PARA–ORTHOGONAL RATIONAL MATRIX–VALUED FUNCTIONS ON THE UNIT CIRCLE

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Abstract. In this paper, we continue previous investigations with the ultimate goal being a Szegő theory for orthogonal rational matrix functions. We implement here the concept of para-orthogonal functions on the unit circle in the context of rational matrix functions and present some fundamental properties of the para-orthogonal functions in question. We discuss, among other things, the relationship between these functions and orthogonal rational matrix functions as well as existence criteria and some para-orthogonal functions of particular interest.

1. Introduction

Referencing and building on [30] and [31], the present paper continues towards a Szegő theory of orthogonal rational matrix functions on the unit circle $\mathbb{T}$. In doing so, we turn to the work of Bultheel, González-Vera, Hendriksen, and Njåstad on orthogonal rational (complex-valued) functions and use the monograph [10] as guide. These authors prepared systematically the topic of a rational generalization of the classical theory of orthogonal polynomials on $\mathbb{T}$ which goes back to Szegő (see, e.g., [53] as well as [38], [47], and [54]). However, first considerations on orthogonal rational functions on $\mathbb{T}$ occur already in the work of Djrbashian (see, e.g., [26]). We will also refer to the paper [55] of Velázquez, where a spectral approach to orthogonal rational functions is pointed out. As a further aside, we mention the paper [44] of Njåstad and Velázquez, where a remarkable formula for orthogonal polynomials due to Khrushchev (see [41, Theorems 2 and 3]) is extended to the rational case.

We develop here the basic concept of para-orthogonal functions in the context of rational matrix-valued functions on $\mathbb{T}$. Note that in the case of para-orthogonal rational (complex-valued) functions, it is possible to obtain quadrature formulas on $\mathbb{T}$ just as in the classical case of polynomials (see, e.g., [8] and concerning explicit expressions and numerical examples [6]). Para-orthogonal polynomials on $\mathbb{T}$ seem to have been first introduced by Jones, Njåstad, and Thron in [40], even though some related aspects had already been presented in a more classical context by Geronimus [37], Grenander and Szegő [38], and also Szegő [53]. (For more information on para-orthogonal polynomials and para-orthogonal rational functions on $\mathbb{T}$, see also [4], [16], [18], [19], [22], [39], [42], [47]–[52], [56] and [7]–[14], [23], respectively.)

Classical Gaussian quadrature formulas are exact on sets of polynomials and optimal in a sense (for a survey, see [35]). The Szegő quadrature formulas are the analogs for these on the unit circle $\mathbb{T}$. In that context the formulas are exact on sets of Laurent polynomials. (As an aside, there are slightly modified techniques of computing integrals with respect to Borel measures supported on $\mathbb{T}$ via quadrature formulas as well; see, e.g., [3], [21], and [46].)

Since Laurent polynomials are rational functions with poles at the origin and at infinity, the step towards a more general situation seems natural, where the poles are at several other fixed positions. This gave rise to a discussion of orthogonal rational functions and para-orthogonal rational functions with arbitrary, but fixed poles. In this case (see, e.g., [8]), the nodes are the zeros of corresponding para-orthogonal rational functions and the quadrature formula is an integral of the rational Lagrange interpolant in these nodes, so that the weights can be obtained as the integral of rational Lagrange basis functions. (In view of some numerical aspects of these rational Szegő quadrature formulas we refer to [25], where similar to the case of rational quadrature formulas on an interval studied in [24] and [36] error bounds are given and compared with other bounds appearing in the literature.)

An alternative approach to a rational kind of Szegő quadrature formulas, by using the Hermite interpolation, is pointed out in [9]. Based on the recently obtained matricial representation for orthogonal rational functions on $\mathbb{T}$ in [55], a further way to calculate the nodes and weights to rational quadrature formulas is presented in [5] (see also [15]). This exposes particularly an interrelation between para-orthogonal rational functions on $\mathbb{T}$ and eigenvalue problems for special matrices.

As another extension of the classical considerations on para-orthogonal polynomials on $\mathbb{T}$ and related quadrature formulas, one can already find this topic with respect to matrix polynomials in the literature (see, e.g., [20], [50], and [51]). In particular, we present in this paper some results which can be regarded also as a generalization of well-known facts in the theory of matrix polynomials to the rational case.

The considerations in the paper at hand are also motivated by those in [32] dealing with particular solutions of the matricial Carathéodory problem in the nondegenerate case that are extremal in several directions (see also [33], [34], and [43]). In particular, [32] is concerned with an extremal problem using an approach based on a general result due to Arov (see [1]). More specifically, this problem is that of determining a matrix-valued Carathéodory function such that its Riesz–Herglotz measure produces the maximal value of the mass $F(\{u\})$ for some fixed point $u \in \mathbb{T}$ if F varies over the Riesz–Herglotz measures of all solutions of the problem. In the classical case for complex-valued functions, this problem can be handled in analyzing para-orthogonal polynomials on $\mathbb{T}$. Having in mind discussions regarding similar questions (as, for instance, presented in [32, Section 9]) concerning a moment problem for rational matrix functions, we will point out some fundamental properties of para-orthogonal functions on $\mathbb{T}$ in the context of rational matrix-valued functions. These will later be useful.

The approach can be outlined as follows. At first, in Section 2, we review symbols and notation drawn from previous papers on orthogonal rational matrix functions.

We present basic properties of the para-orthogonal functions in question in Section 3. In particular, we discuss some connections between para-orthogonal systems and orthonormal systems (or, alternatively, reproducing kernels) of rational matrix functions. Theorem 3.5 (or Corollary 3.8) shows that the relationship in the matrix case is quite similar to the well-known scalar case of rational (complex-valued) functions.

In Sections 4 and 5, we focus on the fundamental existence problem for para-orthogonal systems of rational matrix functions. We present a thorough investigation into the existence of these para-orthogonal functions in terms of the underlying nonnegative Hermitian matrix-valued Borel measure on $\mathbb{T}$ (which is used to define the associated left and right matrix-valued inner products). We will see that dealing with the existence question for para-orthogonal rational matrix functions is somewhat more complicated than in the scalar case. Via Theorem 4.4, we obtain a sufficient condition for existence. Though this is, in general, not necessarily fulfilled. Thus, the condition does not completely agree with the scalar case for rational functions (where this is sufficient and necessary; cf. [14, Theorem 3.5]). We do find, however, that under certain stronger requirements, the situation does closely correspond to the scalar case (see, e.g., Theorems 4.6 and 5.5). Along the way, we verify some auxiliary results on reproducing kernels of rational matrix functions in Section 4. In Section 5, we present some results on molecular matrix-valued Borel measures on $\mathbb{T}$.

Finally, in Section 6, we consider particular para-orthogonal systems of rational matrix functions. These para-orthogonal systems can be used to obtain rational Szegő quadrature formulas in the scalar case of complex-valued functions. The essential result, in the scalar case, is that the zeros of these para-orthogonal rational functions are simple and that all of them are located on $\mathbb{T}$. In Theorem 6.5, we present a matrix version of this result. In particular, this can be regarded as a starting point for obtaining an extension of the Gaussian quadrature formula presented in [50, Theorem 3.3], where instead of matricial Laurent polynomials then (more general) rational matrix functions appear. We intend, however, to return to this topic at a later time.

2. Preliminaries

For convenience, we now review some notation introduced in [30] and [31].

Let $\mathbb{N}_0$ and $\mathbb{N}$ be the set of all nonnegative integers and the set of all positive integers, respectively. For each $k \in \mathbb{N}_0$ and each $\tau \in \mathbb{N}_0$ or $\tau = \infty$, let $\mathbb{N}_{k,\tau}$ be the set of all integers n for which $k \leqslant n \leqslant \tau$. Furthermore, let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit disk and the unit circle in the complex plane $\mathbb{C}$. We will use the notation $\mathbb{C}_0$ for the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Throughout this paper, p and q will be positive integers, unless otherwise indicated. If $\mathfrak{X}$ is a nonempty set, then $\mathfrak{X}^{p \times q}$ stands for the set of all $p \times q$ matrices with elements in $\mathfrak{X}$. If $\mathbf{A} \in \mathbb{C}^{p \times q}$, then $\mathbf{A}^*$ is the adjoint matrix of $\mathbf{A}$. The null space of a matrix $\mathbf{A}$ will be denoted by $\mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})$ will be used for the range of $\mathbf{A}$. The zero matrix in $\mathbb{C}^{p \times q}$ will be denoted by $0_{p \times q}$. (In cases, where there is no chance for confusion, the indices might be omitted.) It will be used $\mathbf{I}_q$ to denote the identity matrix

in $\mathbb{C}^{q \times q}$. For $\mathbf{A} \in \mathbb{C}^{q \times q}$, we use $\det \mathbf{A}$ to denote the determinant of $\mathbf{A}$.

Let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $(\alpha_j)_{j=1}^{\tau}$ be a sequence, where $\alpha_j \in \mathbb{C} \setminus \mathbb{T}$, and let $n \in \mathbb{N}_{0,\tau}$. If $n = 0$, then let $\pi_{\alpha,0}$ be the constant function on $\mathbb{C}_0$ with value 1. Let $\mathscr{R}_{\alpha,0}$ denote the set of all constant complex-valued functions defined on $\mathbb{C}_0$ and let $\mathbb{P}_{\alpha,0} := \emptyset$. If $n \in \mathbb{N}$, then let $\pi_{\alpha,n} : \mathbb{C} \to \mathbb{C}$ be the polynomial defined by

$$\pi_{\alpha,n}(u) := \prod_{j=1}^{n}(1 - \overline{\alpha_j}u) \tag{2.1}$$

and let $\mathscr{R}_{\alpha,n}$ denote the set of all rational functions f that admit a representation

$$f = \frac{1}{\pi_{\alpha,n}}P$$

with some (complex-valued) polynomial P of degree not greater than n. Further, let

$$\mathbb{P}_{\alpha,n} := \bigcup_{j=1}^{n}\left\{\frac{1}{\overline{\alpha_j}}\right\}$$

(using $\frac{1}{0} := \infty$). We also use $\mathscr{O}$ to denote the constant function on $\mathbb{C}_0$ with value $0_{p \times q}$ (for any choice of $p, q \in \mathbb{N}$), where the size $p \times q$ will be clear from the context.

Let $\mathfrak{B}_{\mathbb{T}}$ and $\mathfrak{B}$ be the σ-algebra of all Borel subsets of $\mathbb{T}$ and $\mathbb{C}$, respectively. Furthermore, let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, where $\mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ stands for the set of all nonnegative Hermitian $q \times q$ measures defined on $\mathfrak{B}_{\mathbb{T}}$. As in [28]–[31], the right $\mathbb{C}^{p \times p}$-module $\mathscr{R}_{\alpha,n}^{q \times p}$ (resp., left $\mathbb{C}^{p \times p}$-module $\mathscr{R}_{\alpha,n}^{p \times q}$) is equipped with a complex $p \times p$ matrix inner product which is given by

$$(X,Y)_{F,r} := \int_{\mathbb{T}}(X(z))^{*}F(\mathrm{d}z)Y(z) \quad \left(\text{resp.,} \ (X,Y)_{F,l} := \int_{\mathbb{T}}X(z)F(\mathrm{d}z)(Y(z))^{*}\right)$$

for all $X, Y \in \mathscr{R}_{\alpha,n}^{q \times p}$ (resp., $X, Y \in \mathscr{R}_{\alpha,n}^{p \times q}$). For details on integration theory with respect to nonnegative Hermitian matrix measures, we refer to Rosenberg [45]. Note that

$$(X,Y)_{F,r} = (Y,X)_{F,r}^{*} \quad \left(\text{resp.,} \ (X,Y)_{F,l} = (X,Y)_{F,l}^{*}\right) \tag{2.2}$$

for all $X, Y \in \mathscr{R}_{\alpha,n}^{q \times p}$ (resp., $X, Y \in \mathscr{R}_{\alpha,n}^{p \times q}$). We will pay special attention to the case in which some nondegeneracy condition is fulfilled.

Recall that a nonnegative Hermitian $q \times q$ measure F on $\mathfrak{B}_{\mathbb{T}}$ is called *nondegenerate of order n* if the $q \times q$ block Toeplitz matrix

$$\mathbf{T}_n^{(F)} := \left(\mathbf{c}_{j-k}^{(F)}\right)_{j,k=0}^{n}$$

is nonsingular, where

$$\mathbf{c}_{\ell}^{(F)} := \int_{\mathbb{T}}z^{-\ell}F(\mathrm{d}z)$$

for some integer ℓ. We will write $\mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ for the subset of $\mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ consisting of all nondegenerate measures of order n. Furthermore, we set

$$\mathscr{M}_{\geqslant}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) := \bigcap_{m=0}^{\infty}\mathscr{M}_{\geqslant}^{q,m}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}).$$

Let $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Because of [29, Theorem 5.8] and [28, Theorem 10] one can find that $(\mathscr{R}_{\alpha,n}^{q \times q}, (\cdot, \cdot)_{F,r})$ forms a right $\mathbb{C}^{q \times q}$-Hilbert module with reproducing kernel $K_{n;r}^{(\alpha,F)}$ and that $(\mathscr{R}_{\alpha,n}^{q \times q}, (\cdot, \cdot)_{F,l})$ forms a left $\mathbb{C}^{q \times q}$-Hilbert module with reproducing kernel $K_{n;l}^{(\alpha,F)}$. In fact (cf. [28, Remark 12]), if $X_0, X_1, \ldots, X_n$ is a basis for the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$ (resp., if $Y_0, Y_1, \ldots, Y_n$ is a basis for the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$), then the reproducing kernel $K_{n;r}^{(\alpha,F)}$ (resp., $K_{n;l}^{(\alpha,F)}$) can be represented as follows

$$K_{n;r}^{(\alpha,F)}(v,w) = \Xi_n(v) \left(\left(\int_{\mathbb{T}} (X_j(z))^* F(\mathrm{d}z) X_k(z) \right)_{j,k=0}^n \right)^{-1} (\Xi_n(w))^*$$

$$\left(\text{resp., } K_{n;l}^{(\alpha,F)}(w,v) = (\Upsilon_n(w))^* \left(\left(\int_{\mathbb{T}} X_j(z) F(\mathrm{d}z) (X_k(z))^* \right)_{j,k=0}^n \right)^{-1} \Upsilon_n(v) \right) \tag{2.3}$$

for all $v, w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$, where

$$\Xi_n := \left(X_0, X_1, \ldots, X_n \right) \quad \left(\text{resp., } \Upsilon_n := \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix} \right). \tag{2.4}$$

Furthermore, for each $w \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n}$, let the matrix function $A_{n,w}^{(\alpha,F)} : \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n} \to \mathbb{C}^{q \times q}$ (resp., $C_{n,w}^{(\alpha,F)} : \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n} \to \mathbb{C}^{q \times q}$) be defined by

$$A_{n,w}^{(\alpha,F)}(v) := K_{n;r}^{(\alpha,F)}(v,w) \quad \left(\text{resp., } C_{n,w}^{(\alpha,F)}(v) := K_{n;l}^{(\alpha,F)}(w,v) \right). \tag{2.5}$$

Having regard to [30], we focus on the situation in which the elements of the underlying sequence $(\alpha_j)_{j=1}^n$ are, in a sense, well-positioned concerning the unit circle $\mathbb{T}$. We will denote by $\mathscr{T}_1$ the set of all sequences $(\alpha_j)_{j=1}^{\infty}$ of complex numbers which satisfy $\overline{\alpha_j} \alpha_k \neq 1$ for all $j, k \in \mathbb{N}$. Clearly, if $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$, then $\alpha_j \notin \mathbb{T}$ for each $j \in \mathbb{N}$.

Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. For each $j \in \mathbb{N}$, let

$$\eta_j := \begin{cases} -1 & \text{if } \alpha_j = 0 \\ \dfrac{\overline{\alpha_j}}{|\alpha_j|} & \text{if } \alpha_j \neq 0 \end{cases}$$

and let $b_{\alpha_j} : \mathbb{C}_0 \setminus \left\{ \frac{1}{\overline{\alpha_j}} \right\} \to \mathbb{C}$ be given by

$$b_{\alpha_j}(u) := \begin{cases} \eta_j \dfrac{\alpha_j - u}{1 - \overline{\alpha_j} u} & \text{if } u \in \mathbb{C} \setminus \left\{ \frac{1}{\overline{\alpha_j}} \right\} \\ \dfrac{1}{|\alpha_j|} & \text{if } u = \infty. \end{cases} \tag{2.6}$$

If $B_{\alpha,0}^{(q)}$ stands for the constant function on $\mathbb{C}_0$ with value $\mathbf{I}_q$ and if, for each $k \in \mathbb{N}_{1,n}$

$$B_{\alpha,k}^{(q)} := \left(\prod_{j=1}^{k} b_{\alpha_j} \right) \mathbf{I}_q,$$

then the system $B_{\alpha,0}^{(q)}, B_{\alpha,1}^{(q)}, \ldots, B_{\alpha,n}^{(q)}$ is a basis for the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$ and a basis for the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,n}^{q \times q}$ (cf. [29, Remark 2.4]). More universal (cf. [29, Remark 2.3]), if $X \in \mathscr{R}_{\alpha,n}^{p \times q}$, then there are unique matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ belonging to $\mathbb{C}^{p \times q}$ such that

$$X = \sum_{j=0}^{n} \mathbf{A}_j B_{\alpha,j}^{(q)}.$$

Thereby, the *reciprocal rational* (*matrix-valued*) *function* $X^{[\alpha,n]}$ of X with respect to $(\alpha_j)_{j=1}^{\infty}$ *and* n is given by

$$X^{[\alpha,n]} := \sum_{j=0}^{n} B_{\beta,j}^{(q)} \mathbf{A}_{n-j}^*, \tag{2.7}$$

where $(\beta_j)_{j=1}^{\infty}$ is the sequence defined by $\beta_k := \alpha_{n+1-k}$ for each $k \in \mathbb{N}_{1,n}$ and $\beta_j := \alpha_j$ otherwise (cf. [30, Section 2]). If $\alpha_j = 0$ for each $j \in \mathbb{N}_{1,n}$, a function X belonging to $\mathscr{R}_{\alpha,n}^{p \times q}$ is a $p \times q$ matrix polynomial of degree not greater than n and $X^{[\alpha,n]}$ is just the reciprocal matrix polynomial $\tilde{X}^{[n]}$ of X with respect to $\mathbb{T}$ and formal degree n (as used, e.g., in [27]). In general (see [30, Remark 2.4]), we have $X^{[\alpha,n]} \in \mathscr{R}_{\alpha,n}^{q \times p}$ and

$$\left(X^{[\alpha,n]} \right)^{[\alpha,n]} = X. \tag{2.8}$$

Furthermore (see [30, Remark 4.2]), for all $X, Y \in \mathscr{R}_{\alpha,n}^{p \times q}$, it follows that

$$\left(X^{[\alpha,n]}, Y^{[\alpha,n]} \right)_{F,r} = \left(X, Y \right)_{F,l}. \tag{2.9}$$

Finally, we revisit the concept of orthonormal systems introduced in [30]. Here and in the following, δ_{jk} stands for the Kronecker delta, i.e. $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$.

DEFINITION 2.1. Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}_0$ or $\tau = \infty$. Furthermore, let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. A sequence $(X_k)_{k=0}^{\tau}$ (resp., $(Y_k)_{k=0}^{\tau}$) of matrix functions is called a *left* (resp., *right*) *orthonormal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F if:

(I) For each $k \in \mathbb{N}_{0,\tau}$, the function X_k (resp., Y_k) belongs to $\mathscr{R}_{\alpha,k}^{q \times q}$.

(II) For all $j, k \in \mathbb{N}_{0,\tau}$, the equality $\left(X_j, X_k \right)_{F,l} = \delta_{jk} \mathbf{I}_q$ (resp., $\left(Y_j, Y_k \right)_{F,r} = \delta_{jk} \mathbf{I}_q$) holds.

If $\tau = 0$ or if $\alpha_j = 0$ for all $j \in \mathbb{N}_{1,\tau}$, then a left (resp., right) orthonormal system $(X_k)_{k=0}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and some $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ simply consists of complex $q \times q$ matrix polynomials. In that particular situation we will also refer to $(X_k)_{k=0}^{\tau}$ as left (resp., right) *orthonormal matrix polynomial system corresponding to* F (cf. [27, Section 3.6]).

3. Some basics on para-orthogonal rational matrix functions

In this section, we begin to implement the concept of para-orthogonal functions in the context of rational matrix functions and will point out some fundamental properties of the para-orthogonal functions in question. In particular, we will study the relationship between para-orthogonal and orthonormal systems of rational matrix-valued functions on the unit circle $\mathbb{T}$. We start by presenting the central notion of this paper.

DEFINITION 3.1. Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. A sequence $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) of matrix functions is called a *left* (resp., *right*) *para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F if, for each $j \in \mathbb{N}_{1,\tau}$, the following holds:

(I) The function P_j (resp., R_j) belongs to $\mathscr{R}_{\alpha,j}^{q \times q}$.

(II) The matrices $(P_j, B_{\alpha,0}^{(q)})_{F,l}$ and $(P_j, B_{\alpha,j}^{(q)})_{F,l}$ (resp., $(B_{\alpha,0}^{(q)}, R_j)_{F,r}$ and $(B_{\alpha,j}^{(q)}, R_j)_{F,r}$) are not equal to $0_{q \times q}$.

(III) If $Z \in \mathscr{R}_{\alpha,j}^{q \times q}$ such that the identities $Z(\alpha_j) = 0_{q \times q}$ and $Z^{[\alpha,j]}(\alpha_j) = 0_{q \times q}$ are fulfilled, then $(P_j, Z)_{F,l} = 0_{q \times q}$ (resp., $(Z, R_j)_{F,r} = 0_{q \times q}$).

In the particular case that, for each $j \in \mathbb{N}_{1,\tau}$, the conditions (I), (III), and

(II') The matrices $(P_j, B_{\alpha,0}^{(q)})_{F,l}$ and $(P_j, B_{\alpha,j}^{(q)})_{F,l}$ (resp., $(B_{\alpha,0}^{(q)}, R_j)_{F,r}$ and $(B_{\alpha,j}^{(q)}, R_j)_{F,r}$) are nonsingular.

are satisfied, we call $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) a *left* (resp., *right*) *strictly para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

Similar to the case for orthonormal systems (i.e. relating to Definition 2.1), if $\alpha_j = 0$ for all $j \in \mathbb{N}_{1,\tau}$, then we will also refer to a $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) as *left* (resp., *right*) *para-orthogonal matrix polynomial system corresponding to* F when the conditions (I), (II), and (III) hold and as a *left* (resp., *right*) *strictly para-orthogonal matrix polynomial system corresponding to* F when (I), (II'), and (III) are satisfied.

The terminology "para-orthogonal system" introduced in Definition 3.1 is an adaptation of the scalar case $q = 1$. Such rational (complex-valued) functions are studied, e.g., in [8] (see also [7, Section 15], [10, Chapter 5], and [40]). The phrase "para-orthogonal" is chosen because the orthogonality properties for the relevant sequences fall short of providing proper orthogonality (cf. Definition 2.1). Comparing (III) of Definition 3.1 and the corresponding part in the definition given in [8], we note that [30, Equation (2.10)] implies, by setting

$$\widehat{\mathscr{R}}_{\alpha,j}^{p \times q} := \left\{ Z \in \mathscr{R}_{\alpha,j}^{p \times q} : Z(\alpha_j) = 0_{p \times q} \text{ and } Z^{[\alpha,j]}(\alpha_j) = 0_{q \times p} \right\}$$

(and we will use the notation $\widehat{\mathscr{R}}_{\alpha,j}^{p \times q}$ throughout this paper), that

$$\widehat{\mathscr{R}}_{\alpha,j}^{p \times q} = \left\{ Z \in \mathscr{R}_{\alpha,j-1}^{p \times q} : Z(\alpha_j) = 0_{p \times q} \right\} \tag{3.1}$$

for each $j \in \mathbb{N}_{1,\tau}$. Furthermore, the para-orthogonal systems of rational functions in [8] include the index $j = 0$ (for which some constant functions result). For the purposes in the current context, this would prove to be unwieldy, because it would be necessary to repeatedly differentiate between cases (trivial, but mainly tedious). Consequently, the systems here exclude the index $j = 0$ and begin with $j = 1$.

We now present some elementary properties of para-orthogonal systems of rational matrix functions. These properties are closely related to properties of orthogonal systems of rational matrix functions on $\mathbb{T}$ (cf. [30, Section 3]). Unless otherwise indicated, let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$ as well as suppose that $F \in \mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

REMARK 3.2. Let $(P_j)_{j=1}^{\tau}$ be a sequence of complex $q \times q$ matrix functions.

(a) Let $(\mathbf{C}_j)_{j=1}^{\tau}$ be a sequence of nonsingular complex $q \times q$ matrices. Then $(P_j)_{j=1}^{\tau}$ is a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if $(\mathbf{C}_j P_j)_{j=1}^{\tau}$ is a left (resp., $(P_j \mathbf{C}_j)_{j=1}^{\tau}$ is a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. (The special case for strictly para-orthogonal systems is analogous.)

(b) $(P_j)_{j=1}^{\tau}$ is a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if $(P_j^T)_{j=1}^{\tau}$ is a right (resp., left) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F^T. (The special case for strictly para-orthogonal systems is analogous.)

(c) Suppose that $P_j \in \mathcal{R}_{\alpha,j}^{q \times q}$ for each $j \in \mathbb{N}_{1,\tau}$. Because of (2.2), (2.8), (2.9), and [30, Remark 2.9] it follows that $(P_j)_{j=1}^{\tau}$ is a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if $(P_j^{[\alpha,j]})_{j=1}^{\tau}$ is a right (resp., left) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. (The special case for strictly para-orthogonal systems is analogous.)

REMARK 3.3. Let $\mathbf{A} \in \mathbb{C}^{q \times q}$ be nonsingular. Then $F_{\mathbf{A}} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q \times q}$ given by $F_{\mathbf{A}}(B) := \mathbf{A}^* F(B) \mathbf{A}$ for all $B \in \mathfrak{B}_{\mathbb{T}}$ belongs to $\mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, since $F \in \mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, a sequence $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) of matrix functions is a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if $(P_j \mathbf{A}^{-*})_{j=1}^{\tau}$ is a left (resp., $(\mathbf{A}^{-1} R_j)_{j=1}^{\tau}$ is a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and $F_{\mathbf{A}}$. (The case for strictly para-orthogonal systems is analogous.)

In the following, we study the relationship between the concepts of Definitions 2.1 and 3.1. We will be using the notation $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ for a *pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F, where $(X_k)_{k=0}^{\tau}$ is a left (resp., $(Y_k)_{k=0}^{\tau}$ is a right) orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. If $\tau = 0$ or if $\alpha_j = 0$ for all $j \in \mathbb{N}_{1,\tau}$, then we will also refer to $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ as a *pair of orthonormal matrix polynomial systems corresponding to* F (cf. [27, Section 3.6]).

LEMMA 3.4. *Suppose that* $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ *is a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Then:*

(a) *For each $j \in \mathbb{N}_{1,\tau}$, the matrix $(X_j, B_{\alpha,j}^{(q)})_{F,l}$ (resp., $(B_{\alpha,j}^{(q)}, Y_j)_{F,r}$) is nonsingular and, if $Z \in \widehat{\mathcal{R}}_{\alpha,j}^{q \times q}$, then $(X_j, Z)_{F,l} = 0_{q \times q}$ (resp., $(Z, Y_j)_{F,r} = 0_{q \times q}$). But, it holds*

$$(X_j, B_{\alpha,0}^{(q)})_{F,l} = 0_{q \times q} \quad \left(resp., \ (B_{\alpha,0}^{(q)}, Y_j)_{F,r} = 0_{q \times q}\right) \tag{3.2}$$

for each $j \in \mathbb{N}_{1,\tau}$. In particular, $(X_j)_{j=1}^{\tau}$ is not a left (resp., $(Y_j)_{j=1}^{\tau}$ is not a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.

(b) *For all $j \in \mathbb{N}_{1,\tau}$, the matrix $(Y_j^{[\alpha,j]}, B_{\alpha,0}^{(q)})_{F,l}$ (resp., $(B_{\alpha,0}^{(q)}, X_j^{[\alpha,j]})_{F,r}$) is nonsingular and $(Y_j^{[\alpha,j]}, Z)_{F,l} = 0_{q \times q}$ (resp., $(Z, X_j^{[\alpha,j]})_{F,r} = 0_{q \times q}$) when $Z \in \widehat{\mathcal{R}}_{\alpha,j}^{q \times q}$. But, it holds*

$$(Y_j^{[\alpha,j]}, B_{\alpha,j}^{(q)})_{F,l} = 0_{q \times q} \quad \left(resp., \ (B_{\alpha,j}^{(q)}, X_j^{[\alpha,j]})_{F,r} = 0_{q \times q}\right)$$

for each $j \in \mathbb{N}_{1,\tau}$. In particular, $(Y_j^{[\alpha,j]})_{j=1}^{\tau}$ is not a left (resp., $(X_j^{[\alpha,j]})_{j=1}^{\tau}$ is not a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.

Proof. (a) Taking (2.2) and (3.1) into account, we see that the first part (up to and including (3.2)) of (a) is a consequence of [30, Lemma 3.6]. Because of (3.2) and (II) in Definition 3.1 it follows that $(X_j)_{j=1}^{\tau}$ is not a left (resp., $(Y_j)_{j=1}^{\tau}$ is not a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.

(b) By part (c) of Remark 3.2 and part (a) we see that $(X_j^{[\alpha,j]})_{j=1}^{\tau}$ (resp., $(Y_j^{[\alpha,j]})_{j=1}^{\tau}$) cannot be a right (resp., left) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. The rest follows from part (a) along with (2.2), (2.8), and (2.9). $\square$

Even though Lemma 3.4 suggests the opposite, there is a relationship between para-orthogonal and orthogonal systems of rational matrix functions. Similar to the scalar case $q = 1$ (cf. [8, Theorem 2]), this relationship can be described as follows.

THEOREM 3.5. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F and let $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) be a sequence of complex $q \times q$ matrix-valued functions. Then the following statements are equivalent:*

(i) *$(P_j)_{j=1}^{\tau}$ is a left (resp., $(R_j)_{j=1}^{\tau}$ is a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

(ii) *For each $j \in \mathbb{N}_{1,\tau}$, there are complex $q \times q$ matrices $\mathbf{A}_j$ and $\mathbf{B}_j$, both not equal to the zero matrix, such that P_j (resp., R_j) admits the representation*

$$P_j = \mathbf{A}_j X_j + \mathbf{B}_j Y_j^{[\alpha,j]} \quad \left(resp., \ R_j = Y_j \mathbf{A}_j + X_j^{[\alpha,j]} \mathbf{B}_j\right).$$

If (i) *holds, then* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *in* (ii) *are uniquely determined for* $j \in \mathbb{N}_{1,\tau}$, *where*

$$\mathbf{A}_j = \big(P_j, B^{(q)}_{\alpha,j}\big)_{F,l} X^{[\alpha,j]}_j(\alpha_j) \quad \text{and} \quad \mathbf{B}_j = \big(P_j, B^{(q)}_{\alpha,0}\big)_{F,l} \big(Y^{[\alpha,j]}_j(\alpha_j)\big)^*$$

$$\Big(\text{resp.,} \ \mathbf{A}_j = Y^{[\alpha,j]}_j(\alpha_j) \big(B^{(q)}_{\alpha,j}, R_j\big)_{F,r} \quad \text{and} \quad \mathbf{B}_j = \big(X^{[\alpha,j]}_j(\alpha_j)\big)^* \big(B^{(q)}_{\alpha,0}, R_j\big)_{F,r}\Big).$$

In particular, $(P_j)^{\tau}_{j=1}$ *is a left* (*resp.,* $(R_j)^{\tau}_{j=1}$ *is a right*) *strictly para-orthogonal system corresponding to* $(\alpha_j)^{\infty}_{j=1}$ *and* F *if and only if, for* $j \in \mathbb{N}_{1,\tau}$, *there are nonsingular matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *such that* P_j (*resp.,* R_j) *admits the representation in* (ii).

