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CENTRAL AND ALMOST CONSTRAINED SUBSPACES OF BANACH SPACES

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Abstract. In this paper we continue the study of central subspaces initiated in [2] and its infinite version called almost constrained subspaces. We are interested in studying situations where these intersection properties of balls lead to the existence of a linear projection of norm one. We show that every finite dimensional subspace is a central subspace only in Hilbert spaces. By considering direct sums of Banach space we give examples where central subspaces are almost constrained or one-complemented. We show that a M-ideal can fail to be a central subspace, answering a question raised in [2].

1. Introduction

Let X be a real Banach space. Let $B(x,r)$ denote the closed ball centered at x and radius $r>0$. Let X_1 denote the closed unit ball. We say that a finite or infinite collection $\{B(x_i,r_i)\}$ of closed balls intersect in X if $\bigcap B(x_i,r_i) \neq \emptyset$. We say that a collection of balls almost intersect, if for any $\varepsilon > 0$, there is a $x_\varepsilon \in \bigcap B(x_i, r_i+\varepsilon)$.

In this paper we study three forms of intersection properties of balls for closed subspaces $Y \subset X$ that are related to the existence of Chebyshev centres for finite sets and one-complementability, i.e, there exists an onto linear projection $P : X \to Y$ such that $\|P\| = 1$.

DEFINITION 1. A closed subspace $Y \subset X$ is said to be a central subspace if for any finite set $\{y_i\}_{1 \leqslant i \leqslant n} \subset Y$ and $x_0 \in X$, there exists a $y_0 \in Y$ such that $\|y_0 - y_i\| \leqslant \|x_0 - y_i\|$ for $1 \leqslant i \leqslant n$.

It is easy to see that this is equivalent to every finite collection of balls with centres from Y that intersect in X, also intersect in Y.

This notion was motivated by existence of Chebyshev centres for finite sets. We recall that for a finite set $\{x_1,..,x_n\}$, x_0 is said to be a Chebyshev centre (relative to X), if $\max_{1 \leqslant i \leqslant n} \|x_0 - x_i\| = \inf_{x \in X} \max_{1 \leqslant i \leqslant n} \|x - x_i\|$.

It is easy to see that if $Y \subset X$ is a central subspace, then finite subsets of Y that have Chebyshev centres in X have Chebyshev centres (relative to Y) in Y.

A particularly interesting situation occurs when X is a central subspace of X^{**} under the canonical embedding. Using the weak*-compactness of balls in X^{**}, it was

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proved in [2] (Proposition 2.9) that X is a central subspace of X^{**} if and only if every finite family of almost intersecting balls in X intersect. Clearly this happens in dual spaces and is preserved under projections of norm one. Spaces that have this property were called as spaces with generalized centres (GC for short) and were investigated by Veselý in [20] and [19].

Lindenstrauss in [14](page 61) gave a renorming of ℓ^∞ in which there is a closed subspace with three balls that almost intersect but do not intersect. In an equivalent formulation, Konyagin [12] showed that any non-reflexive Banach space has a renorming in which there is a hyperplane where a set of three elements that fails to have Chebyshev centre.

An infinite version of central subspaces called almost constrained subspaces was investigated in [3] and [4].

DEFINITION 2. $Y \subset X$ is said to be an almost constrained subspace if every family of closed balls in Y that intersect in X also intersect in Y.

It is easy to see that any reflexive (or a weak*-closed subspace when X is a dual space) that is a central subspace is an almost constrained subspace. It is easy to see that c_0 is a central subspace of ℓ^∞. By taking the sequence of balls $\{B(e_n, \frac{1}{2})\}$ one can see that they have empty intersection in c_0, where as the constant sequence $\{\frac{1}{2}\}$ is in their intersection. On the other hand, it is easy to show that any finite dimensional central subspace of c_0 or ℓ^∞ is isometric to $\ell^\infty(k)$ for some k and hence is the range of a projection of norm one. An important problem is to find conditions on a Banach spaces X or local conditions on the subspace $Y \subset X$ so that $Y \subset X$ is a central subspace implies $Y^{\perp\perp}$ (which is isometric to Y^{**}, ignoring the canonical isometry, we treat the spaces as the same) is a central subspace of X^{**}. We note that since $Y^{\perp\perp}$ is a weak*-closed subspace, that it is a central subspace of X^{**} implies that it is an almost constrained subspace.

A prime motivation for our study is a consequence of a result of Lindenstrauss (see [14] Theorem 5.9) which says that if $Y \subset X$ is an almost constrained subspace, then for any $x_0 \notin Y$ there exists an onto projection $P : \text{span}\{Y, x_0\} \to Y$, $\|P\| = 1$. See [15], where he also gives an example (in the language of projections) of an almost constrained subspace that is not the range of a norm-one projection. In this equivalent formulation this property was also investigated recently in [11] where they exhibit several classes of function spaces in which almost constrained subspaces are one-complemented. In particular they show that any almost constrained subspace of ℓ^1 is one-complemented. Since any finite dimensional central subspace is almost constrained we get that, for any discrete set Γ, any finite dimensional central subspace of $\ell^1(\Gamma)$ is one-complemented, and thus isometric to $\ell^1(k)$ for some k. We do not know a complete description of central subspaces of $\ell^1(\Gamma)$.

