $\mathscr{A}$ is PEP, we claim that $\mathscr{A}$ must be MPEP. Assume that there exists a PEP proper subpattern $\mathscr{B}$ of $\mathscr{A}$. By Lemma 2.2, $\mathscr{B}$ is also PPP. Then the assumption that $\mathscr{A}$ is MPPP is contradicted.*

*Case* 2. *If $-\mathscr{A}$ is PEP, we claim that $-\mathscr{A}$ must be MPEP. If there exists a PEP proper subpattern $\mathscr{B}$ of $-\mathscr{A}$. By Lemma 2.2, $\mathscr{B}$ is also PPP. We claim that $-\mathscr{B}$ is PPP for $B^k > 0$ implies $(-B)^{2k} > 0$ for some real matrix $B \in Q(\mathscr{B})$. We obtain that $-\mathscr{A}$ is not MPPP. It follows that $\mathscr{A}$ is not MPPP by Proposition 2.6. This is a contradiction. Thus, $-\mathscr{A}$ must be MPEP.* $\square$

THEOREM 2.10. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern. If both $\mathscr{A}$ and $-\mathscr{A}$ are MPEP, then $\mathscr{A}$ is MPPP.*

*Proof.* Since $\mathscr{A}$ is MPEP, $\mathscr{A}$ is PPP. Assume that $\mathscr{A}$ is not MPPP. Then there exists a proper subpattern $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}$ is PPP. By Lemma 2.2 and the assumption that $\mathscr{A}$ is MPEP, $-\mathscr{B}$ is PEP and PPP. Hence, $-\mathscr{A}$ is not MPPP since $-\mathscr{B}$ is a proper subpattern of $-\mathscr{A}$, which is a contradiction. So $\mathscr{A}$ is MPPP. $\square$

We doubt whether there exists a sign pattern $\mathscr{A}$ such that both $\mathscr{A}$ and $-\mathscr{A}$ are MPEP. It is apparent from Section 3 that the order of such a sign pattern (if any) is greater than 3.

THEOREM 2.11. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern. If $\mathscr{A}$ contains at most $n+1$ negative entries, then $\mathscr{A}$ is MPPP if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is MPEP.*

*Proof.* The necessity is shown by Theorem 2.9. For the sufficiency, without loss of generality assume that $\mathscr{A}$ is MPEP. Then $\mathscr{A}$ is PPP. Suppose $\mathscr{A}$ is not MPPP. Then there exists a proper subpattern $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}$ is PPP. Then $-\mathscr{B}$ is PEP for $\mathscr{A}$ is MPEP. It is a contradiction for $-\mathscr{B}$ contains at most $n$ positive entries. So $\mathscr{A}$ is MPPP. $\square$

## 3. Classification of MPPP sign patterns of order $n \leqslant 3$

In this section, we use results stated in Section 2 to classify the $n$-by-$n$ MPPP sign patterns of order $n \leqslant 3$. Following [1], we use the notation $?$ to denote one of $0,+,-$, $\ominus$ to denote one of $0$, $-$, and $\oplus$ to denote one of $0$, $+$.

PROPOSITION 3.1. *Let $\mathscr{A}$ be a 1-by-1 sign pattern, then the following statements are equivalent:*
*(1) $\mathscr{A}$ is PPP;*
*(2) $\mathscr{A} = [+]$ or $[-]$;*
*(3) $\mathscr{A}$ is MPPP.*

*Proof.* Proposition 3.1 can be verified directly. $\square$

THEOREM 3.2. *Let $\mathscr{A}$ be a 2-by-2 sign pattern. Then $\mathscr{A}$ is MPPP if and only if $\mathscr{A}$ is equivalent to*

$$\begin{pmatrix} + & + \\ + & 0 \end{pmatrix} \text{ or } \begin{pmatrix} - & - \\ - & 0 \end{pmatrix}.$$

*Proof.* By Lemma 2.3,

$$\begin{pmatrix} + & + \\ + & 0 \end{pmatrix} \text{ and } \begin{pmatrix} - & - \\ - & 0 \end{pmatrix}$$

are PPP. Since each of the above two sign patterns contains 3 nonzero entries,

$$\begin{pmatrix} + & + \\ + & 0 \end{pmatrix} \text{ and } \begin{pmatrix} - & - \\ - & 0 \end{pmatrix}$$

are MPPP by Theorem 2.7. Conversely, suppose that a 2-by-2 sign pattern $\mathscr{A}$ is PPP. Then $\mathscr{A}$ must contain at least 3 nonzero entries. Up to equivalence,

$$A = \begin{pmatrix} * & * \\ * & ? \end{pmatrix} \text{ and } \begin{pmatrix} * & ? \\ * & * \end{pmatrix},$$

where $*$ denotes nonzero entries. By Lemma 2.2 above and Theorem 5.2 in [1], we obtain that $\mathscr{A}$ is equivalent to

$$\begin{pmatrix} + & + \\ + & ? \end{pmatrix} \text{ or } \begin{pmatrix} - & - \\ - & ? \end{pmatrix}.$$

Since $A$ is MPPP, the entry denoted by $?$ must be 0. It follows that $\mathscr{A}$ is equivalent to

$$\begin{pmatrix} + & + \\ + & 0 \end{pmatrix} \text{ or } \begin{pmatrix} - & - \\ - & 0 \end{pmatrix}. \qquad \square$$

LEMMA 3.3. *The 3-by-3 sign patterns $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ and $\mathscr{A}_5$ are MPPP, where*

$$\mathscr{A}_1 = \begin{pmatrix} + & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{pmatrix}, \mathscr{A}_2 = \begin{pmatrix} + & + & + \\ + & 0 & 0 \\ + & 0 & 0 \end{pmatrix}, \mathscr{A}_3 = \begin{pmatrix} 0 & + & 0 \\ + & 0 & + \\ + & 0 & 0 \end{pmatrix},$$

$$\mathscr{A}_4 = \begin{pmatrix} + & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{pmatrix}, \mathscr{A}_5 = \begin{pmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{pmatrix}.$$