Proof. We will only show the assertion for the sequence $(P_j)^{\tau}_{j=1}$. A similar argument can be used for the remaining case. We first suppose that, for each $j \in \mathbb{N}_{1,\tau}$, there exist complex $q \times q$ matrices $\mathbf{A}_j$ and $\mathbf{B}_j$, both not equal to the zero matrix, such that

$$P_j = \mathbf{A}_j X_j + \mathbf{B}_j Y^{[\alpha,j]}_j \tag{3.3}$$

holds. Let $j \in \mathbb{N}_{1,\tau}$. Because of (3.3), Definition 2.1, and (2.8) we have

$$P_j \in \mathscr{R}^{q \times q}_{\alpha,j}.$$

Furthermore, (3.3) and part (a) of Lemma 3.4 imply that

$$\big(P_j, B^{(q)}_{\alpha,0}\big)_{F,l} = \mathbf{A}_j \big(X_j, B^{(q)}_{\alpha,0}\big)_{F,l} + \mathbf{B}_j \big(Y^{[\alpha,j]}_j, B^{(q)}_{\alpha,0}\big)_{F,l} = \mathbf{B}_j \big(Y^{[\alpha,j]}_j, B^{(q)}_{\alpha,0}\big)_{F,l}. \tag{3.4}$$

Combining (3.3) with part (b) of Lemma 3.4 we obtain

$$\big(P_j, B^{(q)}_{\alpha,j}\big)_{F,l} = \mathbf{A}_j \big(X_j, B^{(q)}_{\alpha,j}\big)_{F,l} + \mathbf{B}_j \big(Y^{[\alpha,j]}_j, B^{(q)}_{\alpha,j}\big)_{F,l} = \mathbf{A}_j \big(X_j, B^{(q)}_{\alpha,j}\big)_{F,l}. \tag{3.5}$$

Since $\mathbf{B}_j \neq 0_{q \times q}$ and since part (b) of Lemma 3.4 yields that $\big(Y^{[\alpha,j]}_j, B^{(q)}_{\alpha,0}\big)_{F,l}$ is a nonsingular matrix, it follows from (3.4) that the relation

$$\big(P_j, B^{(q)}_{\alpha,0}\big)_{F,l} \neq 0_{q \times q}$$

holds. Similarly, $\mathbf{A}_j \neq 0_{q \times q}$ along with part (a) of Lemma 3.4 and (3.5) leads to

$$\big(P_j, B^{(q)}_{\alpha,j}\big)_{F,l} \neq 0_{q \times q}.$$

If $Z \in \widehat{\mathscr{R}}^{q \times q}_{\alpha,j}$, then by (3.3) and Lemma 3.4 we get

$$\big(P_j, Z\big)_{F,l} = \mathbf{A}_j \big(X_j, Z\big)_{F,l} + \mathbf{B}_j \big(Y^{[\alpha,j]}_j, Z\big)_{F,l} = 0_{q \times q}.$$

Therefore, we have shown that $(P_j)^{\tau}_{j=1}$ is a left para-orthogonal system corresponding to $(\alpha_j)^{\infty}_{j=1}$ and F. Conversely, we are now starting from a left para-orthogonal system $(P_j)^{\tau}_{j=1}$ corresponding to $(\alpha_j)^{\infty}_{j=1}$ and F. We will prove that, for each $j \in \mathbb{N}_{1,\tau}$, there

exist complex $q \times q$ matrices $\mathbf{A}_j$ and $\mathbf{B}_j$, both not equal to the zero matrix, such that (3.3) holds. Let $j \in \mathbb{N}_{1,\tau}$. Recalling part (b) of Lemma 3.4, we set

$$\mathbf{B}_j := \left(P_j, B_{\alpha,0}^{(q)}\right)_{F,l} \left(Y_j^{[\alpha,j]}, B_{\alpha,0}^{(q)}\right)_{F,l}^{-1}, \qquad \mathbf{A}_j := \left(P_j, X_j\right)_{F,l} - \mathbf{B}_j \left(Y_j^{[\alpha,j]}, X_j\right)_{F,l},$$

and

$$H_j := P_j - \mathbf{A}_j X_j - \mathbf{B}_j Y_j^{[\alpha,j]}. \tag{3.6}$$

Obviously, $\mathbf{A}_j$ and $\mathbf{B}_j$ belong to $\mathbb{C}^{q \times q}$, where $\mathbf{B}_j \neq 0_{q \times q}$. Because of (3.6) and (2.7) the function H_j belongs to $\mathscr{R}_{\alpha,j}^{q \times q}$. Thus, [30, Remark 3.4] implies that H_j admits a representation

$$H_j = \sum_{k=0}^{j} \mathbf{C}_k X_k, \tag{3.7}$$

where $\mathbf{C}_k \in \mathbb{C}^{q \times q}$ for each $k \in \mathbb{N}_{0,j}$. We will verify that H_j is the constant function on $\mathbb{C}_0$ with value $0_{q \times q}$, i.e. that $H_j = \mathscr{O}$. Since $\left(X_j, X_j\right)_{F,l} = \mathbf{I}_q$, from (3.6) we see that

$$\left(H_j, X_j\right)_{F,l} = \left(P_j, X_j\right)_{F,l} - \mathbf{A}_j - \mathbf{B}_j \left(Y_j^{[\alpha,j]}, X_j\right)_{F,l} = 0_{q \times q}.$$

Hence, in view of (3.7) and the orthogonality of $(X_k)_{k=0}^{\tau}$ we obtain

$$\mathbf{C}_j = \mathbf{C}_j \left(X_j, X_j\right)_{F,l} = \sum_{k=0}^{j} \mathbf{C}_k \left(X_k, X_j\right)_{F,l} = \left(H_j, X_j\right)_{F,l} = 0_{q \times q}. \tag{3.8}$$

Furthermore, from part (a) of Lemma 3.4 we know that the identity $\left(X_s, B_{\alpha,0}^{(q)}\right)_{F,l} = 0_{q \times q}$ holds for each $s \in \mathbb{N}_{1,\tau}$. Consequently, because of (3.6) we have

$$\left(H_j, B_{\alpha,0}^{(q)}\right)_{F,l} = \left(P_j, B_{\alpha,0}^{(q)}\right)_{F,l} - \mathbf{B}_j \left(Y_j^{[\alpha,j]}, B_{\alpha,0}^{(q)}\right)_{F,l} = 0_{q \times q}.$$

Using again that $\left(X_s, B_{\alpha,0}^{(q)}\right)_{F,l} = 0_{q \times q}$ for each $s \in \mathbb{N}_{1,\tau}$ and (3.7), this leads to

$$\mathbf{C}_0 \left(X_0, B_{\alpha,0}^{(q)}\right)_{F,l} = \sum_{k=0}^{j} \mathbf{C}_k \left(X_k, B_{\alpha,0}^{(q)}\right)_{F,l} = \left(H_j, B_{\alpha,0}^{(q)}\right)_{F,l} = 0_{q \times q}.$$

Since the matrix $\left(X_0, B_{\alpha,0}^{(q)}\right)_{F,l}$ is nonsingular (see, e.g., [30, Lemma 3.6]), we obtain

$$\mathbf{C}_0 = 0_{q \times q}. \tag{3.9}$$

In particular, for the case $j = 1$, due to (3.7)–(3.9) we get that $H_j = \mathscr{O}$. Let $j \geqslant 2$. From (3.7)–(3.9) we find

$$H_j = \sum_{k=1}^{j-1} \mathbf{C}_k X_k. \tag{3.10}$$

Let $s \in \mathbb{N}_{1,j-1}$ and let $p_{s,j} : \mathbb{C} \to \mathbb{C}$ denote the polynomial given by

$$p_{s,j}(u) = \begin{cases} (\alpha_j - u) & \text{if } s = 1 \\ (\alpha_j - u) \displaystyle\prod_{k=1}^{s-1} (\alpha_k - u) & \text{if } s > 1. \end{cases}$$

Let $\pi_{\alpha,s}$ be defined as in (2.1) and let

$$U_{s,j} := \frac{p_{s,j}}{\pi_{\alpha,s}} \mathbf{I}_q.$$

It is immediately apparent that

$$U_{s,j} \in \mathscr{R}_{\alpha,s}^{q\times q} \subseteq \mathscr{R}_{\alpha,j-1}^{q\times q} \quad \text{and} \quad U_{s,j}(\alpha_j) = 0_{q\times q}.$$

From [30, Proposition 2.13] one can infer that there is some $\eta \in \mathbb{T}$ such that

$$U_{s,j}^{[\alpha,s]}(v) = \eta \frac{1 - \overline{\alpha_j}v}{1 - \overline{\alpha_s}v} \mathbf{I}_q$$

for all $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,s}$. Thus, the matrix $U_{s,j}^{[\alpha,s]}(\alpha_s)$ is nonsingular. From [30, Lemma 3.6], (2.2), (3.10), (3.6), (III) in Definition 3.1, (3.1), and Lemma 3.4 we obtain

$$\mathbf{C}_1\big(X_1, U_{1,j}\big)_{F,l} = \sum_{k=1}^{j-1} \mathbf{C}_k\big(X_k, U_{1,j}\big)_{F,l} = \big(H_j, U_{1,j}\big)_{F,l}$$
$$= \big(P_j, U_{1,j}\big)_{F,l} - \mathbf{A}_j\big(X_j, U_{1,j}\big)_{F,l} - \mathbf{B}_j\big(Y_j^{[\alpha,j]}, U_{1,j}\big)_{F,l} = 0_{q\times q}.$$

Accordingly, since [30, Lemma 3.6 and Equation (2.10)] and $\det U_{1,j}^{[\alpha,1]}(\alpha_1) \neq 0$ imply that the matrix $\big(X_1, U_{1,j}\big)_{F,l}$ is nonsingular, it follows that $\mathbf{C}_1 = 0_{q\times q}$. Iterating this argument we obtain $\mathbf{C}_s = 0_{q\times q}$ for each $s \in \mathbb{N}_{1,j-1}$. Therefore, (3.10) implies that $H_j = \mathscr{O}$ in the case $j \geqslant 2$ as well. Again, suppose that $j \in \mathbb{N}_{1,\tau}$. Since $H_j = \mathscr{O}$, the definition of H_j in (3.6) leads to (3.3). As explained above, (3.3) results in (3.4) and (3.5). In particular, from (3.4), (3.5), and Lemma 3.4 one can see that $\mathbf{A}_j$ and $\mathbf{B}_j$ are uniquely determined by (3.3). Furthermore, by using (3.4) and (3.5) along with [30, Equation (2.10), Remark 2.9, and Lemma 3.6], (2.2), (2.8), and (2.9) it follows that

$$\mathbf{A}_j = \big(P_j, B_{\alpha,j}^{(q)}\big)_{F,l} X_j^{[\alpha,j]}(\alpha_j) \quad \text{and} \quad \mathbf{B}_j = \big(P_j, B_{\alpha,0}^{(q)}\big)_{F,l} \big(Y_j^{[\alpha,j]}(\alpha_j)\big)^*. \qquad \square$$

Note that, for each $j \in \mathbb{N}_{1,\tau}$, the complex $q \times q$ matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ in (ii) of Theorem 3.5 depend on the choice of the pair of orthonormal systems $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$.

COROLLARY 3.6. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ *and let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$. *Let* $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Furthermore, let* $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ *be a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Suppose that* $(P_j)_{j=1}^{\tau}$ *is a left (resp., let* $(R_j)_{j=1}^{\tau}$ *is a right) para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Let* $j \in \mathbb{N}_{1,\tau}$. *There are complex* $q \times q$ *matrices* $\tilde{\mathbf{A}}_j$ *and* $\tilde{\mathbf{B}}_j$ *(resp.,* $\tilde{\mathbf{D}}_j$ *and* $\tilde{\mathbf{E}}_j$) *such that, for each* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$,

$$P_j(v) = \frac{1 - \overline{\alpha_{j-1}}v}{1 - \overline{\alpha_j}v}\big(b_{\alpha_{j-1}}(v)\tilde{\mathbf{A}}_j X_{j-1}(v) + \tilde{\mathbf{B}}_j Y_{j-1}^{[\alpha,j-1]}(v)\big)$$

$$\Big(resp.,\ R_j(v) = \frac{1 - \overline{\alpha_{j-1}}v}{1 - \overline{\alpha_j}v}\big(b_{\alpha_{j-1}}(v)Y_{j-1}(v)\tilde{\mathbf{D}}_j + X_{j-1}^{[\alpha,j-1]}(v)\tilde{\mathbf{E}}_j\big)\Big)$$

(3.11)

holds, where $\tilde{\mathbf{A}}_j$ *and* $\tilde{\mathbf{B}}_j$ *(resp.,* $\tilde{\mathbf{D}}_j$ *and* $\tilde{\mathbf{E}}_j$) *are uniquely determined.*

Proof. By using [30, Corollary 4.4, Corollary 4.7, and Remark 6.2] and the recurrence relations of [31, Propositions 2.5 and 2.6], Theorem 3.5 leads to (3.11). That the complex $q \times q$ matrices $\tilde{\mathbf{A}}_j$ and $\tilde{\mathbf{B}}_j$ (resp., $\tilde{\mathbf{D}}_j$ and $\tilde{\mathbf{E}}_j$) in (3.11) are uniquely determined follows from $b_{\alpha_{j-1}}(\alpha_{j-1}) = 0$ along with [30, Remark 6.2, Theorem 6.7, and Theorem 6.9]. □

For each $j \in \mathbb{N}_{1,\tau}$, based on Theorem 3.5 and [31, Proposition 3.14] it is possible to obtain explicit expressions for the matrices $\tilde{\mathbf{A}}_j$ and $\tilde{\mathbf{B}}_j$ (resp., $\tilde{\mathbf{D}}_j$ and $\tilde{\mathbf{E}}_j$) in (3.11). In particular, it follows that one of these matrices could be the zero matrix.

Because of Theorem 3.5 there is also a relationship between para-orthogonal systems of rational matrix functions and the matrix functions defined by (2.3)–(2.5). The following corollaries serve to further clarify this fact.

COROLLARY 3.7. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Suppose that $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) be a sequence of complex $q \times q$ matrix-valued functions. Then the following statements are equivalent:*

(i) *$(P_j)_{j=1}^{\tau}$ is a left (resp., $(R_j)_{j=1}^{\tau}$ is a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

(ii) *For each $j \in \mathbb{N}_{1,\tau}$, there are complex $q \times q$ matrices $\check{\mathbf{A}}_j$ and $\check{\mathbf{B}}_j$, both not equal to the zero matrix, such that P_j (resp., R_j) admits the representation*

$$P_j = \check{\mathbf{A}}_j \big(A_{j,\alpha_j}^{(\alpha,F)}\big)^{[\alpha,j]} + \check{\mathbf{B}}_j C_{j,\alpha_j}^{(\alpha,F)} \quad \Big(\text{resp., } R_j = \big(C_{j,\alpha_j}^{(\alpha,F)}\big)^{[\alpha,j]} \check{\mathbf{A}}_j + A_{j,\alpha_j}^{(\alpha,F)} \check{\mathbf{B}}_j\Big).$$

If (i) holds, then $\check{\mathbf{A}}_j$ and $\check{\mathbf{B}}_j$ in (ii) are uniquely determined for $j \in \mathbb{N}_{1,\tau}$, where

$$\check{\mathbf{A}}_j = \big(P_j, B_{\alpha,j}^{(q)}\big)_{F,l} \quad and \quad \check{\mathbf{B}}_j = \big(P_j, B_{\alpha,0}^{(q)}\big)_{F,l}$$

$$\Big(\text{resp., } \check{\mathbf{A}}_j = \big(B_{\alpha,j}^{(q)}, R_j\big)_{F,r} \quad and \quad \check{\mathbf{B}}_j = \big(B_{\alpha,0}^{(q)}, R_j\big)_{F,r}\Big).$$

In particular, $(P_j)_{j=1}^{\tau}$ is a left (resp., $(R_j)_{j=1}^{\tau}$ is a right) strictly para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if, for $j \in \mathbb{N}_{1,\tau}$, there are nonsingular matrices $\check{\mathbf{A}}_j$ and $\check{\mathbf{B}}_j$ such that P_j (resp., R_j) admits the representation in (ii).

Proof. Use Theorem 3.5 along with [30, Corollary 4.4 and Theorem 4.5]. □

COROLLARY 3.8. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and let $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Furthermore, for each $j \in \mathbb{N}_{1,\tau}$, let $z_j \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ and let*

$$P_j := \big(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\big) C_{j-1,z_j}^{(\alpha,F)} \quad \Big(\text{resp., } R_j := \big(1 - b_{\alpha_j} \overline{b_{\alpha_j}(z_j)}\big) A_{j-1,z_j}^{(\alpha,F)}\Big). \tag{3.12}$$

(a) *The following statements are equivalent:*

(i) $(P_j)_{j=1}^{\tau}$ *is a left* (*resp.,* $(R_j)_{j=1}^{\tau}$ *is a right*) *para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(ii) *For each* $j \in \mathbb{N}_{1,\tau}$, *one of the values* $X_j(z_j)$ *and* $Y_j(z_j)$ *as well as one of the values* $X_j^{[\alpha,j]}(z_j)$ *and* $Y_j^{[\alpha,j]}(z_j)$ *is not equal to the zero matrix.*

(iii) *For each* $j \in \mathbb{N}_{1,\tau}$, *the values* $X_j(z_j)$, $Y_j(z_j)$, $X_j^{[\alpha,j]}(z_j)$, *and* $Y_j^{[\alpha,j]}(z_j)$ *are not equal to the zero matrix.*

(b) *The following statements are equivalent:*

(iv) $(P_j)_{j=1}^{\tau}$ *is a left* (*resp.,* $(R_j)_{j=1}^{\tau}$ *is a right*) *strictly para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(v) *For each* $j \in \mathbb{N}_{1,\tau}$, *one of the matrices* $X_j(z_j)$ *and* $Y_j(z_j)$ *as well as one of the matrices* $X_j^{[\alpha,j]}(z_j)$ *and* $Y_j^{[\alpha,j]}(z_j)$ *is nonsingular.*

(vi) *For each* $j \in \mathbb{N}_{1,\tau}$, *the values* $X_j(z_j)$, $Y_j(z_j)$, $X_j^{[\alpha,j]}(z_j)$, *and* $Y_j^{[\alpha,j]}(z_j)$ *are nonsingular matrices.*

(c) *If* $z_j \in \mathbb{T}$ *for each* $j \in \mathbb{N}_{1,\tau}$, *then* $(P_j)_{j=1}^{\tau}$ *is a left* (*resp.,* $(R_j)_{j=1}^{\tau}$ *is a right*) *strictly para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

Proof. Let $j \in \mathbb{N}_{1,\tau}$. Because of [30, Remark 6.2, Lemma 6.5, Theorem 6.7, and Theorem 6.10] one can realize that (ii) and (iii) (resp., (v) and (vi)) are equivalent. By using this along with the fact that from [30, Lemma 5.1 and Theorem 5.4] we obtain

$$\left(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\right) C_{j-1,z_j}^{(\alpha,F)} = \left(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\right) \sum_{k=0}^{j-1} \left(X_k(z_j)\right)^* X_k$$
$$= \left(Y_j^{[\alpha,j]}(z_j)\right)^* Y_j^{[\alpha,j]} - \left(X_j(z_j)\right)^* X_j$$

$$\left(\text{resp.,} \ \left(1 - b_{\alpha_j} \overline{b_{\alpha_j}(z_j)}\right) A_{j-1,z_j}^{(\alpha,F)} = X_j^{[\alpha,j]} \left(X_j^{[\alpha,j]}(z_j)\right)^* - Y_j \left(Y_j(z_j)\right)^* \right),$$

we see that applying Theorem 3.5 yields the assertion of parts (a) and (b). Furthermore, if $z_j \in \mathbb{T}$, then [30, Remark 2.6 and Corollary 4.7] imply that $X_j(z_j)$, $Y_j(z_j)$, $X_j^{[\alpha,j]}(z_j)$, and $Y_j^{[\alpha,j]}(z_j)$ are nonsingular matrices. Thus, part (c) is a consequence of (b). $\square$

If we consider Corollary 3.8, it is not hard to accept that, for some left (resp., right) para-orthogonal system $(P_j)_{j=1}^{\tau}$ of rational matrix functions and $s \in \mathbb{N}_{1,\tau}$, the matrix function P_s might belong only to $\mathscr{R}_{\alpha,s-1}^{q \times q}$ (cf. (I) in Definition 3.1 and Remark 3.11).