In this paper we investigate conditions under which the notions of central, almost constrained and norm-one complemented coincide. We show that a Banach space is almost constrained in every super space if and only if it is isometric to $C(K)$ for some compact extremally disconnected space and thus it is one-complemented in every super space. We show that any Banach space of dimension greater than or equal to 3, that is not isometric to a Hilbert space has a closed subspace that is not a central subspace.

In sequence spaces, ranges of norm one projections that are of finite co-dimension have been well investigated in [5] and [6]. The notion of generalized centres for hyperplanes and relation with one-complemented subspaces in c_0 was studied in [20]. Motivated by this we show that in a c_0-direct sum of reflexive spaces any factor reflexive (i.e, the quotient space is reflexive), proximinal subspace that is a central subspace is an almost constrained subspace. We also show that in this set up, a proximinal factor reflexive, subspace Y, is a central subspace if and only if its bidual is a central subspace. When the collection of spaces is infinite, we show that no factor reflexive subspace of the c_0-direct sum is almost constrained in the bidual. This result extends the known result that no infinite dimensional subspace of c_0 is almost constrained in ℓ^∞.

A closed subspace $Y \subset X$ is said to be an ideal ([8]), if there is a projection $P : X^* \to X^*$ such that $\ker(P) = Y^\perp$ and $\|P\| = 1$. In this paper we always consider a Banach space X as canonically embedded in its bidual X^{**}. We show that if $Y \subset X$ is an ideal and has the property GC, then Y is a central subspace of X^{**}. We recall from [9] that an ideal $Y \subset X$ is said to be a M-ideal, if the projection P further satisfies, $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. Y is said to be a M-summnad, if $X = Y \bigoplus_\infty Z$ for some closed subspace $Z \subset X$. Any M-summand is a M-ideal. These subspaces can be characterized in terms of intersection properties of balls. See Chapter I of [9]. Classical examples include $c_0 \subset \ell^\infty$ or more generally the c_0-direct sum of a family of Banach spaces in the corresponding ℓ^∞-direct sum.

Using simple geometric properties of M-ideals we show that any Banach space that is a M-ideal in its bidual (under the canonical embedding), if it is almost constrained, then it is reflexive. We also give a simple proof using intersection properties of balls that a M-ideal that is an almost constrained subspace is a M-summand (see [4]). We give an example to show that intersection of two central subspaces can fail to be a central subspace.

We show that for an extremally disconnected compact Hausdorff space Ω and for any finite dimensional central subspace $Y \subset X$, $C(\Omega, Y)$ is an almost constrained subspace of $C(\Omega, X)$.

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2. Classification of Banach spaces

A real Banach space X is said to be a L^1-predual, if X^* is isometric to $L^1(\mu)$ for a positive measure μ. It follows from Lindenstrauss' Characterization of these spaces in terms of intersection properties of balls (see [14] Theorem 6.1) that such a space has the property GC. It also follows from Theorem 6.1 in [14] that X is a L^1-predual space if and only if X^{**} is isometric to a $C(K)$ space for a compact extremally disconnected space K.

It follows from [2] Theorem 3.3, that X is a L^1-predual space if and only if it is a central subspace of every Banach space that contains it. In particular L^1-preduals have the property GC. Our first result addresses this question for almost constrained subspaces. For a compact Hausdorff space K, let $C(K)$-denote the space of real-valued continuous functions, with the supremum norm. See Chapter 3, Section 11 of [13] for properties of $C(K)$ spaces for extremally disconnected compact spaces K.

PROPOSITION 3. *A Banach space X is almost constrained in every super space that contains it if and only if it is isometric to $C(K)$ for some compact extremally disconnected space K. Thus X is one-complemented in every super space.*

Proof. Suppose X is almost constrained in every super space. As already noted the hypothesis in particular implies that X is a central subspace of every super space and hence X is a L^1-predual.

We next show that any family of balls $\{B(x_i,r_i)\}$ in X that pair-wise intersect have non-empty intersection. It then follows from [14] that X is isometric to a $C(K)$ space for a compact extremally disconnected space K.

Consider X as canonically embedded in X^{**}. Since X is a L^1-predual space by Theorem 6.1 of [14], any finite collection of balls from $\{B(x_i.r_i)\}$ have non-empty intersection. Now considering them as balls in X^{**}, as they are also weak*-compact set we get that $\{B(x_i,r_i)\}$ intersects in X^{**}. Since X is almost constrained in X^{**} we get that $\{B(x_i,r_i)\}$ intersect in X. Therefore X is isometric to $C(K)$ for some compact extremally disconnected space K.