*Proof.* Example 2.8 indicates that $\mathscr{A}_5$ is MPPP. Note that a nonnegative sign pattern is PPP if and only if it is primitive. Clearly, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_4$ are nonnegative and primitive. Hence, they are PPP. It is easily observed that no proper subpattern of $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ or $\mathscr{A}_4$ is primitive. Thus, $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_4$ are MPPP. $\square$

THEOREM 3.4. *Let $\mathscr{A}$ be a 3-by-3 sign pattern. Then $\mathscr{A}$ is MPPP if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is equivalent to one of $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ and $\mathscr{A}_5$ as defined in Lemma 3.3.*

*Proof.* We show the sufficiency first. If $\mathscr{A}$ is equivalent to one of $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$ and $A_5$, then $\mathscr{A}$ is MPPP by Lemma 3.3. If $\mathscr{A}$ is equivalent to one of $-\mathscr{A}_1$, $-\mathscr{A}_2$, $-\mathscr{A}_3$, $-\mathscr{A}_4$ and $-\mathscr{A}_5$, then $-\mathscr{A}$ is MPPP. It follows that $\mathscr{A}$ is MPPP by Proposition 2.6.

We next show the necessity. If $\mathscr{A}$ is MPPP, then either $\mathscr{A}$ or $-\mathscr{A}$ is MPEP by Theorem 2.9. Without loss of generality we assume that $\mathscr{A}$ is MPEP. The $\mathscr{A}$ is PEP and no proper subpattern is PEP. By Theorem 6.4 in [1], either the positive part of $\mathscr{A}$ is primitive or $\mathscr{A}$ is equivalent to a sign pattern of the form

$$\mathscr{U} = \begin{pmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{pmatrix}.$$

We complete this proof by discussing the following two cases.

*Case* 1. Suppose that $\Gamma(\mathscr{A}^+)$ is primitive. For the minimality, $\Gamma(\mathscr{A}^+)$ must contain one 3-cycle and one 1-cycle or one 2-cycle, or two 2-cycles and one 1-cycle. If $\Gamma(\mathscr{A}^+)$ contains one 3-cycle and one 1-cycle, then $\mathscr{A}^+$ is equivalent to $\mathscr{A}_4$. Since $\mathscr{A}$ is minimal PPP and $\mathscr{A}_4$ is MPPP by Lemma 3.3, without loss of generality, let

$\mathscr{A} = \mathscr{A}^+$. Then $\mathscr{A}$ is equivalent to $\mathscr{A}_4$. Similarly, if $\Gamma(\mathscr{A}^+)$ contains one 3-cycle and one 2-cycle, then $\mathscr{A}$ is equivalent to $\mathscr{A}_3$. If $\Gamma(\mathscr{A}^+)$ contains two 2-cycles and one 1-cycle, then $\mathscr{A}$ is equivalent to $\mathscr{A}_1$ or $\mathscr{A}_2$, up to equivalence.

*Case* 2. Suppose that $\mathscr{A}$ is equivalent to a sign pattern of the form $\mathscr{U}$. Then $\mathscr{A}_5$ is a proper subpattern of $\mathscr{U}$. The fact that $\mathscr{A}$ is equivalent to $\mathscr{A}_5$ follows from that $\mathscr{A}_5$ is MPPP by Lemma 3.3. $\square$

The following two results follow readily from Theorem 3.4.

COROLLARY 3.5. *Let $\mathscr{A}$ be a 3-by-3 sign pattern such that neither $\Gamma(\mathscr{A}^+)$ nor $\Gamma(\mathscr{A}^-)$ are primitive. Then $\mathscr{A}$ is MPPP if and only if $\mathscr{A}$ is equivalent to*

$$\mathfrak{A} = \begin{pmatrix} + & - & 0 \\ + & 0 & - \\ - & + & + \end{pmatrix}, \textit{ or its negation } \begin{pmatrix} - & + & 0 \\ - & 0 & + \\ + & - & - \end{pmatrix}.$$

COROLLARY 3.6. *Let $\mathscr{A}$ be an $n$-by-$n$ sign pattern with $n \leqslant 3$. Then $\mathscr{A}$ is MPPP if and only if either $\mathscr{A}$ or $-\mathscr{A}$ is MPEP.*

It is easily verified that up to equivalence, the only 2-by-2 MPEP sign pattern is $\begin{pmatrix} + & + \\ + & 0 \end{pmatrix}$ and the 3-by-3 MPEP sign patterns are the sign patterns displayed in Lemma 3.3. It follows that for $n \leqslant 3$, there is no $n$-by-$n$ MPEP sign pattern $\mathscr{A}$ such that $-\mathscr{A}$ is also MPEP.

## 4. Concluding remarks

We have explored the connections between MPPP sign patterns and MPEP sign patterns in this paper. Some properties of MPPP sign patterns have been established. We also classified the MPPP sign patterns of order $n \leqslant 3$. It is shown that there are only two 2-by-2 MPPP sign patterns and ten 3-by-3 MPPP sign patterns, up to equivalence. However, identification of the sufficient and necessary conditions for an $n$-by-$n$ sign pattern ($n \geqslant 4$) to be MPPP is still open. We conjecture that every $n$-by-$n$ MPPP sign pattern contains at least $n-1$ entries equal to 0. Also open is the existence of MPEP sign patterns $\mathscr{A}$ such that $-\mathscr{A}$ is also MPEP.

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