We now make use of Theorem 3.5 and keep the results of [33] and [34] in mind. Based on (2.6), for some $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_{0,m}$, we will use the settings

$$b_{n,m}^{(\alpha)} := \begin{cases} B_{\alpha,0}^{(1)} & \text{if } n = m \text{ or } \alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \in \mathbb{D} \\ \prod_{j \in \{k \in \mathbb{N}_{n+1,m} : \alpha_k \notin \mathbb{D}\}} b_{\alpha_j} & \text{if } \alpha_k \notin \mathbb{D} \text{ for some } k \in \mathbb{N}_{n+1,m} \end{cases}$$

and

$$
\widetilde{b}_{n,m}^{(\alpha)} := \begin{cases} B_{\alpha,0}^{(1)} & \text{if } n = m \text{ or } \alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+r} \notin \mathbb{D} \\ \displaystyle\prod_{j \in \{k \in \mathbb{N}_{n+1,m} : \alpha_k \in \mathbb{D}\}} b_{\alpha_j} & \text{if } \alpha_k \in \mathbb{D} \text{ for some } k \in \mathbb{N}_{n+1,m}, \end{cases}
$$

where $B_{\alpha,0}^{(1)}$ stands for the constant function on $\mathbb{C}_0$ with value 1 (as in Section 2).

COROLLARY 3.9. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $n \in \mathbb{N}_0$ *and let* $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Suppose that* $[(X_k)_{k=0}^n, (Y_k)_{k=0}^n]$ *is a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Let* $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ *and let* $A_{n,w}^{(\alpha,F)}$ *and* $C_{n,w}^{(\alpha,F)}$ *be the rational matrix functions given by* (2.3)–(2.5). *Furthermore, let the nonnegative Hermitian measure* $F_{n,w}^{(\alpha)} : \mathfrak{B}_{\mathbb{T}} \to \mathbb{C}^{q \times q}$ *be defined by*

$$
F_{n,w}^{(\alpha)}(B) := \frac{1}{2\pi} \int_B \frac{1 - |w|^2}{|z - w|^2} \left(A_{n,w}^{(\alpha,F)}(z)\right)^{-*} A_{n,w}^{(\alpha,F)}(w) \left(A_{n,w}^{(\alpha,F)}(z)\right)^{-1} \underline{\lambda}(dz),
$$

where $\underline{\lambda}$ *stands for the linear Lebesgue measure defined on* $\mathfrak{B}_{\mathbb{T}}$. *Let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$ *and suppose that* $(P_j)_{j=1}^{\tau}$ *(resp.,* $(R_j)_{j=1}^{\tau}$ *) is a sequence of rational matrix functions. Then* $F_{n,w}^{(\alpha)} \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and the following statements are equivalent:*

(i) $(P_j)_{j=1}^{\tau}$ *is a left (resp.,* $(R_j)_{j=1}^{\tau}$ *is a right) para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* $F_{n,w}^{(\alpha)}$.

(ii) *For each* $j \in \mathbb{N}_{1,\tau}$, *there are complex* $q \times q$ *matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$, *both not equal to the zero matrix, such that* P_j *(resp.,* R_j *) admits the representation*

$$
P_j = \mathbf{A}_j X_j + \mathbf{B}_j Y_j^{[\alpha,j]} \quad \left(\text{resp., } R_j = Y_j \mathbf{A}_j + X_j^{[\alpha,j]} \mathbf{B}_j\right), \quad j \leqslant n,
$$

and, if $j > n$ *and* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, *then the value* $P_j(v)$ *(resp.,* $R_j(v)$ *) is given by*

$$
P_j(v) = \frac{1 - \overline{w}v}{1 - \overline{\alpha_j}v} \left(b_w(v) \widetilde{b}_{n,j-1}^{(\alpha)}(v) \mathbf{A}_j \left(A_{n,w}^{(\alpha,F)}\right)^{[\alpha,n]}(v) + b_{n,j-1}^{(\alpha)}(v) \mathbf{B}_j C_{n,w}^{(\alpha,F)}(v)\right)
$$

$$
\left(\text{resp., } R_j(v) = \frac{1 - \overline{w}v}{1 - \overline{\alpha_j}v} \left(b_w(v) \widetilde{b}_{n,j-1}^{(\alpha)}(v) \left(C_{n,w}^{(\alpha,F)}\right)^{[\alpha,n]}(v) \mathbf{A}_j + b_{n,j-1}^{(\alpha)}(v) A_{n,w}^{(\alpha,F)}(v) \mathbf{B}_j\right)\right).
$$

If (i) *holds, then the matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *in* (ii) *are uniquely determined for each* $j \in \mathbb{N}_{1,\tau}$. *In particular,* $(P_j)_{j=1}^{\tau}$ *is a left (resp.,* $(R_j)_{j=1}^{\tau}$ *is a right) strictly para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* $F_{n,w}^{(\alpha)}$ *if and only if, for each* $j \in \mathbb{N}_{1,\tau}$, *there are nonsingular matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *such that the equalities in* (ii) *hold.*

Proof. Recalling [33, Remark 3.6] and the rules for working with reciprocal rational matrix function presented in [30, Section 2], one can apply Theorem 3.5 along with [34, Theorem 4.5 and Remark 4.7] to obtain the assertion. □

For each $s \in \mathbb{N}_{n+1,\tau}$, the formulas (in [34, Theorem 4.5 and Remark 4.7]) for the elements X_s and Y_s of a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_{n,w}^{(\alpha)}$ require that we differentiate between the case in which α_s belongs to $\mathbb{D}$ and the remaining case in which it does not. This case differentiation does not, however, appear in the formulas of Corollary 3.9 for left (resp., right) para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_{n,w}^{(\alpha)}$.

REMARK 3.10. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Furthermore, let $[(X_k)_{k=0}^\tau, (Y_k)_{k=0}^\tau]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. If $j \in \mathbb{N}_{1,\tau}$ and if $\mathbf{A}_j, \mathbf{B}_j \in \mathbb{C}^{q \times q}$ such that $\mathbf{A}_j X_j + \mathbf{B}_j Y_j^{[\alpha,j]} = \mathcal{O}$ (resp., $Y_j \mathbf{A}_j + X_j^{[\alpha,j]} \mathbf{B}_j = \mathcal{O}$) holds, then based on Theorem 3.5 one can see that $\mathbf{A}_j = 0$ and $\mathbf{B}_j = 0$ follows (see also Remark 6.3).

The excluded case $j = 0$ in Remark 3.10 does, indeed, not hold in general.

In view of (I) in Definition 3.1, we now emphasize a situation different from the one for orthogonal rational matrix functions (cf. [30, Remark 3.4]).

REMARK 3.11. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $(P_j)_{j=1}^\tau$ be a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^\infty$ and some measure $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Let $j \in \mathbb{N}_{1,\tau}$. By Theorem 3.5 along with (2.8) and [30, Remark 2.7, Remark 2.8, and Corollary 4.4] it seems reasonable that $P_j(\alpha_j) = 0$ or $P_j^{[\alpha,j]}(\alpha_j) = 0$ might hold. Thus, from [30, Equation (2.10)] and (2.8) we see that it may be the case that $P_j^{[\alpha,j]} \in \mathscr{R}_{\alpha,j-1}^{q \times q}$ or $P_j \in \mathscr{R}_{\alpha,j-1}^{q \times q}$.

The statement of Remark 3.11 does not apply to the matrix case. It holds, in particular, for the scalar case $q = 1$ and hence also for left (resp., right) strictly para-orthogonal systems of rational (matrix) functions.

The next result shows that the "or" in Remark 3.11 cannot be replaced by "and".

PROPOSITION 3.12. *Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Furthermore, let $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Suppose that $(P_j)_{j=1}^\tau$ is a left (resp., right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Let $j \in \mathbb{N}_{1,\tau}$. Then $P_j(\alpha_j) \neq 0$ or $P_j^{[\alpha,j]}(\alpha_j) \neq 0$ holds. In particular, $P_j^{[\alpha,j]}$ or P_j belongs to $\mathscr{R}_{\alpha,j}^{q \times q} \setminus \mathscr{R}_{\alpha,j-1}^{q \times q}$. Moreover, if there exists a pair $[(X_k)_{k=0}^j, (Y_k)_{k=0}^j]$ of orthonormal systems corresponding to $(\alpha_s)_{s=1}^\infty$ and F such that $X_j(\alpha_j) = 0$ or $Y_j(\alpha_j) = 0$, then $P_j(\alpha_j) \neq 0$, $P_j^{[\alpha,j]}(\alpha_j) \neq 0$, as well as P_j and $P_j^{[\alpha,j]}$ belong to $\mathscr{R}_{\alpha,j}^{q \times q} \setminus \mathscr{R}_{\alpha,j-1}^{q \times q}$.*

Proof. Suppose that both equalities $P_j(\alpha_j) = 0$ and $P_j^{[\alpha,j]}(\alpha_j) = 0$ hold. Recalling (I) and (III) of Definition 3.1, we see that

$$\left(P_j, P_j\right)_{F,l} = 0_{q \times q} \quad \left(\text{resp., } \left(P_j, P_j\right)_{F,r} = 0_{q \times q}\right).$$

But, this implies

$$\left(P_j, B_{\alpha,0}^{(q)}\right)_{F,l} = 0_{q\times q} \quad \left(\text{resp.,} \ \left(B_{\alpha,0}^{(q)}, P_j\right)_{F,r} = 0_{q\times q}\right),$$

which contradicts (II) of Definition 3.1. Therefore, it follows that $P_j(\alpha_j)$ or $P_j^{[\alpha,j]}(\alpha_j)$ must not be equal to the zero matrix. Hence, because of [30, Equation (2.10)] and (2.8) we obtain

$$P_j^{[\alpha,j]} \in \mathscr{R}_{\alpha,j}^{q\times q} \setminus \mathscr{R}_{\alpha,j-1}^{q\times q} \quad \text{or} \quad P_j \in \mathscr{R}_{\alpha,j}^{q\times q} \setminus \mathscr{R}_{\alpha,j-1}^{q\times q}.$$

We now consider the case for which there exists a pair $[(X_k)_{k=0}^j, (Y_k)_{k=0}^j]$ of orthonormal systems corresponding to $(\alpha_s)_{s=1}^\infty$ and F such that

$$X_j(\alpha_j) = 0 \quad \text{or} \quad Y_j(\alpha_j) = 0$$

holds. Based on [30, Remark 6.2, part (a) of Lemma 6.5, Theorem 6.7, and Theorem 6.10] we find that $X_j(\alpha_j) = 0$ and $Y_j(\alpha_j) = 0$ and also that $X_j^{[\alpha,j]}(\alpha_j)$ and $Y_j^{[\alpha,j]}(\alpha_j)$ are nonsingular matrices. Consequently, Theorem 3.5 along with (2.8) and [30, Remarks 2.7 and 2.8] implies that $P_j(\alpha_j) \neq 0$ and $P_j^{[\alpha,j]}(\alpha_j) \neq 0$. Thus, from [30, Equation (2.10)] and (2.8) we get that $P_j^{[\alpha,j]}$ and P_j belong to $\mathscr{R}_{\alpha,j}^{q\times q} \setminus \mathscr{R}_{\alpha,j-1}^{q\times q}$. $\quad\square$

With regard to the special case considered at the end of Proposition 3.12, in which $X_j(\alpha_j) = 0$ or $Y_j(\alpha_j) = 0$, it should be noted that it is closely related to the case studied in Corollary 3.9 (see also [33, Proposition 6.2] and [34, Theorem 4.5]).

4. On the existence of para-orthogonal rational matrix functions

Because of Theorem 3.5 and [30, Corollary 4.4] it is clear that, for $\tau \in \mathbb{N}$ or $\tau = \infty$, there is a left (resp., right) para-orthogonal system $(P_j)_{j=1}^\tau$ corresponding to some $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and $F \in \mathscr{M}_\geqslant^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ if the underlying measure F belongs to $\mathscr{M}_\geqslant^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. In this section, we study the question as to what extent this property is necessary. In particular, we will see that, for $\tau \in \mathbb{N}$, the condition $F \in \mathscr{M}_\geqslant^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ is already sufficient. This condition is, furthermore, necessary and sufficient for the existence of a left (resp., right) strictly para-orthogonal system $(P_j)_{j=1}^\tau$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F.

As the following result emphasizes, we can essentially restrict the considerations on existence criteria to the case of left para-orthogonal systems.

REMARK 4.1. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_\geqslant^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. By part (c) of Remark 3.2 we see that there is a left para-orthogonal system $(P_j)_{j=1}^\tau$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if there is a right para-orthogonal system $(R_j)_{j=1}^\tau$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F. (The case for strictly para-orthogonal systems is analogous.)

We now give some information on the special case $\tau = 1$. (In a sense, this case is somewhat peculiar, since (III) of Definition 3.1 does not need to be regarded.)

REMARK 4.2. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $F \in \mathscr{M}_\geqslant^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Because of Definition 3.1 and (3.1) a sequence $(P_j)_{j=1}^1$ is a left para-orthogonal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $P_1 \in \mathscr{R}_{\alpha,1}^{q \times q}$ such that $(P_1, B_{\alpha,0}^{(q)})_{F,l} \neq 0_{q \times q}$ and $(P_1, B_{\alpha,1}^{(q)})_{F,l} \neq 0_{q \times q}$ hold. Thus, one can see that there exists a left para-orthogonal system $(P_j)_{j=1}^1$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $F(\mathbb{T}) \neq 0_{q \times q}$. Similarly, there is a left strictly para-orthogonal system $(P_j)_{j=1}^1$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $\det F(\mathbb{T}) \neq 0$ (i.e. $F \in \mathscr{M}_\geqslant^{q,0}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$).

Remark 4.2 leads us to suspect that, for $\tau \in \mathbb{N}$, there is a left (strictly) para-orthogonal system $(P_j)_{j=1}^\tau$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F if the measure F belongs to $\mathscr{M}_\geqslant^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. The following considerations are aimed at verifying this suspicion. The proof relies on Corollary 3.8. We first point out some auxiliary results on the reproducing kernels of rational matrix functions given by (2.3)–(2.5).

LEMMA 4.3. *Let* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$. *Let* $n \in \mathbb{N}_0$ *and suppose that* $F \in \mathscr{M}_\geqslant^{q,n}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. *Furthermore, let* $z \in \mathbb{T}$. *Then:*

(a) $A_{n,z}^{(\alpha,F)} = (B_{\alpha,n}^{(q)}(z))^* (C_{n,z}^{(\alpha,F)})^{[\alpha,n]}$ *and* $C_{n,z}^{(\alpha,F)} = (B_{\alpha,n}^{(q)}(z))^* (A_{n,z}^{(\alpha,F)})^{[\alpha,n]}$.