It is well known that for a compact extremally disconnected space K, $C(K)$ is one-complemented in every super space (see Chapter 3 of [13]). $\square$

COROLLARY 4. *Let X be a L^1-predual space. $Y \subset X$ is a central subspace if and only if $Y^{\perp\perp} \subset X^{**}$ is a one complemented subspace.*

Proof. Suppose $Y \subset X$ is a central subspace. Since X is a L^1-predual space, as noted before, Y is a L^1-predual space and hence Y^{**} is isometric to a $C(K)$ space for K, compact and extremally disconnected (see [13] Chapter 3, Section 11). As $Y^{\perp\perp}$ is isometric to Y^{**} we have that, $Y^{\perp\perp}$ is one complemented in X^{**}.

Conversely suppose that $Y^{\perp\perp}$ is a one complemented subspace of X^{**}. Since X^{**} is isometric to a $C(K)$ space for a compact and extremally disconnected space K, again by results from Chapter 3, section 11 of [13], we have that $Y^{\perp\perp}$ is isometric to $C(K')$ for some compact extremally disconnected space K'. Thus Y is a L^1-predual space. It now follows from Theorem 3.3 of [2] that Y is a central subspace. $\square$

The subspace question is easier to settle as one only need to consider finite dimensional spaces. This result is formulated for complex Banach spaces. See the monograph by D. Amir, [1] for several characterizations of Hilbert spaces.

THEOREM 5. *Let X be a complex Banach space of dimension* $\dim(X) \geqslant 3$. *X is isometric to a Hilbert space if and only if every finite dimensional subspace is a central subspace.*

Proof. Suppose every finite dimensional subspace $F \subset X$ is a central subspace. Let $F \subset X$ be such that $\dim(F) \geqslant 3$. We need to show that F is the Euclidean space. Let $G \subset F \subset X$. Since G is a finite dimensional central subspace, clearly G is an almost constrained subspace of X and hence in F. Let $x \in F$ and $x \notin G$. From our remarks in the Introduction we have an onto projection $P : \mathrm{span}\{G,x\} \to G$ such that $\|P\| = 1$. Now since $\mathrm{span}\{G,x\}$ is an almost constrained subspace, this process can be continued to get an onto projection $Q : F \to G$ such that $\|Q\| = 1$. Since this is true for every subspace of F, we conclude that F is the Euclidean space. $\square$

3. Intersection properties in direct sums of Banach spaces

As remarked earlier, any central subspace Y of c_0 is a L^1-predual. Thus if it is finite dimensional, it is isometric to $\ell^\infty(k)$. By a simple application of the Hahn-Banach theorem, it is easy to see that an isometric copy of $\ell^\infty(k)$ is one-complemented in any super space. Thus in c_0 or ℓ^∞, any finite dimensional central subspace is one complemented. We do not know if finite dimensional central subspaces are always one complemented.

In this section we consider these concepts for c_0, ℓ^∞ and ℓ^1 direct sums of Banach spaces and certain subspaces of the direct sums. It is easy to see that being a central subspace is preserved by finite ℓ^∞ or ℓ^1-direct sums. In particular having the property GC is preserved under finite ℓ^∞-direct sums. In the following theorem, we give an easy proof of the fact that the property GC gets preserved by c_0-sums (see [20], Theorem 4.7). We recall that c_0 is not an almost constrained subspace of ℓ^∞. In what follows we omit writing the index set for the direct sums.

THEOREM 6. *Let $\{X_i\}_{\{i \in I\}}$ be a family of Banach spaces having the property GC. Then $X = \bigoplus_{c_0} X_i$ is a central subspace of $\bigoplus_\infty X_i$. In particular for any family of dual spaces $\{X_i^*\}_{\{i \in I\}}$, $\bigoplus_{c_0} X_i^*$ is a central subspace of $\bigoplus_\infty X_i^*$.*

Proof. We will show that any finite family of almost intersecting balls in $X = \bigoplus_{c_0} X_i$ intersect. It then follows from Proposition 2.9 in [2] that X has the property GC. As $(\bigoplus_{c_0} X_i)^{**} = \bigoplus_\infty X_i^{**}$ and since $\bigoplus_{c_o} X_i \subset \bigoplus_\infty X_i \subset \bigoplus_\infty X_i^{**}$ under the canonical embedding, it then follows that $\bigoplus_{c_0} X_i$ is a central subspace of $\bigoplus_\infty X_i$.

Let $\{B(x^i, r_i)\}_{1 \leqslant i \leqslant n}$ be any finite family of almost intersecting balls in $\bigoplus_{c_0} X_i$. Let $\delta = \min\{r_i\}$. Choose N large such that $\|x^i(j)\| \leqslant \delta$ for all $j \notin A$ for some finite set $A \subset I$ and for $1 \leqslant i \leqslant n$. Since the property GC is preserved under finite sums, again by Proposition 2. 9 of [2], there exists $x'^j \in X_j$ such that $\|x^i(j) - x'^j\| \leqslant r_i$ for $j \in A$ and for $1 \leqslant i \leqslant n$. Thus by taking $x^0(j) = x'^j$ for $j \in A$ and $x^0(j) = 0$ for $j \notin A$, we have, $x^0 \in \bigoplus_{c_0} X_i$. We therefore have, $\|x^i - x^0\| \leqslant r_i$ for $1 \leqslant i \leqslant n$. $\square$

LEMMA 7. *Let* $X = Z_1 \bigoplus_\infty Z_2$ *and* $Y \subset X$ *be such that* $Y = Y \cap Z_1 \bigoplus_\infty Z_2$. Y *is a central subspace if and only if* $Y \cap Z_1$ *is cental subspace of* Z_1 *and hence of* X.

Proof. Suppose Y is a central subspace of X. Let $z \in Z_1$ and $z_1,..,z_n \in Y \cap Z_1$. By hypothesis there exists a $y_0 \in Y$ such that $\|z_i - y_0\| \leqslant \|z_i - z\|$ for all i. Now let $y_0 = y_{01} + z_{02}$ where $y_{01} \in Y \cap Z_1$, $z_{02} \in Z_2$. Since $z_i - y_{01} \in Z_1$, we have,

$$\|z_i - y_{01}\| \leqslant \max\{\|z_i - y_{01}\|, \|z_{02}\|\} = \|z_i - y_0\| \leqslant \|z_i - z\|.$$

Therefore $Y \cap Z_1$ is a central subspace of Z_1. Since Z_1 is the range of a projection of norm one in X, it is a central subspace of X and thus transitivity $Y \cap Z_1$ is a central subspace of X.

Conversely, suppose $Y \cap Z_1$ is a central subspace of Z_1. Then by our earlier remarks $Y = Y \cap Z_1 \bigoplus_\infty Z_2$ is a central subspace of $X = Z_1 \bigoplus_\infty Z_2$. □

We recall that a closed subspace $Y \subset X$ is said to be proximinal if for every $x \in X$, there exists a $y \in Y$ such that $d(x,Y) = \|x - y\|$.

THEOREM 8. *Let* $\{X_i\}_{\{i \in I\}}$ *be an infinite family of reflexive Banach spaces. Let* $X = \bigoplus_{c_0} X_i$. *Let* $Y \subset X$ *be a factor reflexive, proximinal subspace.* Y *is a central subspace if and only if it is almost constrained.*

Proof. For any $f \in Y^\perp = (X/Y)^*$, by reflexivity, f attains its norm and since Y is proximinal, it is easy to see that f attains its norm on X. Thus if $NA(X)$ denotes the set of all norm attaining functionals in X^*, then $Y^\perp \subset NA(X)$. Since $X^* = \bigoplus_{\ell^1} X_i^*$, it is easy to see that for any $f \in NA(X)$ there exist a finite index set $A \subset I$ such that $f(i) = 0$ for $i \notin A$. As $Y^\perp$ is a Banach space, a simple Baire category argument gives that there is a finite index set A such that $f(i) = 0$ for all $i \notin A$ and for all $f \in Y^\perp$. Thus under the canonical identification, $Y^\perp \subset \bigoplus_{i \in A} X_i^* \subset X^*$. Also note that $X = \bigoplus_{i \in A} X_i \bigoplus_\infty \bigoplus_{i \notin A} X_i$ (the last summand is a c_0-direct sum). Since $\bigoplus_{i \notin A} X_i \subset Y$ we get that $Y = Y \cap \bigoplus_{i \in A} X_i \bigoplus_\infty \bigoplus_{i \notin A} X_i$.

Since A is a finite set, $\bigoplus_{i \in A} X_i$ is a reflexive space. Since Y is a central subspace, by Lemma 7, $Y \cap \bigoplus_{i \in A} X_i$ is a central subspace of $\bigoplus_{i \in A} X_i$. By reflexivity, this is an almost constrained subspace of $\bigoplus_{i \in A} X_i$. It is easy to see that Y is an almost constrained subspace of X. □

The following corollaries are easy to prove using the arguments given during the proof of the above theorem and the fact that any central subspace of $\ell^\infty(k)$ is one-complemented. See Theorem 2 in [20].

COROLLARY 9. *Let* Γ *be a discrete set. Any proximinal subspace* $Y \subset c_0(\Gamma)$ *of finite co-dimension that is a central subspace is one-complemented.*

Similar arguments yield the following corollary. We note that any ℓ^∞-sum of reflexive spaces X_i is the dual of the corresponding ℓ^1-sum of X_i^*.