(b) *The following statements are equivalent:*

   (i) $(B_{\alpha,n+1}^{(q)}, C_{n,z}^{(\alpha,F)})_{F,l} - B_{\alpha,n+1}^{(q)}(z)$ *is a singular* (*resp., the zero*) *matrix.*

   (ii) $((1 - \overline{b_{\alpha_{n+1}}(z)} b_{\alpha_{n+1}}) C_{n,z}^{(\alpha,F)}, B_{\alpha,n+1}^{(q)})_{F,l}$ *is a singular* (*resp., the zero*) *matrix.*

   (iii) $(B_{\alpha,0}^{(q)}, (1 - b_{\alpha_{n+1}} \overline{b_{\alpha_{n+1}}(z)}) A_{n,z}^{(\alpha,F)})_{F,r}$ *is a singular* (*resp., the zero*) *matrix.*

   Moreover, there is only a set $\Delta_{n+1}^{(l)}$ *of at most* $(n+1)q$ (*resp., a set* $\widetilde{\Delta}_{n+1}^{(l)}$ *of at most* $n+1$) *pairwise different points belonging to* $\mathbb{T}$ *such that* (i) *is satisfied. In particular,* $\widetilde{\Delta}_{n+1}^{(l)} \subseteq \Delta_{n+1}^{(l)}$ *holds,* $\Delta_{n+1}^{(l)}$ *consists of at most* $(n+1-k)q+k$ *pairwise different points if* $\widetilde{\Delta}_{n+1}^{(l)}$ *includes k pairwise different points, and* $\Delta_{n+1}^{(l)} = \emptyset$ *in the case of* $F \in \mathscr{M}_\geqslant^{q,n+1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$.

(c) *The following statements are equivalent:*

   (iv) $(A_{n,z}^{(\alpha,F)}, B_{\alpha,n+1}^{(q)})_{F,r} - B_{\alpha,n+1}^{(q)}(z)$ *is a singular* (*resp., the zero*) *matrix.*

   (v) $(B_{\alpha,n+1}^{(q)}, (1 - b_{\alpha_{n+1}} \overline{b_{\alpha_{n+1}}(z)}) A_{n,z}^{(\alpha,F)})_{F,r}$ *is a singular* (*resp., the zero*) *matrix.*

   (vi) $((1 - \overline{b_{\alpha_{n+1}}(z)} b_{\alpha_{n+1}}) C_{n,z}^{(\alpha,F)}, B_{\alpha,0}^{(q)})_{F,l}$ *is a singular* (*resp., the zero*) *matrix.*

   Moreover, there is only a set $\Delta_{n+1}^{(r)}$ *of at most* $(n+1)q$ (*resp., a set* $\widetilde{\Delta}_{n+1}^{(r)}$ *of at most* $n+1$) *pairwise different points belonging to* $\mathbb{T}$ *such that* (iv) *is satisfied. In particular,* $\widetilde{\Delta}_{n+1}^{(r)} \subseteq \Delta_{n+1}^{(r)}$ *holds,* $\Delta_{n+1}^{(r)}$ *consists of at most* $(n+1-k)q+k$ *pairwise*

different points if $\widetilde{\Delta}_{n+1}^{(r)}$ *includes* k *pairwise different points, and* $\Delta_{n+1}^{(r)} = \emptyset$ *in the case of* $F \in \mathcal{M}_{\geq}^{q,n+1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

Proof. (a) Recalling $\left(B_{\alpha,n}^{(q)}(z)\right)^* B_{\alpha,n}^{(q)}(z) = \mathbf{I}_q$ and [30, Lemma 2.2], by using [30, Lemmas 5.1 and 5.2] we obtain the assertion of (a) (see also [14, Lemma 3.2]).

(b) Based on (2.6), (2.2), and [28, Theorem 10], for each $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n+1}$, we get

$$\left((1 - \overline{b_{\alpha_{n+1}}(v)} b_{\alpha_{n+1}}) C_{n,v}^{(\alpha,F)}, B_{\alpha,n+1}^{(q)}\right)_{F,l} = \left(C_{n,v}^{(\alpha,F)}, B_{\alpha,n+1}^{(q)}\right)_{F,l} - \overline{b_{\alpha_{n+1}}(v)}\left(C_{n,v}^{(\alpha,F)}, B_{\alpha,n}^{(q)}\right)_{F,l}$$

$$= \left(B_{\alpha,n+1}^{(q)}, C_{n,v}^{(\alpha,F)}\right)_{F,l}^* - \left(B_{\alpha,n+1}^{(q)}(v)\right)^*.$$

Furthermore, by (2.6), (2.2), [28, Theorem 10], $B_{\alpha,0}^{(q)}(z) = \mathbf{I}_q$, (a), (2.8), (2.9), [30, Remark 2.9], and the structure of $B_{\alpha,n+1}^{(q)}$ (in particular, $\left(B_{\alpha,n+1}^{(q)}(z)\right)^* B_{\alpha,n+1}^{(q)}(z) = \mathbf{I}_q$) we obtain the equality

$$\left(B_{\alpha,0}^{(q)}, (1 - b_{\alpha_{n+1}} \overline{b_{\alpha_{n+1}}(z)}) A_{n,z}^{(\alpha,F)}\right)_{F,r} = \left(B_{\alpha,0}^{(q)}, A_{n,z}^{(\alpha,F)}\right)_{F,r} - \left(B_{\alpha,0}^{(q)}, b_{\alpha_{n+1}} \overline{b_{\alpha_{n+1}}(z)} A_{n,z}^{(\alpha,F)}\right)_{F,r}$$

$$= \mathbf{I}_q - \left(B_{\alpha,0}^{(q)}, b_{\alpha_{n+1}} \left(B_{\alpha,n+1}^{(q)}(z)\right)^* \left(C_{n,z}^{(\alpha,F)}\right)^{[\alpha,n]}\right)_{F,r}$$

$$= \mathbf{I}_q - \left(B_{\alpha,n+1}^{(q)}(z)\right)^* \left(B_{\alpha,0}^{(q)}, b_{\alpha_{n+1}} \left(C_{n,z}^{(\alpha,F)}\right)^{[\alpha,n]}\right)_{F,r}$$

$$= \left(B_{\alpha,n+1}^{(q)}(z)\right)^* \left(B_{\alpha,n+1}^{(q)}(z) - \left(B_{\alpha,n+1}^{(q)}, C_{n,z}^{(\alpha,F)}\right)_{F,l}\right),$$

where $B_{\alpha,n+1}^{(q)}(z)$ is a nonsingular matrix. Thus, we get that the statements (i), (ii), and (iii) are equivalent. For some $m \in \mathbb{N}_0$, let

$$\mathbf{H}_m^{(\alpha,F)} := \left(\left(B_{\alpha,j}^{(q)}, B_{\alpha,k}^{(q)}\right)_{F,l}\right)_{j,k=0}^m$$

and let $\left(\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n+1}\right)$ denote the last $q \times (n+2)q$ block row of $\mathbf{H}_{n+1}^{(\alpha,F)}$, where $\mathbf{h}_k \in \mathbb{C}^{q \times q}$ for each $k \in \mathbb{N}_{0,n+1}$. By [29, Remarks 2.4 and 3.7] and (2.3)–(2.5), for each $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n+1}$, it follows that

$$\left(B_{\alpha,n+1}^{(q)}, C_{n,v}^{(\alpha,F)}\right)_{F,l} = \left(0_{q \times q}, \ldots, 0_{q \times q}, \mathbf{I}_q\right) \mathbf{H}_{n+1}^{(\alpha,F)} \left(\left(\begin{pmatrix} B_{\alpha,0}^{(q)}(v) \\ \vdots \\ B_{\alpha,n}^{(q)}(v) \end{pmatrix}^* \left(\mathbf{H}_n^{(\alpha,F)}\right)^{-1}\right)^*\right)$$
$$0_{q \times q}$$

$$= \left(\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n+1}\right) \left(\left(\mathbf{H}_n^{(\alpha,F)}\right)^{-1} \begin{pmatrix} B_{\alpha,0}^{(q)}(v) \\ \vdots \\ B_{\alpha,n}^{(q)}(v) \end{pmatrix}\right)$$
$$0_{q \times q}$$

$$= \left(\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_n\right) \left(\mathbf{H}_n^{(\alpha,F)}\right)^{-1} \begin{pmatrix} B_{\alpha,0}^{(q)}(v) \\ \vdots \\ B_{\alpha,n}^{(q)}(v) \end{pmatrix}.$$

Hence (note again [29, Remark 2.4]), we see that $H : \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,n+1} \to \mathbb{C}^{q \times q}$ given by

$$H(v) := \left(B_{\alpha,n+1}^{(q)}, C_{n,v}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,n+1}^{(q)}(v)$$

defines a function belonging to $\mathscr{R}_{\alpha,n+1}^{q \times q}$, where (2.7) implies $H^{[\alpha,n+1]}(\alpha_{n+1}) = -\mathbf{I}_q$. It therefore follows from [30, Remark 2.6] and the Fundamental Theorem of Algebra that there is a set $\Delta_{n+1}^{(l)}$ of at most $(n+1)q$ (resp., a set $\widetilde{\Delta}_{n+1}^{(l)}$ of at most $n+1$) pairwise different points belonging to $\mathbb{T}$ such that $\left(B_{\alpha,n+1}^{(q)}, C_{n,u}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,n+1}^{(q)}(u)$ is a singular matrix for each $u \in \Delta_{n+1}^{(l)}$ (resp., the zero matrix for each $u \in \widetilde{\Delta}_{n+1}^{(l)}$). Furthermore, $\widetilde{\Delta}_{n+1}^{(l)} \subseteq \Delta_{n+1}^{(l)}$ holds and the set $\Delta_{n+1}^{(l)}$ consists of at most $(n+1-k)q + k$ pairwise different points if the set $\widetilde{\Delta}_{n+1}^{(l)}$ includes k pairwise different points. Finally, part (c) of Corollary 3.8 shows that $\Delta_{n+1}^{(l)} = \emptyset$ if F belongs to $\mathscr{M}_{\geqslant}^{q,n+1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.
(c) Use part (b) along with [28, Remark 8]. $\square$

If $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, then (with some $\tau \in \mathbb{N}$ or $\tau = \infty$) we will call a pair $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$, where $(P_j)_{j=1}^{\tau}$ is a left (resp., $(R_j)_{j=1}^{\tau}$ is a right) para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, a *pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F*. (If $\alpha_j = 0$ for all $j \in \mathbb{N}_{1,\tau}$, then we will also refer to $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ as a *pair of para-orthogonal matrix polynomial systems corresponding to F*.) We will use analogous terms in the case of strictly para-orthogonal systems.

THEOREM 4.4. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *and* $F \in \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *For each* $j \in \mathbb{N}_{1,\tau}$, *let* $z_j \in \mathbb{T}$ *and let the matrix functions* P_j *and* R_j *be given by* (3.12). *Then:*

(a) *There is a set* $\widetilde{\Delta}_{\tau}$ *of at most* 2τ *pairwise different points belonging to* $\mathbb{T}$ *such that* $\left(B_{\alpha,\tau}^{(q)}, C_{\tau-1,z}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,\tau}^{(q)}(z)$ *and* $\left(A_{\tau-1,z}^{(\alpha,F)}, B_{\alpha,\tau}^{(q)}\right)_{F,r} - B_{\alpha,\tau}^{(q)}(z)$ *are nonzero matrices for all* $z \in \mathbb{T} \setminus \widetilde{\Delta}_{\tau}$. *Moreover,* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *is a pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *if and only if* $z_{\tau} \in \mathbb{T} \setminus \widetilde{\Delta}_{\tau}$. *In particular, there is a left (resp., right) para-orthogonal system* $(Q_j)_{j=1}^{\tau}$ *corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(b) *There is a set* Δ_{τ} *of at most* $2\tau q$ *pairwise different points belonging to* $\mathbb{T}$ *such that* $\left(B_{\alpha,\tau}^{(q)}, C_{\tau-1,z}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,\tau}^{(q)}(z)$ *and* $\left(A_{\tau-1,z}^{(\alpha,F)}, B_{\alpha,\tau}^{(q)}\right)_{F,r} - B_{\alpha,\tau}^{(q)}(z)$ *are nonsingular matrices for all* $z \in \mathbb{T} \setminus \Delta_{\tau}$. *Moreover,* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *is a pair of strictly para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *if and only if* $z_{\tau} \in \mathbb{T} \setminus \Delta_{\tau}$. *In particular, there exists a left (resp., right) strictly para-orthogonal system* $(Q_j)_{j=1}^{\tau}$ *corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

Proof. Because of Lemma 4.3 it follows that there is a set Δ_τ of at most $2\tau q$ (resp., a set $\widetilde{\Delta}_\tau$ of at most 2τ) pairwise different points belonging to $\mathbb{T}$ such that

$$\det\left(\left(B_{\alpha,\tau}^{(q)}, C_{\tau-1,z}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,\tau}^{(q)}(z)\right) \neq 0 \quad\text{and}\quad \det\left(\left(A_{\tau-1,z}^{(\alpha,F)}, B_{\alpha,\tau}^{(q)}\right)_{F,r} - B_{\alpha,\tau}^{(q)}(z)\right) \neq 0$$

$$\left(\text{resp., } \left(B_{\alpha,\tau}^{(q)}, C_{\tau-1,z}^{(\alpha,F)}\right)_{F,l} \neq B_{\alpha,\tau}^{(q)}(z) \quad\text{and}\quad \left(A_{\tau-1,z}^{(\alpha,F)}, B_{\alpha,\tau}^{(q)}\right)_{F,r} \neq B_{\alpha,\tau}^{(q)}(z)\right)$$

for all $z \in \mathbb{T}\setminus\Delta_\tau$ (resp., for all $z \in \mathbb{T}\setminus\widetilde{\Delta}_\tau$). Let $z_\tau \in \mathbb{T}\setminus\Delta_\tau$ (resp., $z \in \mathbb{T}\setminus\widetilde{\Delta}_\tau$). Based on (3.12) and Lemma 4.3 we see that $\left(P_\tau, B_{\alpha,\tau}^{(q)}\right)_{F,l}$, $\left(P_\tau, B_{\alpha,0}^{(q)}\right)_{F,l}$, $\left(B_{\alpha,\tau}^{(q)}, R_\tau\right)_{F,r}$ and $\left(B_{\alpha,0}^{(q)}, R_\tau\right)_{F,r}$ are all nonsingular matrices (resp., nonzero matrices). In particular, for the case $\tau = 1$, it follows that $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a pair of strictly para-orthogonal systems (resp., a pair of para-orthogonal systems) corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Now let $\tau \geqslant 2$. Noting that $F \in \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$, from Corollary 3.8 we find that $[(P_j)_{j=1}^{\tau-1}, (R_j)_{j=1}^{\tau-1}]$ is a pair of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Let $Z \in \widehat{\mathscr{R}}_{\alpha,\tau}^{q\times q}$. In view of (3.1) the function Z admits

$$Z = \frac{p_\tau}{\pi_{\alpha,\tau-1}}Q$$

with some complex $q \times q$ matrix polynomial Q of degree not greater than $\tau-2$, where $\pi_{\alpha,\tau-1}$ is given by (2.1) and $p_\tau(v) := \alpha_\tau - v$ for each $v \in \mathbb{C}$. If $q_\tau(v) := \overline{\eta_\tau}(1 - \overline{\alpha_\tau}v)$ for each $v \in \mathbb{C}$ with a view to (2.6), then (2.2) and [28, Theorem 10] imply

$$\left(P_\tau, Z\right)_{F,l} = \left(C_{\tau-1,z_\tau}^{(\alpha,F)}, \frac{p_\tau}{\pi_{\alpha,\tau-1}}Q\right)_{F,l} - \overline{b_{\alpha_\tau}(z_\tau)}\left(b_{\alpha_\tau}C_{\tau-1,z_\tau}^{(\alpha,F)}, \frac{p_\tau}{\pi_{\alpha,\tau-1}}Q\right)_{F,l}$$

$$= \left(\frac{p_\tau(z_\tau)}{\pi_{\alpha,\tau-1}(z_\tau)}Q(z_\tau)\right)^* - \overline{b_{\alpha_\tau}(z_\tau)}\left(C_{\tau-1,z_\tau}^{(\alpha,F)}, \frac{q_\tau}{\pi_{\alpha,\tau-1}}Q\right)_{F,l}$$

$$= \left(\frac{p_\tau(z_\tau)}{\pi_{\alpha,\tau-1}(z_\tau)}Q(z_\tau)\right)^* - \overline{b_{\alpha_\tau}(z_\tau)}\left(\frac{q_\tau(z_\tau)}{\pi_{\alpha,\tau-1}(z_\tau)}Q(z_\tau)\right)^* = 0_{q\times q}.$$

Similarly, we obtain

$$\left(Z, R_\tau\right)_{F,r} = 0_{q\times q}.$$

Thus, for $\tau \geqslant 2$, we again have that $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a pair of strictly para-orthogonal systems (resp., a pair of para-orthogonal systems) corresponding to $(\alpha_j)_{j=1}^\infty$ and F. $\quad\square$

Because of Theorem 4.4 and the Christoffel–Darboux formulas for orthonormal systems of rational matrix functions we obtain the following (cf. Corollary 3.6).