COROLLARY 10. *Let* $\{X_i\}_{\{i\in I\}}$ *be an infinite family of reflexive Banach spaces. Let* $X = \bigoplus_\infty X_i$ *. Let* $Y \subset X$ *be a factor reflexive subspace such that any* $f \in Y^\perp$ *has at most finitely many non-zero coordinates . Then* Y *is a central subspace if and only if it is almost constrained. In this case* Y *is also a dual space.*

We do not know if the proximinality assumption can be dropped in Theorem 8. In the following corollary we once again use the explicit description of proximinal factor reflexive subspaces.

COROLLARY 11. *Let* $\{X_i\}_{\{i\in I\}}$ *be an infinite family of reflexive Banach spaces. Let* $X = \bigoplus_{c_0} X_i$ *. Let* $Y \subset X$ *be a factor reflexive, proximinal subspace. Then* $Y^{\perp\perp}$ *is an almost constrained subspace of* X^{**} *if and only if* Y *is a central subspace of* X*.*

Proof. As in the proof of Theorem 8 from hypothesis we have that $Y = Y \cap \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i$, for some finite set $A \subset I$, where the second summand is a c_0-direct sum . Now since X_i' as well as $Y \cap \bigoplus_{i\in A} X_i$ are reflexive,

$$Y^{**} = Y^{\perp\perp} = Y \cap \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i.$$

Here the second summand is a ℓ^∞-direct sum. Also $X^{**} = \bigoplus_{i\in A} X_i \bigoplus_\infty \bigoplus_{i\notin A} X_i$, where the second direct sum is a ℓ^∞-direct sum. We note that $Y^{\perp\perp} \cap \bigoplus_{i\in A} X_i = Y \cap \bigoplus_{i\in A} X_i$. Thus by Lemma 7 again we have that $Y^{\perp\perp}$ is a central subspace if and only if Y is a central subspace. Being weak*-closed, $Y^{\perp\perp}$ is an almost constrained subspace of X^{**}. □

In the following proposition we consider quotient spaces.

PROPOSITION 12. *Let* $\{X_i\}_{\{i\in I\}}$ *be an infinite family of reflexive Banach spaces. Let* $X = \bigoplus_{c_0} X_i$ *. Let* $Y \subset X$ *be a proximinal subspace. Then the quotient space* $X|Y$ *is a central subspace of* $X^{**}|Y$*. If* Y *is also factor reflexive,* $X|Y$ *is an almost constrained subspace of* $X^{**}|Y$*.*

Proof. By Theorem 6 we have that X is a central subspace of X^{**}. It follows from Proposition 4. 4 in [2] that $X|Y$ is a central subspace of $X^{**}|Y$. In particular if Y is factor reflexive, $X|Y$ is an almost constrained subspace of $X^{**}|Y$. □

4. Norm one projections in dual spaces

An interesting tool in the study of geometric properties is the notion of an ideal, introduced in [8] and developed in [16].

DEFINITION 13. A closed subspace $Y \subset X$ is said to be an ideal if there exists a projection $P : X^* \to X^*$ such that $\ker(P) = Y^\perp$ and $\|P\| = 1$.

For any Banach space X, since the canonical projection $Q : X^{***} \to X^{***}$ defined by restricting functionals to the canonical image of X, is a projection of norm with $\ker(Q) = X^{\perp}$, we have that X is an ideal in X^{**}.

Clearly the range of a projection of norm one is an ideal. Prime examples of ideals that are, in general are not complemented subspaces, include the space of compact operators $\mathscr{K}(X,Y)$ in the space of bounced operators $\mathscr{L}(X,Y)$ under the assumption that X^* or Y has the metric approximation property. More generally the space of vector-valued continuous functions $C(K,X)$, is an ideal in the space $WC(K,X)$, of functions that are continuous when X has the weak topology (see [16]). Since any M-ideal is an ideal, the following proposition extends Proposition 2.8 in [2].

PROPOSITION 14. *Let $Y \subset X$ be a closed subspace with the property GC. Suppose Y is an ideal in X. Then Y is a central subspace of X^{**} and hence a central subspace of X.*

Proof. Since Y is an ideal, it is easy to see that $P^* : X^{**} \to X^{**}$ is a projection of norm one, with range $Y^{\perp\perp} = Y^{**}$. Now let $\{y_i\}_{1 \leqslant i \leqslant n} \subset Y$ and let $\Lambda \in X^{**}$. Since Y is a central subspace of Y^{**}, for $P^*(\Lambda) \in Y^{**}$, there exists a $y_0 \in Y$ such that $\|y_i - y_0\| \leqslant \|y_i - P^*(\Lambda)\|$. Since P^* is a projection of norm one and $P^*(y_i) = y_i$ for all i, $\|y_i - y_0\| \leqslant \|y_i - P^*(\Lambda)\| = \|P^*(y_i - \Lambda)\| \leqslant \|y_i - \Lambda\|$. Thus Y is a central subspace of X^{**}. As $Y \subset X \subset X^{**}$, we also have that Y is a central subspace of X. $\square$

REMARK 15. In [19] the author exhibits several classes of functions for which $C(K,X)$ has the property GC. It now follows that in all these cases, $C(K,X)$ is a central subspace of $WC(K,X)$.

It is easy to see that if $Y \subset Z \subset X$ and Y is a M-ideal in X then it is a M-ideal in Z. Intersection of finitely many M-ideals is a M-ideal and a finite sum is a closed space and a M-ideal. See [9] Chapter I. These spaces are of particular interest when X under the canonical embedding is a M-ideal in X^{**}. See Chapter III of [9] for geometric properties of these spaces and several examples from function theory and the theory of operators. In particular for any infinite family $\{X_i\}_{\{i \in I\}}$ of reflexive Banach spaces, $\bigoplus_{c_0} X_i$ is a M-ideal in its bidual, $\bigoplus_{\infty} X_i$.

The next two results use the geometric structure of M-ideals in specific situations, to give simple proofs of results indicating their relation with the intersection properties we are considering. These two results can also be deduced from Proposition 20 below, but we prefer to give a direct proof. We recall that X_1 denotes the closed unit ball.

THEOREM 16. *Let $X \subset X^{**}$ be a M-ideal. If X is an almost constrained subspace of X^{**}, then X is reflexive.*

Proof. Let $\Lambda \in X^{**}$ be a unit vector that attains its norm. Thus there exists a $x^* \in X_1^*$ such that $\Lambda(x^*) = 1$. Let $x_0 \in X_1$ be such that $x^*(x_0) > \frac{1}{2}$. Suppose $\Lambda \notin X$. Since X is an almost constrained subspace, there exists a projection P : $\operatorname{span}\{\Lambda, X\} \to X$ with $\|P\| = 1$. Also the hypothesis implies that X is a M-ideal in

$\text{span}\{\Lambda, X\}$. Thus it follows from Corollary I.1.3 in [9] that P is a M-projection (i.e., $\|\tau\| = \max\{\|P(\tau)\|, \|\tau - P(\tau)\|\}$). Now $\frac{1}{2} + 1 < x^*(x_0) + \Lambda(x^*) = P(\Lambda + x_0)(x^*) \leqslant \|\Lambda + x_0\| = \max\{\|\Lambda\|, \|x_0\|\} = 1$. This contradiction shows that $\Lambda \in X$. Since X contains all the norm attaining vectors in X^{**}, we get by the Bishop-Phelps theorem (see [10]) that $X^{**} = X$. Therefore X is reflexive. $\square$

The following corollary illustrates the limitations of Theorem 8 and generalizes the fact that a infinite dimensional subspace of $c_0(\Gamma)$ (for an infinite discrete set Γ) is not almost constrained in $\ell^\infty(\Gamma)$.

COROLLARY 17. *Let $\{X_i\}_{\{i \in I\}}$ be an infinite family of reflexive Banach spaces. Then a central subspace $Y \subset X = \bigoplus_{c_0} X_i$ is almost constrained in X^{**} if and only if it is reflexive. Hence no factor reflexive subspace $Y \subset X$ is almost constrained in X^{**}.*

Proof. Suppose Y is an almost constrained subspace of X^{**}. Since $Y \subset Y^{**} \subset X^{**}$, we have that Y is an almost constrained subspace of Y^{**}. As the property, being a M-ideal in its bidual, is hereditary (Theorem III.1.6 of [9]) we get that Y is a M-ideal in its bidual and thus by the above result Y is reflexive. Conversely suppose Y is a reflexive, central subspace of X. By Theorem 6 we have that X is a central subspace of X^{**} and thus by transitivity, Y is a central subspace of X^{**}. Since Y is reflexive, it is an almost constrained subspace of X^{**} as well. Now since the collection of spaces is infinite, Y can not be both reflexive and factor reflexive. Thus a factor reflexive subspace Y can not be almost constrained in X^{**}. $\square$

The next result concerns M-ideals in the space of operators $\mathscr{L}(X,Y)$ and is related to the well studied problem of possible one-complementability of the space of compact operators. For $x^* \in X^*$ and $y \in Y$, by $x^* \otimes y$ we denote the rank one operator, $(x^* \otimes y)(x) = x^*(x)y$.