COROLLARY 4.5. *Let* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *and* $F \in \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. *Let* Δ_τ *and* $\widetilde{\Delta}_\tau$ *be the subsets of* $\mathbb{T}$ *as in* Theorem 4.4 *and let* $[(X_k)_{k=0}^{\tau-1}, (Y_k)_{k=0}^{\tau-1}]$ *be a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F. *Furthermore, let* $z_j \in \mathbb{T}$ *and let* $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\left(X_{j-1}(z_j)\right)^*$, $\tilde{\mathbf{B}}_j := \left(Y_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$, $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\left(Y_{j-1}(z_j)\right)^*$, *and* $\tilde{\mathbf{E}}_j := \left(X_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$ *for each* $j \in \mathbb{N}_{1,\tau}$. *Then:*

(a) *There exists a pair $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that, for each $j \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, both equalities in (3.11) are satisfied if and only if $z_\tau \in \mathbb{T} \setminus \widetilde{\Delta}_\tau$.*

(b) *There is a pair $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that, for each $j \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, both equalities in (3.11) are satisfied if and only if $z_\tau \in \mathbb{T} \setminus \Delta_\tau$.*

Proof. Let $j \in \mathbb{N}_{1,\tau}$ and let $z \in \mathbb{T}$. Furthermore, let f_j be the rational function given by

$$f_j(v) := \frac{(1-|\alpha_j|^2)(1-\overline{z}\alpha_{j-1})(1-\overline{\alpha_{j-1}}v)}{(1-|\alpha_{j-1}|^2)(1-\overline{z}\alpha_j)(1-\overline{\alpha_j}v)}, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}.$$

Thus, from (2.6) and [30, Corollary 4.4, Lemma 5.1, Remark 5.3, and Corollary 5.5] it follows

$$
\begin{aligned}
\left(1 - \overline{b_{\alpha_j}(z)}b_{\alpha_j}\right)C_{j-1,z}^{(\alpha,F)} &= \frac{1 - \overline{b_{\alpha_j}(z)}b_{\alpha_j}}{1 - \overline{b_{\alpha_{j-1}}(z)}b_{\alpha_{j-1}}}\left(1 - \overline{b_{\alpha_{j-1}}(z)}b_{\alpha_{j-1}}\right)\sum_{k=0}^{j-1}(X_k(z))^* X_k \\
&= f_j\left(\left(Y_{j-1}^{[\alpha,j-1]}(z)\right)^* Y_{j-1}^{[\alpha,j-1]} - \overline{b_{\alpha_{j-1}}(z)}b_{\alpha_{j-1}}(X_{j-1}(z))^* X_{j-1}\right) \\
&= f_j\left(b_{\alpha_{j-1}}\tilde{\mathbf{A}}_j X_{j-1} + \tilde{\mathbf{B}}_j Y_{j-1}^{[\alpha,j-1]}\right)
\end{aligned}
$$

and similarly

$$\left(1 - b_{\alpha_j}\overline{b_{\alpha_j}(z)}\right)A_{j-1,z}^{(\alpha,F)} = f_j\left(b_{\alpha_{j-1}}Y_{j-1}\tilde{\mathbf{D}}_j + X_{j-1}^{[\alpha,j-1]}\tilde{\mathbf{E}}_j\right).$$

Applying this along with Theorem 4.4 and part (a) of Remark 3.2 we get the assertion. □

As an aside (cf. Corollaries 3.6 and 3.8), it should be noted that the statements of Theorem 4.4 and Corollary 4.5 are valid for $\tau = \infty$ (with $\tau - 1 = \infty$ and $\widetilde{\Delta}_\tau = \Delta_\tau = \emptyset$) as well. Furthermore, in the scalar case $q = 1$ (see [14, Theorem 3.5]), a converse to Theorem 4.4 (resp., Corollary 4.5) holds. Thus, if $\tau \in \mathbb{N}$, then $F \in \mathcal{M}_{\geqslant}^{1,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is necessary and sufficient for the existence of a left (resp., right) para-orthogonal system $(P_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Because of Remark 4.2 we see that this is particular to the case $q = 1$. Even for $\tau = 1$, Remark 4.2 shows that in the case $q \geqslant 2$ the existence of a left para-orthogonal system $(P_j)_{j=1}^{1}$ corresponding to some $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $F \in \mathcal{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ does not yield $F \in \mathcal{M}_{\geqslant}^{q,0}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ in general. Concerning this, the remaining question is if the existence of a left (resp., right) strictly para-orthogonal system $(P_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F leads to $F \in \mathcal{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ for some $\tau \in \mathbb{N}$. As the following result emphasizes, this guess is true.

THEOREM 4.6. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. Furthermore, let $\tau \in \mathbb{N}$ and $F \in \mathcal{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Then the following statements are equivalent:*

(i) *F belongs to $\mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$.*

(ii) *There exists a sequence $(z_j)_{j=1}^{\tau}$ of points belonging to $\mathbb{T}$ such that, by using both declarations in (3.12) for each $j \in \mathbb{N}_{1,\tau}$, the pair $[(P_j)_{j=1}^{\tau},(R_j)_{j=1}^{\tau}]$ is a pair of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

(iii) *There is a pair $[(P_j)_{j=1}^{\tau},(R_j)_{j=1}^{\tau}]$ of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that with some pair $[(X_k)_{k=0}^{\tau-1},(Y_k)_{k=0}^{\tau-1}]$ of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, for each $j \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, both equalities in (3.11) hold, where $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\big(X_{j-1}(z_j)\big)^*$, $\tilde{\mathbf{B}}_j := \big(Y_{j-1}^{[\alpha,j-1]}(z_j)\big)^*$, $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\big(Y_{j-1}(z_j)\big)^*$, and $\tilde{\mathbf{E}}_j := \big(X_{j-1}^{[\alpha,j-1]}(z_j)\big)^*$ with some $z_j \in \mathbb{T}$.*

(iv) *There is a left (resp., right) strictly para-orthogonal system $(Q_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

Proof. Because of Theorem 4.4 it follows that (i) leads to (ii). Furthermore, the equivalence of (ii) and (iii) is a consequence of the Christoffel–Darboux formulas for orthonormal systems of rational matrix functions (cf. the proof of Corollary 4.5). We see next that, (ii) clearly yields (iv). Suppose now that (iv) holds. We will show by induction that (i) follows from (iv). By Remark 4.1, without loss of generality, we can restrict the considerations to the case in which there exists a left strictly para-orthogonal system $(P_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. If $\tau = 1$, then Remark 4.2 yields $F \in \mathscr{M}_{\geqslant}^{q,0}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. Now suppose that, for some $n \in \mathbb{N}$, the existence of a strictly para-orthogonal system $(P_j)_{j=1}^{n}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F implies that $F \in \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. We then show that the same implication holds for $n+1$. Let $(P_j)_{j=1}^{n+1}$ be a strictly para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. In particular, $(P_j)_{j=1}^{n}$ is a strictly para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. From the induction hypothesis, it thus follows that $F \in \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. We verify below, by using an argument by contradiction, that $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. We first assume that F belongs to $\mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T},\mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T},\mathfrak{B}_{\mathbb{T}})$. From that assumption along with [29, Theorem 5.8] one can see that there is a function $X \in \mathscr{R}_{\alpha,n}^{q\times q} \setminus \mathscr{R}_{\alpha,n-1}^{q\times q}$ such that

$$\big(X,X\big)_{F,l} = 0_{q\times q}. \tag{4.1}$$

Obviously, (4.1) implies

$$\big(P_{n+1},X\big)_{F,l} = 0_{q\times q}.$$

Hence, since (3.1) gives us that $Y := X(\alpha_{n+1}) - X$ belongs to $\widehat{\mathscr{R}}_{\alpha,n+1}^{q\times q}$, we get

$$\begin{aligned} 0_{q\times q} = \big(P_{n+1},Y\big)_{F,l} &= \big(P_{n+1},B_{\alpha,0}^{(q)}\big)_{F,l}\big(X(\alpha_{n+1})\big)^* - \big(P_{n+1},X\big)_{F,l} \\ &= \big(P_{n+1},B_{\alpha,0}^{(q)}\big)_{F,l}\big(X(\alpha_{n+1})\big)^*. \end{aligned}$$

Because $(P_j)_{j=1}^{n+1}$ is a strictly para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F it follows that $X(\alpha_{n+1}) = 0_{q \times q}$. Therefore, there is a complex $q \times q$ matrix polynomial P of degree not greater than $n-1$ such that $P \neq \mathcal{O}$ and so that X admits the representation

$$X = \frac{p_{n+1}}{\pi_{\alpha,n}} P,$$

where $p_{n+1} : \mathbb{C} \to \mathbb{C}$ is the polynomial given by $p_{n+1}(v) := \alpha_{n+1} - v$ and where $\pi_{\alpha,n}$ is the polynomial defined as in (2.1). We note that the rational (complex-valued) function

$$g := \frac{p_{n+1}}{\pi_{\alpha,n}}$$

does not have any poles or zeros on $\mathbb{T}$ (which also means that the restriction of g to $\mathbb{T}$ is a $\mathfrak{B}_{\mathbb{T}}$-$\mathfrak{B}$-measurable function and that there are positive real numbers L_1 and L_2 such that $L_1 \leqslant |g(z)| \leqslant L_2$ holds for all $z \in \mathbb{T}$). Recalling this, the assumption $F \in \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, and [29, Remarks 1.1, 1.3, and 5.9], by setting

$$H(B) := \int_B \big(g(z)\mathbf{I}_q\big)^* F(\mathrm{d}z)\, g(z)\mathbf{I}_q, \quad B \in \mathfrak{B}_{\mathbb{T}},$$

we obtain a measure belonging to $\mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, the choice of H, [29, Remark 1.1], and (4.1) imply

$$\int_{\mathbb{T}} P(z) H(\mathrm{d}z) \big(P(z)\big)^* = \int_{\mathbb{T}} |g(z)|^2 P(z) F(\mathrm{d}z) \big(P(z)\big)^* = \big(X, X\big)_{F,l} = 0_{q \times q}.$$

Consequently, by using [29, Theorem 5.8], we get $H \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. This is a contradiction to $H \in \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Therefore, the assumption that the matrix measure F belongs to $\mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ was false and it follows that $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Thus, (iv) implies (i). $\square$

COROLLARY 4.7. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Then the following statements are equivalent:*

(i) *F belongs to $\mathscr{M}_{\geqslant}^{q,\infty}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.*

(ii) *There exists a sequence $(z_j)_{j=1}^{\infty}$ of points belonging to $\mathbb{T}$ such that, by using both declarations in (3.12), for each $j \in \mathbb{N}$, the pair $[(P_j)_{j=1}^{\infty}, (R_j)_{j=1}^{\infty}]$ is a pair of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

(iii) *There is a pair $[(P_j)_{j=1}^{\infty}, (R_j)_{j=1}^{\infty}]$ of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that with some pair $[(X_k)_{k=0}^{\infty}, (Y_k)_{k=0}^{\infty}]$ of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, for each $j \in \mathbb{N}$ and each point $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, both equalities in (3.11) hold, where $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\big(X_{j-1}(z_j)\big)^*$, $\tilde{\mathbf{B}}_j := \big(Y_{j-1}^{[\alpha,j-1]}(z_j)\big)^*$, $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\big(Y_{j-1}(z_j)\big)^*$, and $\tilde{\mathbf{E}}_j := \big(X_{j-1}^{[\alpha,j-1]}(z_j)\big)^*$ with some $z_j \in \mathbb{T}$.*

(iv) *There is a left (resp., right) strictly para-orthogonal system $(Q_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

Proof. The assertion is an immediate consequence of Theorem 4.6. □

COROLLARY 4.8. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$. Furthermore, let $F \in \mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Then the following statements are equivalent:*

(i) *For each sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, there is a pair $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.*

(ii) *There is a left (resp., right) strictly para-orthogonal system $(P_j)_{j=1}^{\tau}$ corresponding to some $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and F.*

(iii) *There exists a left (resp., right) strictly para-orthogonal matrix polynomial system $(P_j)_{j=1}^{\tau}$ corresponding to F.*

Proof. Since (i) in Theorem 4.6 (resp., (i) in Corollary 4.7) does not depend on the exact choice of the underlying sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ (i.e. the location of the poles for the relevant rational matrix functions is not important), the assertion follows. □

Because of Lemma 4.3 the finite sets Δ_{τ} and $\widetilde{\Delta}_{\tau}$ in Theorem 4.4 (resp., Corollary 4.5) are empty when the underlying measure F belongs to $\mathcal{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Conversely (cf. [14, Proposition 3.9]), if $q = 1$ and if $\widetilde{\Delta}_{\tau} = \emptyset$, then $F \in \mathcal{M}_{\geqslant}^{1,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. As the following example illustrates, this a special feature of the scalar case $q = 1$.

EXAMPLE 4.9. *Let $\alpha_1 := 0$. Suppose that $\mathbf{c}_0 := \mathbf{I}_2$ and*

$$\mathbf{c}_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then there is an $F \in \mathcal{M}_{\geqslant}^2(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that the identities $\mathbf{c}_0^{(F)} = \mathbf{c}_0$ and $\mathbf{c}_1^{(F)} = \mathbf{c}_1$ hold, where $F \in \mathcal{M}_{\geqslant}^{2,0}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathcal{M}_{\geqslant}^{2,1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and where $(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)})_{F,l} - B_{\alpha,1}^{(2)}(z)$ and $(A_{0,z}^{(\alpha,F)}, B_{\alpha,1}^{(2)})_{F,r} - B_{\alpha,1}^{(2)}(z)$ are nonsingular matrices for each $z \in \mathbb{T}$.

Proof. The choice of $\mathbf{c}_0$ and $\mathbf{c}_1$ implies

$$\mathbf{c}_0 - \mathbf{c}_1 \mathbf{c}_0^{-1} \mathbf{c}_1^* = \mathbf{I}_2 - \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, from $\mathbf{c}_0 = \mathbf{I}_2$ and [27, Lemma 1.1.9 and Theorem 3.4.2] it follows that there is a measure $F \in \mathcal{M}_{\geqslant}^2(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that $\mathbf{c}_0^{(F)} = \mathbf{c}_0$ and $\mathbf{c}_1^{(F)} = \mathbf{c}_1$ hold, where $F \in \mathcal{M}_{\geqslant}^{2,0}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathcal{M}_{\geqslant}^{2,1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $z \in \mathbb{T}$. Byu $\alpha_1 = 0$ and (2.3)–(2.5) we get

$$\left(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)} \right)_{F,l} - B_{\alpha,1}^{(2)}(z) = \int_{\mathbb{T}} z F(\mathrm{d}z) \left(\mathbf{c}_0^{(F)} \right)^{-*} - z \mathbf{I}_2 = \left(\mathbf{c}_1^{(F)} \right)^* - z \mathbf{I}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} - z & 0 \\ \frac{1}{\sqrt{2}} & z \end{pmatrix}.$$

In particular, $\left(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(z)$ is a nonsingular matrix. Similarly, it follows that $\left(A_{0,z}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(z)$ is a nonsingular matrix. $\quad\square$

5. The case of molecular nonnegative Hermitian matrix Borel measures on $\mathbb{T}$

We now study the existence of para-orthogonal systems of rational matrix functions for the special case in which we have underlying molecular Borel measures on $\mathbb{T}$.

Recall that, for some $n \in \mathbb{N}$, a measure $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is called *molecular of order at most* n if there is a sequence $(u_j)_{j=1}^n$ of n points belonging to $\mathbb{T}$ so that $F(\mathbb{T} \setminus \{u_1, u_2, \ldots, u_n\}) = 0_{q \times q}$. For some $n \in \mathbb{N}$, the notation $\mathscr{M}_{\geqslant,n}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is used for the set of all $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ that are molecular of order at most n. Furthermore, $\mathscr{M}_{\geqslant,0}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ denotes the singleton consisting of the zero measure belonging to $\mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. We also use $\varepsilon_{u,\mathfrak{B}_{\mathbb{T}}}$ for the Dirac measure defined on $\mathfrak{B}_{\mathbb{T}}$ with unit mass located at some point $u \in \mathbb{T}$.

LEMMA 5.1. *Let* $n \in \mathbb{N}$ *and suppose that* $F \in \mathscr{M}_{\geqslant,n}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Furthermore, let* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ *and let* $X \in \mathscr{R}_{\alpha,n+1}^{q \times q}$. *Then the following statements are equivalent:*

(i) $\left(X,Y\right)_{F,l} = 0_{q \times q}$ *(resp.,* $\left(Y,X\right)_{F,r} = 0_{q \times q}$*) for each* $Y \in \mathscr{R}_{\alpha,n+1}^{q \times q}$.

(ii) $\left(Y, X^{[\alpha,n+1]}\right)_{F,r} = 0_{q \times q}$ *(resp.,* $\left(X^{[\alpha,n+1]}, Y\right)_{F,l} = 0_{q \times q}$*) for each* $Y \in \mathscr{R}_{\alpha,n+1}^{q \times q}$.

(iii) $\left(X,Z\right)_{F,l} = 0_{q \times q}$ *(resp.,* $\left(Z,X\right)_{F,r} = 0_{q \times q}$*) for each* $Z \in \widehat{\mathscr{R}}_{\alpha,n+1}^{q \times q}$.

Proof. The equivalence of (i) and (ii) is a simple consequence of (2.2), (2.8), and (2.9). Furthermore, (i) clearly implies (iii). Conversely, we now assume that (iii) is satisfied. We first suppose that, for each $Z \in \widehat{\mathscr{R}}_{\alpha,n+1}^{q \times q}$, the equality

$$\left(X,Z\right)_{F,l} = 0_{q \times q} \tag{5.1}$$

holds. Because of the choice of F there exist a sequence $(u_j)_{j=1}^n$ of pairwise different points belonging to $\mathbb{T}$ and a sequence $(\mathbf{A}_j)_{j=1}^n$ of nonnegative Hermitian $q \times q$ matrices such that

$$F = \sum_{j=1}^n \varepsilon_{u_j, \mathfrak{B}_{\mathbb{T}}} \mathbf{A}_j. \tag{5.2}$$

For each $Y \in \mathscr{R}_{\alpha,n+1}^{q \times q}$, from (5.2) we obtain

$$\left(X,Y\right)_{F,l} = \int_{\mathbb{T}} X(z) F(\mathrm{d}z) \left(Y(z)\right)^* = \sum_{j=1}^n X(u_j) \mathbf{A}_j \left(Y(u_j)\right)^*. \tag{5.3}$$

Let $k \in \mathbb{N}_{1,n}$. Let $p_{n+1} : \mathbb{C} \to \mathbb{C}$ be the polynomial given by $p_{n+1}(v) := \alpha_{n+1} - v$ and let $Q_k : \mathbb{C} \to \mathbb{C}^{q \times q}$ be the matrix polynomial of degree $n - 1$ given by

$$Q_k(v) := \begin{cases} \mathbf{I}_q & \text{if } n = 1 \\ \displaystyle\prod_{j \in \mathbb{N}_{1,n} \setminus \{k\}} \frac{u_j - v}{u_j - u_k} \mathbf{I}_q & \text{if } n > 1 . \end{cases} \tag{5.4}$$

Furthermore, let

$$Z_k := \frac{\pi_{\alpha,n}(u_k) \, p_{n+1}}{(\alpha_{n+1} - u_k) \, \pi_{\alpha,n}} Q_k,$$

where $\pi_{\alpha,n}$ is the polynomial defined as in (2.1). In view of (3.1) and (5.4) one can see that the function Z_k belongs to $\widehat{\mathscr{R}}^{q \times q}_{\alpha,n+1}$, where the additional equality $Z_k(u_j) = \delta_{jk} \mathbf{I}_q$ holds for each $j \in \mathbb{N}_{1,n}$. Moreover, from (5.3) and (5.1) it follows that

$$X(u_k)\mathbf{A}_k = X(u_k)\mathbf{A}_k \big(Z_k(u_k)\big)^* = \sum_{j=1}^n X(u_j)\mathbf{A}_j \big(Z_k(u_j)\big)^* = \big(X, Z\big)_{F,l} = 0_{q \times q}.$$

This yields along with (5.3) the equality

$$\big(X, Y\big)_{F,l} = \sum_{j=1}^n X(u_j)\mathbf{A}_j \big(Y(u_j)\big)^* = 0_{q \times q}, \quad Y \in \mathscr{R}^{q \times q}_{\alpha,n+1}.$$

Similarly, one can show that, if $(Z, X)_{F,r} = 0$ for each $Z \in \widehat{\mathscr{R}}^{q \times q}_{\alpha,n+1}$, then $(Y, X)_{F,r} = 0$ for each $Y \in \mathscr{R}^{q \times q}_{\alpha,n+1}$. Consequently, (iii) implies (i). $\square$

PROPOSITION 5.2. *Let* $n, \tau \in \mathbb{N}$ *and* $F \in \mathscr{M}^{q,\mathrm{mol}}_{\geqslant,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *If* $(P_j)_{j=1}^\tau$ *is a left* (*resp., right*) *para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ *and* F, *then* $\tau \leqslant n$.

Proof. We present an indirect proof. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$. Furthermore, taking Definition 3.1 into account, we suppose that $(P_j)_{j=1}^{n+1}$ is a left para-orthogonal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Hence, we have $\big(P_{n+1}, B^{(q)}_{\alpha,n+1}\big)_{F,l} \neq 0_{q \times q}$ and

$$\big(P_{n+1}, Z\big)_{F,l} = 0_{q \times q}, \quad Z \in \widehat{\mathscr{R}}^{q \times q}_{\alpha,n+1}.$$

But, this contradicts Lemma 5.1. Thus, there does not exist a left para-orthogonal system $(P_j)_{j=1}^{n+1}$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Therefore, if $(P_j)_{j=1}^\tau$ is a left para-orthogonal system corresponding to $(\alpha_j)_{j=1}^\infty$ and F, then $\tau \leqslant n$. Having dealt with the left case, we note that a similar proof can be used for the remaining case (see also Remark 4.1). $\square$

If $n \in \mathbb{N}$ and if $F \in \mathscr{M}^{1,\mathrm{mol}}_{\geqslant,n}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}^{1,\mathrm{mol}}_{\geqslant,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, then there exists a left (*resp., right*) para-orthogonal system $(P_j)_{j=1}^n$ corresponding to some $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$

and F (see, e.g., [14, Equation (2.1) and Theorem 3.5]). This is, however, a special feature of the scalar case $q = 1$. To emphasize this, we provide the following simple example for $q = 2$ and $n = 2$.

EXAMPLE 5.3. *Let $u_1 \in \mathbb{T}$ and $u_2 \in \mathbb{T} \setminus \{u_1\}$. Let $F := \varepsilon_{u_1, \mathfrak{B}_{\mathbb{T}}} \mathbf{A}_1 + \varepsilon_{u_2, \mathfrak{B}_{\mathbb{T}}} \mathbf{A}_2$, where*

$$\mathbf{A}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad \mathbf{A}_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $F \in \mathcal{M}_{\geqslant,2}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathcal{M}_{\geqslant,1}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, but there does not exist a left (resp., right) para-orthogonal system $(P_j)_{j=1}^2$ corresponding to some $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$ and F.

Proof. The choice of F immediately gives us that F is a matrix measure belonging to $\mathcal{M}_{\geqslant,2}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathcal{M}_{\geqslant,1}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $(\alpha_j)_{j=1}^\infty \in \mathcal{T}_1$. Suppose that there is a left para-orthogonal system $(P_j)_{j=1}^2$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F, where

$$P_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with some $a, b, c, d \in \mathscr{R}_{\alpha,2}$. Furthermore, let $p_2 : \mathbb{C} \to \mathbb{C}$ be the polynomial given by $p_2(v) := \alpha_2 - v$ and let $\pi_{\alpha,1}$ be defined by (2.1). Since (3.1) implies that the function

$$Z := \frac{p_2}{\pi_{\alpha,1}} \mathbf{I}_q$$

belongs to $\widehat{\mathscr{R}}_{\alpha,2}^{q \times q}$, from Definition 3.1 it follows that

$$0_{2\times2} = (P_2, Z)_{F,l} = P_2(u_1)\mathbf{A}_1 (Z(u_1))^* + P_2(u_2)\mathbf{A}_2 (Z(u_2))^*$$
$$= \begin{pmatrix} a(u_1)\overline{\left(\frac{\alpha_2 - u_1}{1 - \overline{\alpha_1}u_1}\right)} & b(u_2)\overline{\left(\frac{\alpha_2 - u_2}{1 - \overline{\alpha_1}u_2}\right)} \\ c(u_1)\overline{\left(\frac{\alpha_2 - u_1}{1 - \overline{\alpha_1}u_1}\right)} & d(u_2)\overline{\left(\frac{\alpha_2 - u_2}{1 - \overline{\alpha_1}u_2}\right)} \end{pmatrix}.$$

Consequently, we obtain that

$$\begin{pmatrix} a(u_1) & b(u_2) \\ c(u_1) & d(u_2) \end{pmatrix} = 0_{2\times2}.$$

However, this is in contradiction to

$$\begin{pmatrix} a(u_1) & b(u_2) \\ c(u_1) & d(u_2) \end{pmatrix} = (P_2, B_{\alpha,0}^{(q)})_{F,l} \neq 0_{2\times2}.$$

Hence, there does not exist a left para-orthogonal system $(P_j)_{j=1}^2$ corresponding to $(\alpha_j)_{j=1}^\infty$ and F. A similar proof can be used for the remaining (right) case. $\square$

As an addendum to Proposition 5.2, the following result concerning the existence of left (resp., right) strictly para-orthogonal systems of rational matrix functions holds.

PROPOSITION 5.4. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $n \in \mathbb{N}$ *and* $F \in \mathscr{M}_{\geqslant,n}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *If there are* τ *pairwise different points* $u_1, u_2, \ldots, u_\tau$ *belonging to* $\mathbb{T}$ *with some* $\tau \in \mathbb{N}_{1,n}$, *such that* $F(\{u_j\})$ *is a nonsingular matrix for* $j \in \mathbb{N}_{1,\tau}$, *then there exists a left* (*resp., right*) *strictly para-orthogonal system* $(P_j)_{j=1}^{\tau}$ *corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Moreover, there exists a left* (*resp., right*) *strictly para-orthogonal system* $(P_j)_{j=1}^{n}$ *corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *if and only if there are* n *pairwise different points* $u_1, u_2, \ldots, u_n$ *belonging to* $\mathbb{T}$ *such that* $F(\{u_j\})$ *is a nonsingular matrix for* $j \in \mathbb{N}_{1,n}$.

Proof. Applying Theorem 4.6 along with [29, Theorem 6.11] yields the assertion. $\square$

For measures belonging to $\mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ even the following equivalence holds. This is quite similar to a characterization of the existence of para-orthogonal systems of rational functions for the scalar case $q = 1$ (cf. [14, Theorem 3.5]).

THEOREM 5.5. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *and* $F \in \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Then the following statements are equivalent:*

(i) F *belongs to* $\mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

(ii) *There is a left* (*resp., right*) *strictly para-orthogonal system* $(P_j)_{j=1}^{\tau}$ *corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(iii) *There is a* $P_\tau \in \mathscr{R}_{\alpha,\tau}^{q \times q}$ *so that* $(P_\tau, y\mathbf{I}_q)_{F,l}$ (*resp.,* $(y\mathbf{I}_q, P_\tau)_{F,r}$) *is a nonsingular matrix for some* $y \in \mathscr{R}_{\alpha,\tau}$ *and* $(P_\tau, Z)_{F,l} = 0_{q \times q}$ (*resp.,* $(Z, P_\tau)_{F,r} = 0_{q \times q}$) *for all* $Z \in \widehat{\mathscr{R}}_{\alpha,\tau}^{q \times q}$.

Moreover, if (i) *holds, then the measure* F *belongs to* $\mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant,\tau-1}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and to* $\mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \setminus \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

Proof. From Theorem 4.4 we get that (i) leads to (ii). Recalling Definition 3.1, we see that (iii) follows from (ii). Suppose now that (iii) holds. Because of (2.8) and (2.9) we can restrict the considerations to the case in which there exists a $P_\tau \in \mathscr{R}_{\alpha,\tau}^{q \times q}$ such that $(P_\tau, y\mathbf{I}_q)_{F,l}$ is nonsingular for some $y \in \mathscr{R}_{\alpha,\tau}$ and such that $(P_\tau, Z)_{F,l} = 0_{q \times q}$ for each $Z \in \widehat{\mathscr{R}}_{\alpha,\tau}^{q \times q}$. Since $F \in \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, there is a sequence $(u_j)_{j=1}^{\tau}$ of pairwise different points belonging to $\mathbb{T}$ and a sequence $(\mathbf{A}_j)_{j=1}^{\tau}$ of nonnegative Hermitian $q \times q$ matrices such that the measure F admits the representation (5.2) with $n = \tau$. Therefore, as in the proof of Lemma 5.1 (cf. (5.3)), we obtain that

$$\det\left(\sum_{j=1}^{\tau} \overline{y(u_j)} P_\tau(u_j)\mathbf{A}_j\right) = \det\left(\sum_{j=1}^{\tau} P_\tau(u_j)\mathbf{A}_j\left(y(u_j)\mathbf{I}_q\right)^*\right) \neq 0 \qquad (5.5)$$

and

$$\sum_{j=1}^{\tau} P_\tau(u_j)\mathbf{A}_j\left(Z(u_j)\right)^* = 0_{q \times q}, \quad Z \in \widehat{\mathscr{R}}_{\alpha,\tau}^{q \times q}. \qquad (5.6)$$

If $\tau = 1$, then it follows directly from (5.5) that $\det \mathbf{A}_1 \neq 0$. Suppose now that $\tau > 1$. We will show that (5.5) and (5.6) imply $\det \mathbf{A}_j \neq 0$ for each $j \in \mathbb{N}_{1,\tau}$. We will use an indirect approach and suppose that $\mathbf{A}_s$ is a singular matrix for some $s \in \mathbb{N}_{1,\tau}$. Thus, there is a matrix $\mathbf{X} \in \mathbb{C}^{q \times q} \setminus \{0_{q \times q}\}$ such that

$$\mathbf{A}_s \mathbf{X} = 0_{q \times q}. \tag{5.7}$$

Let $k \in \mathbb{N}_{1,\tau} \setminus \{s\}$. Let $p_\tau : \mathbb{C} \to \mathbb{C}$ be the polynomial defined by $p_\tau(v)$ $:= \alpha_\tau - v$ and let $Q_{k,s} : \mathbb{C} \to \mathbb{C}^{q \times q}$ be the matrix polynomial of degree $\tau - 2$ defined by

$$Q_{k,s}(v) := \begin{cases} \mathbf{X}^* & \text{if } \tau = 2 \\ \displaystyle\prod_{j \in \mathbb{N}_{1,\tau} \setminus \{k,s\}} \frac{u_j - v}{u_j - u_k} \mathbf{X}^* & \text{if } \tau > 2 . \end{cases}$$

Furthermore, let

$$Z_{k,s} := \frac{\pi_{\alpha,\tau-1}(u_k)\, p_\tau}{(\alpha_\tau - u_k)\, \pi_{\alpha,\tau-1}} Q_{k,s},$$

where $\pi_{\alpha,\tau-1}$ is given by (2.1). From this and (3.1) we see that the function $Z_{k,s}$ belongs to $\widehat{\mathscr{R}}_{\alpha,\tau}^{q \times q}$. Moreover, we have $Z_{k,s}(u_j) = \delta_{jk}\mathbf{X}^*$ for all $j \in \mathbb{N}_{1,\tau} \setminus \{s\}$ and

$$Z_{k,s}(u_s) = \begin{cases} \dfrac{(\alpha_2 - u_s)(1 - \overline{\alpha_1} u_k)}{(\alpha_2 - u_k)(1 - \overline{\alpha_1} u_s)}\mathbf{X}^* & \text{if } \tau = 2 \\ \dfrac{(\alpha_\tau - u_s)\pi_{\alpha,\tau-1}(u_s)}{(\alpha_\tau - u_k)\pi_{\alpha,\tau-1}(u_k)} \displaystyle\prod_{j \in \mathbb{N}_{1,\tau} \setminus \{k,s\}} \dfrac{u_j - u_s}{u_j - u_k}\mathbf{X}^* & \text{if } \tau > 2 . \end{cases}$$

Hence, (5.6) and (5.7) imply

$$P_\tau(u_k)\mathbf{A}_k\mathbf{X} = P_\tau(u_k)\mathbf{A}_k\big(Z_{k,s}(u_k)\big)^* = \sum_{j=1}^{\tau} P_\tau(u_j)\mathbf{A}_j\big(Z_{k,s}(u_j)\big)^* = 0_{q \times q}.$$

Because of (5.7) we then get

$$\left(\sum_{j=1}^{\tau} \overline{y(u_j)}P_\tau(u_j)\mathbf{A}_j\right)\mathbf{X} = \sum_{j=1}^{\tau} \overline{y(u_j)}P_\tau(u_j)\mathbf{A}_j\mathbf{X} = \overline{y(u_s)}P_\tau(u_s)\mathbf{A}_s\mathbf{X} = 0_{q \times q},$$

which is in contradiction to (5.5). Consequently, there does not exist an $s \in \mathbb{N}_{1,\tau}$ such that $\mathbf{A}_s$ is a singular matrix, i.e. $\det \mathbf{A}_j \neq 0$ holds for each $j \in \mathbb{N}_{1,\tau}$. Finally, applying [29, Theorem 6.11] gives us that F belongs to $\mathscr{M}_{\geqslant,\tau}^{q,\text{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant,\tau-1}^{q,\text{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ and to $\mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. In particular, we see that (iii) implies (i). $\square$

With respect to Theorem 4.6 and the finite set Δ_τ of exclusion points of Theorem 4.4 (resp., Corollary 4.5), it appears that Δ_τ consists only of the τ mass points if the underlying measure F belongs to $\mathscr{M}_{\geqslant,\tau}^{q,\text{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. In fact, we get the following.