THEOREM 18. *Let $M \subset \mathscr{L}(X,Y)$ be a M-ideal containing all rank one operators. If M is almost constrained in $\mathscr{L}(X,Y)$, then $M = \mathscr{L}(X,Y)$.*

Proof. Let $T \in \mathscr{L}(X,Y)$, $\|T\| = 1$ and $T \notin M$. Let $\|x\| = 1$ and $\|T(x)\| > \frac{3}{4}$. Let $\|y^*\| = 1$ and $y^*(T(x)) = \|T(x)\|$. Suppose M is almost constrained and as before there is a M-projection $P : \text{span}\{M, T\} \to M$. By hypothesis $T^*(y^*) \otimes T(x) \in M$.

$$\|(T^*(y^*) \otimes T(x) + T)(x)\| = \|(\|T(x)\| + 1)T(x)\| = (1 + \|T(x)\|)\|T(x)\|$$

$$\leqslant \|T^*(y^*) \otimes T(x) + T\| = \max\{\|T^*(y^*) \otimes T(x)\|, \|T\|\} = 1$$

This is a contradiction since $\|T(x)\| > \frac{3}{4}$. Thus $M = \mathscr{L}(X,Y)$ $\square$

REMARK 19. In [3] the authors use analysis of unique norm-preserving extensions to conclude that any M-ideal $Y \subset X$ that is almost constrained is a M-summand (i.e, range of a M-projection). We next give a simple proof of this result using a characterization of M-summands (Proposition II. 3. 4 from [9]) involving intersection properties of balls. If any family of closed balls that intersect in X and each has a point in common with Y, intersects in Y, then Y is a M-summand.

PROPOSITION 20. *Let $Y \subset X$ be a M-ideal. If Y is an almost constrained subspace, then it is a M-summand.*

Proof. Let x be such that $d(x,Y) = 1$. We show as in the proof of $(ii) \Rightarrow (i)$ in the proof of Proposition II.3.4 of [9], $\{B(x+y, 1+\varepsilon)\}_{\|y\|<1, \varepsilon>0}$ intersect in Y.

Since $d(x,Y) = d(x+y,Y) = 1$, clearly each of these balls meet Y and x is in the intersection of all of them. Since $x \notin Y$, and Y is almost constrained, as before there exists a M-projection $P : \text{span}\{Y,x\} \to Y$. Clearly all the balls contain $P(x)$. Therefore Y is a M-summand. $\square$

REMARK 21. Contained in the above proof is the observation that $Y \subset X$ is the range of a M-projection if and only if, it is the range of a M-projection in $\text{span}\{x,Y\}$ for all $x \notin Y$.

We next give an example to show that M-ideals in general need not be central subspaces, and do not inherit the property GC, answering Question 2.7 in [2].

EXAMPLE 22. Let $X = c_0$ and let $Y = \ker(f)$ where $f = (2,1,-1,1,\frac{1}{2},\frac{1}{4},..,..) \in \ell^1$. Since $\|f\|_\infty = 2 < \frac{1}{2}\|f\|_1$, and as all the coordinates are non-zero, one can show that, Y is neither one-complemented nor proximinal. It follows from Theorem 2 in [20] that Y fails to have property GC and hence is not a central subspace of Y^{**}. We recall that c_0 is a M-ideal in its bidual, ℓ^∞ and as this property is hereditary (Theorem III.1.6 of [9]) we get that Y is a M-ideal in its bidual Y^{**}. Also since Y^{**} being a dual space, has the property GC which is not inherited by the subspace Y.

There is a weaker notion of M-ideal called semi-M-ideals studied in [9]. These can be characterized by intersection properties involving only 2 balls. The next proposition clarifies their relation with ideals.

PROPOSITION 23. *Let $Y \subset X$ be a semi-M-ideal. Y is an ideal if and only if it is a M-ideal.*

Proof. Suppose Y is a semi-M-ideal and ideal. Let $P : X^* \to X^*$ be a projection of norm one with $\ker P = Y^\perp$. Since Y is a semi-M-ideal there exists a $Q : X^* \to X^*$ with $\ker Q = Y^\perp$, $\|x^*\| = \|Q(x^*)\| + \|x^* - Q(x^*)\|$, $Q(\lambda x^* + Q(y^*)) = \lambda Q(x^*) + Q(y^*)$ for all $x^*, y^* \in X^*$. It now follows from the arguments given during the proof of Proposition I.1.2 in [9], which does not require the projection Q to be linear, that $P = Q$. Thus Q is a linear L-projection with $\ker Q = Y^\perp$ and hence Y is a M-ideal. $\square$

In [6] the authors study conditions under which intersections of ranges of norm-one projections is again the range of a norm one-projection. Our next example shows that intersection of two finite dimensional central subspaces in ℓ^∞ can fail to be a central subspace.

EXAMPLE 24. Consider $\ell^\infty(4)$. Let $f_1 = (1,0,0,0), f_2 = (\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6})$. It is easy to see that $\ker(f_1)$ and $\ker(f_2)$ are ranges of projections of norm one. Now $\ker(f_1)\cap\ker(f_2) = \{(0,x_1,x_2,-x_1-x_2)\}$. Clearly it is enough to show that $Y = \{(x_1,x_2,-x_1-x_2)\}$ is not a central subspace of $\ell^\infty(3)$. Let $x_0 = (-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ and let $y_1 = (-2,1,1)$, $y_2 = (1,1,-2)$, $y_3 = (1,-2,1)$. Note that $\|x_0 - y_i\| = \frac{3}{2}$ and is the only point in $\cap_1^3 B(y_i,\frac{3}{2})$.

REMARK 25. There are some special situations where intersection of central subspaces is again a central subspace. Let $Y \subset X$ be a M-ideal and central subspace. Suppose $X = Z_1 \bigoplus_\infty Z_2$. Then $Y \cap Z_i$ is a central subspace of X. To see this we note that, by Lemma I.3.5 in [9], $Y = Y\cap Z_1 \bigoplus_\infty Y \cap Z_2$. Thus $Y\cap Z_i$ is a central subspace of Y and hence by transitivity, a central subspace of X .

Let $M,N \subset X$ be two M-ideals, in Proposition 4.10 in [2] the authors assume that $M\cap N$ is a reflexive space to deduce the property GC for the quotient space $M + N/(M\cap N)$. In the following proposition we derive these conclusions assuming that $M\cap N$ is a M-summand in $M+N$. We note that since $M\cap N$ is always a M-ideal, that it is reflexive implies that it is a M-summand in X and hence in $M+N$.

PROPOSITION 26. *Let $M,N \subset X$ be two M-ideals having the property GC. Suppose $M\cap N$ is a M-summand in $M+N$. Then $M+N$ has the property GC.*

Proof. Since $M+N$ is a closed M-ideal, it is easy to see that $M+N/(M\cap N) = M/(M\cap N)\bigoplus_\infty N/(M\cap N)$. Now since $M\cap N$ is a M-summand in $M+N$, it is also a M-summand in M and N. Thus both $M/(M\cap N)$ and $N/(M\cap N)$ have the property GC. Therefore $M+N$ has the property GC. $\square$

Our next result we give a simple proof of the 'if' part of Theorem 5.5 from [5], using our earlier technique of dealing with finite co-dimensional subspaces.

PROPOSITION 27. *Let $Y \subset \ell^1$ be a finite co-dimensional subspace that is an intersection of hyperplanes that are ranges of projections of norm one. Then Y is the range of a projection of norm one.*

Proof. Suppose $Y = \cap_{1\leqslant i\leqslant n}\ker(f_i)$ for $f_i \in \ell^\infty$ such that $\ker(f_i)$ is the range of a projection of norm one for every i. It follows from [7] that each f_i has at most 2 non-zero components. Since permutations and multiplication by ± 1 are isometries we may assume that non-zero coordinates occur consecutively . If the 2-element blocks have no overlap then Y consists of ℓ^1-direct sum of one-dimensional spaces and hence is the range of a projection of norm one. It is easy to see that similar thing happens when the 2-blocks overlap also. To illustrate this, suppose $n = 2$, we may assume w. l. o. g that $f_1 = (x_1,x_2,0,...,0)$ and suppose $f_2 = (0,y_2,y_3,0,...,0)$. Thus as in our earlier arguments the problem is reduced to $\ell^1(3)$ and clearly $Y\cap\ell^1(3)$ being one-dimensional is the range of a norm-one projection. $\square$

We next consider these questions in the space $C(\Omega,X)$, the space of X-valued continuous functions defined on a extremally disconnected Hausdorff spaces Ω, equipped with the supremum norm. The proof of the following theorem is similar to the proof of Theorem 4.11 in [2].

THEOREM 28. *Let Ω be an extremally disconnected compact Hausdorff space. Let $Y \subset X$ be a finite dimensional central subspace. Then $C(\Omega,Y)$ is an almost constrained subspace of $C(\Omega,X)$.*

Proof. Let S be a discrete set. We will first prove the theorem for the Stone-Check compactification $\Omega = \beta(S)$. Since Y is finite dimensional, it is easy to see that $C(\beta(S),Y)$ is isometric to $\bigoplus_\infty Y$ where the sum is taken over the cardinal $|S|$. Similarly one can show that $C(\beta(S),X)$ is isometric to a subspace of $\bigoplus_\infty X$. It is easy to see that $\bigoplus_\infty Y$ is an almost constrained subspace of $\bigoplus_\infty X$ and hence an almost constrained subspace of $C(\beta(S),X)$.

Since Ω is extremally disconnected, there exists a discrete set S such that Ω is homeomorphic to a subset of $\beta(S)$ and there exists a retract $\phi : \beta(S) \to \Omega$. Now considering the balls in $C(\Omega,Y)$ that intersect in $C(\Omega,X)$ and composing with ϕ, one can use the result already established for $\beta(S)$. Now restricting the functions to Ω, as ϕ is a retract, gives that $C(\Omega,Y)$ is an almost constrained subspace of $C(\Omega,X)$. $\square$

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