PROPOSITION 5.6. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *and suppose that* F *is a measure belonging to* $\mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \cap \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Then the subsets* Δ_{τ} *and* $\widetilde{\Delta}_{\tau}$ *of* $\mathbb{T}$ *from* Theorem 4.4 *consists both of exactly* τ *pairwise different points* $u_1, u_2, \ldots, u_{\tau}$. *Moreover,* F *admits* (5.2) *with* $n = \tau$ *and these points, where* $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{\tau}$ *are the positive Hermitian matrices given by* $\mathbf{A}_j = \big(A_{\tau-1,u_j}^{(\alpha,F)}(u_j)\big)^{-1}$ *and* $\mathbf{A}_j = \big(C_{\tau-1,u_j}^{(\alpha,F)}(u_j)\big)^{-1}$ *for each* $j \in \mathbb{N}_{1,\tau}$. *In particular, the four identities*

$$\sum_{j \in \mathbb{N}_{1,\tau} \setminus \{k\}} B_{\alpha,\tau}^{(q)}(u_j)\mathbf{A}_j C_{\tau-1,u_j}^{(\alpha,F)}(u_k) = 0_{q \times q}, \qquad \sum_{j \in \mathbb{N}_{1,\tau} \setminus \{k\}} b_{\alpha\tau}(u_j)C_{\tau-1,u_k}^{(\alpha,F)}(u_j)\mathbf{A}_j = 0_{q \times q},$$

$$\sum_{j \in \mathbb{N}_{1,\tau} \setminus \{k\}} B_{\alpha,\tau}^{(q)}(u_j)A_{\tau-1,u_j}^{(\alpha,F)}(u_k)\mathbf{A}_j = 0_{q \times q}, \qquad \sum_{j \in \mathbb{N}_{1,\tau} \setminus \{k\}} b_{\alpha\tau}(u_j)\mathbf{A}_j A_{\tau-1,u_k}^{(\alpha,F)}(u_j) = 0_{q \times q}$$

are satisfied for each $k \in \mathbb{N}_{1,\tau}$.

Proof. Let $n := \tau$. Because of $F \in \mathscr{M}_{\geqslant,n}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}}) \cap \mathscr{M}_{\geqslant}^{q,n-1}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, Theorem 5.5, and [29, Theorem 6.11] it follows that F admits (5.2) with a sequence $(u_j)_{j=1}^{n}$ of pairwise different points belonging to $\mathbb{T}$ and a sequence $(\mathbf{A}_j)_{j=1}^{n}$ of positive Hermitian $q \times q$ matrices. Let $k \in \mathbb{N}_{1,n}$. Furthermore, let Q_k be the matrix polynomial defined as in (5.4) and let $\pi_{\alpha,n-1}$ be the polynomial given by (2.1). Suppose that

$$X_k := \frac{\pi_{\alpha,n-1}(u_k)}{\pi_{\alpha,n-1}} Q_k.$$

We have that $X_k \in \mathscr{R}_{\alpha,n-1}^{q \times q}$, where $X_k(u_j) = \delta_{jk}\mathbf{I}_q$ holds for each $j \in \mathbb{N}_{1,n}$. Based on [28, Theorem 10], (5.2), and some facts from integration theory (cf. (5.3)) we find that

$$\mathbf{I}_q = \big(X_k, C_{n-1,u_k}^{(\alpha,F)}\big)_{F,l} = \sum_{j=1}^{n} X_k(u_j)\mathbf{A}_j\big(C_{n-1,u_k}^{(\alpha,F)}(u_j)\big)^* = \mathbf{A}_k C_{n-1,u_k}^{(\alpha,F)}(u_k),$$

i.e. that $\mathbf{A}_k = \big(C_{n-1,u_k}^{(\alpha,F)}(u_k)\big)^{-1}$. A similar argument yields $\mathbf{A}_k = \big(A_{n-1,u_k}^{(\alpha,F)}(u_k)\big)^{-1}$. Let $\Delta_n^{(l)}$ and $\widetilde{\Delta}_n^{(l)}$ (resp., $\Delta_n^{(r)}$ and $\widetilde{\Delta}_n^{(r)}$) be the subsets of $\mathbb{T}$ defined in Lemma 4.3 In particular, the set $\widetilde{\Delta}_n^{(l)}$ consists of at most n pairwise different points and

$$\big(B_{\alpha,n}^{(q)}, C_{n-1,z}^{(\alpha,F)}\big)_{F,l} = B_{\alpha,n}^{(q)}(z) \tag{5.8}$$

holds for each $z \in \widetilde{\Delta}_n^{(l)}$. Let $z \in \mathbb{T}$. Because of (5.2) it follows that

$$\big(B_{\alpha,n}^{(q)}, C_{n-1,z}^{(\alpha,F)}\big)_{F,l} = \sum_{j=1}^{n} B_{\alpha,n}^{(q)}(u_j)\mathbf{A}_j\big(C_{n-1,z}^{(\alpha,F)}(u_j)\big)^*.$$

Consequently, we see that (5.8) is fulfilled if and only if

$$\sum_{j=1}^{n} B_{\alpha,n}^{(q)}(u_j)\mathbf{A}_j C_{n-1,u_j}^{(\alpha,F)}(z) = B_{\alpha,n}^{(q)}(z). \tag{5.9}$$

Furthermore, by using part (a) of Lemma 4.3 and [30, Remark 2.5], we obtain

$$\left(B_{\alpha,n}^{(q)}, C_{n-1,z}^{(\alpha,F)}\right)_{F,l} = \sum_{j=1}^{n} B_{\alpha,n}^{(q)}(u_j) \mathbf{A}_j \left((A_{n-1,z}^{(\alpha,F)})^{[\alpha,n-1]}(u_j)\right)^* B_{\alpha,n-1}^{(q)}(z)$$

$$= \sum_{j=1}^{n} b_{\alpha_n}(u_j) \mathbf{A}_j A_{n-1,z}^{(\alpha,F)}(u_j) B_{\alpha,n-1}^{(q)}(z).$$

Thus, (5.8) holds if and only if

$$\sum_{j=1}^{n} b_{\alpha_n}(u_j) \mathbf{A}_j A_{n-1,z}^{(\alpha,F)}(u_j) = b_{\alpha_n}(z) \mathbf{I}_q. \tag{5.10}$$

Taking into account that the function

$$X := \sum_{j=1}^{n} \overline{b_{\alpha_n}(u_j)} X_j$$

belongs to $\mathscr{R}_{\alpha,n-1}^{q \times q}$, where the equality $X(u_j) = \overline{b_{\alpha_n}(u_j)} \mathbf{I}_q$ holds for each $j \in \mathbb{N}_{1,n}$, by applying [28, Theorem 10], (5.2), and (2.2) we see that

$$b_{\alpha_n}(u_j) \mathbf{I}_q = \left(X, A_{n-1,u_k}^{(\alpha,F)}\right)_{F,r} = \sum_{j=1}^{n} \left(X(u_j)\right)^* \mathbf{A}_j A_{n-1,u_k}^{(\alpha,F)}(u_j) = \sum_{j=1}^{n} b_{\alpha_n}(u_j) \mathbf{A}_j A_{n-1,u_k}^{(\alpha,F)}(u_j).$$

Therefore, (5.10) holds if $z = u_k$. Since (5.8) and (5.10) are equivalent, from part (b) of Lemma 4.3, $\tau = n$, and the fact that the set $\widetilde{\Delta}_\tau$ has at most τ elements we get that

$$\Delta_\tau^{(l)} = \widetilde{\Delta}_\tau^{(l)} = \{u_1, u_2, \ldots, u_\tau\}.$$

Recalling this and the form of $(\mathbf{A}_j)_{j=1}^{\tau}$, the equivalence of (5.8) and (5.9) gives us

$$\sum_{j \in \mathbb{N}_{1,\tau} \backslash \{k\}} B_{\alpha,\tau}^{(q)}(u_j) \mathbf{A}_j C_{\tau-1,u_j}^{(\alpha,F)}(u_k) = \sum_{j=1}^{\tau} B_{\alpha,\tau}^{(q)}(u_j) \mathbf{A}_j C_{\tau-1,u_j}^{(\alpha,F)}(u_k) - B_{\alpha,\tau}^{(q)}(u_k) = 0_{q \times q}$$

and the equivalence of (5.8) and (5.10) yields

$$\sum_{j \in \mathbb{N}_{1,\tau} \backslash \{k\}} b_{\alpha_\tau}(u_j) \mathbf{A}_j A_{\tau-1,u_k}^{(\alpha,F)}(u_j) = \sum_{j=1}^{\tau} b_{\alpha_\tau}(u_j) \mathbf{A}_j A_{\tau-1,u_k}^{(\alpha,F)}(u_j) - b_{\alpha_\tau}(u_k) \mathbf{I}_q = 0_{q \times q}.$$

A similar argument, based on part (c) of Lemma 4.3, leads to the remaining two identities (different versions of the last two) with $\Delta_\tau^{(r)} = \widetilde{\Delta}_\tau^{(r)} = \{u_1, u_2, \ldots, u_\tau\}$. In particular, it follows that the subsets Δ_τ and $\widetilde{\Delta}_\tau$ of $\mathbb{T}$ from Theorem 4.4 both consist of the τ pairwise different points $u_1, u_2, \ldots, u_\tau$. $\square$

COROLLARY 5.7. *Let* $\tau \in \mathbb{N}$ *and let* $F \in \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \cap \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. *Furthermore, let* $(z_j)_{j=1}^{\tau}$ *be a sequence of points belonging to* $\mathbb{T}$. *The following statements are equivalent:*

(i) *For every choice of a sequence* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$, *by using both declarations in* (3.12) *for each* $j \in \mathbb{N}_{1,\tau}$, *the pair* $[(P_j)_{j=1}^{\tau},(R_j)_{j=1}^{\tau}]$ *is a pair of strictly para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(ii) *By using* (3.12) *for each* $j \in \mathbb{N}_{1,\tau}$, *the sequence* $(P_j)_{j=1}^{\tau}$ *(resp.,* $(R_j)_{j=1}^{\tau}$*) is a left (resp., right) para-orthogonal system corresponding to some* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ *and* F.

(iii) *By setting*

$$P_j(v) = (1 - \overline{z_j}v)\left(\overline{z_j^{j-1}}\mathbf{I}_q, \overline{z_j^{j-2}}\mathbf{I}_q, \ldots, \overline{z_j^0}\mathbf{I}_q\right)\left(\mathbf{T}_{j-1}^{(F)}\right)^{-1}\begin{pmatrix} v^{j-1}\mathbf{I}_q \\ v^{j-2}\mathbf{I}_q \\ \vdots \\ v^0\mathbf{I}_q \end{pmatrix}$$

$$\left(resp., R_j(v) = (1 - v\overline{z_j})\left(v^0\mathbf{I}_q, v^1\mathbf{I}_q, \ldots, v^{j-1}\mathbf{I}_q\right)\left(\mathbf{T}_{j-1}^{(F)}\right)^{-1}\begin{pmatrix} \overline{z_j^0}\mathbf{I}_q \\ \overline{z_j^1}\mathbf{I}_q \\ \vdots \\ \overline{z_j^{j-1}}\mathbf{I}_q \end{pmatrix}\right)$$

for each $j \in \mathbb{N}_{1,\tau}$ *and* $v \in \mathbb{C}$, *the sequence* $(P_j)_{j=1}^{\tau}$ *(resp.,* $(R_j)_{j=1}^{\tau}$*) is a left (resp., right) para-orthogonal matrix polynomial system corresponding to* F.

(iv) *For all* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$, *there is a pair* $[(P_j)_{j=1}^{\tau},(R_j)_{j=1}^{\tau}]$ *of strictly para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *such that both equalities in* (3.11) *hold for* $j \in \mathbb{N}_{1,\tau}$ *and* $v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,j}$ *with a pair* $[(X_k)_{k=0}^{\tau-1},(Y_k)_{k=0}^{\tau-1}]$ *of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F, *where* $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}$ $\left(X_{j-1}(z_j)\right)^*$, $\tilde{\mathbf{B}}_j := \left(Y_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$, $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\left(Y_{j-1}(z_j)\right)^*$, *and* $\tilde{\mathbf{E}}_j := \left(X_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$.

(v) *There is a left (resp., right) para-orthogonal system* $(P_j)_{j=1}^{\tau}$ *(resp.,* $(R_j)_{j=1}^{\tau}$*) corresponding to some* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ *and* F *such that* (3.11) *holds for each* $j \in \mathbb{N}_{1,\tau}$ *and each* $v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,j}$ *with a pair* $[(X_k)_{k=0}^{\tau-1},(Y_k)_{k=0}^{\tau-1}]$ *of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F, *where* $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\left(X_{j-1}(z_j)\right)^*$ *and* $\tilde{\mathbf{B}}_j := \left(Y_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$ *(resp., where* $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}\left(Y_{j-1}(z_j)\right)^*$ *and* $\tilde{\mathbf{E}}_j := \left(X_{j-1}^{[\alpha,j-1]}(z_j)\right)^*$*).

(vi) *By setting*

$$P_j(v) = \left(\tilde{Y}_{j-1}^{[j-1]}(z_j)\right)^*\tilde{Y}_{j-1}^{[j-1]}(v) - \overline{z_j}v\left(X_{j-1}(z_j)\right)^*X_{j-1}(v)$$

$$\left(resp., R_j(v) = \tilde{X}_{j-1}^{[j-1]}(v)\left(\tilde{X}_{j-1}^{[j-1]}(z_j)\right)^* - v\overline{z_j}Y_{j-1}(v)\left(Y_{j-1}(z_j)\right)^*\right)$$

for $j \in \mathbb{N}_{1,\tau}$ *and* $v \in \mathbb{C}$ *with some pair* $[(X_k)_{k=0}^{\tau-1},(Y_k)_{k=0}^{\tau-1}]$ *of orthonormal matrix polynomial systems corresponding to* F, $(P_j)_{j=1}^{\tau}$ *(resp.,* $(R_j)_{j=1}^{\tau}$*) is a left (resp., right) para-orthogonal matrix polynomial system corresponding to* F.

Proof. Proposition 5.6 gives us that the sets Δ_τ and $\widetilde{\Delta}_\tau$ in Theorem 4.4 coincide and that the location of the poles for the relevant rational functions has no influence on the form of Δ_τ if $F \in \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \cap \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Thus, the assertion is a simple consequence of Proposition 5.6, Theorem 4.4, and Corollary 4.5. (In doing so, with respect to (iii), one has to notice (2.3)–(2.5) and [29, Remark 2.4, Remark 3.9, and Lemma 3.14].) $\square$

COROLLARY 5.8. *Let* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ *and let* $\tau \in \mathbb{N}$. *Let* $F \in \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ *be such that, for every choice of a sequence* $(z_j)_{j=1}^\tau$ *of points belonging to* $\mathbb{T}$, *by using* (3.12) *for each* $j \in \mathbb{N}_{1,\tau}$, *the sequence* $(P_j)_{j=1}^\tau$ *(resp.,* $(R_j)_{j=1}^\tau$) *is a left (resp., right) para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F. *Then*

$$F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}).$$

Proof. Use Theorem 4.4 along with Proposition 5.6. $\square$

COROLLARY 5.9. *Let* $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *and* $F \in \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. *If, for every sequence* $(z_j)_{j=1}^\tau$ *of points belonging to* $\mathbb{T}$, *there is a left (resp., right) para-orthogonal system* $(P_j)_{j=1}^\tau$ *(resp.,* $(R_j)_{j=1}^\tau$) *corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F *such that* (3.11) *holds for all* $j \in \mathbb{N}_{1,\tau}$ *and* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, *where* $\tilde{\mathbf{A}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}(X_{j-1}(z_j))^*$ *and* $\tilde{\mathbf{B}}_j := (Y_{j-1}^{[\alpha,j-1]}(z_j))^*$ *(resp.,* $\tilde{\mathbf{D}}_j := -\overline{b_{\alpha_{j-1}}(z_j)}(Y_{j-1}(z_j))^*$ *and* $\tilde{\mathbf{E}}_j := (X_{j-1}^{[\alpha,j-1]}(z_j))^*$) *with a pair* $[(X_k)_{k=0}^{\tau-1}, (Y_k)_{k=0}^{\tau-1}]$ *of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F, *then*

$$F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}).$$

Proof. Apply Corollary 4.5 along with Proposition 5.6. $\square$

It should be noted that, for the scalar case $q = 1$ with some $\tau \in \mathbb{N}$, Corollary 5.8 (resp., Corollary 5.9) leads in connection with Corollary 3.8 to a characterization of whether a measure F belongs to $\mathscr{M}_{\geqslant}^{1,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ or not (cf. [14, Proposition 3.9]).

REMARK 5.10. Let $\tau \in \mathbb{N}$ and let $\Delta_\tau^{(l)}$ and $\widetilde{\Delta}_\tau^{(l)}$ (resp., $\Delta_\tau^{(r)}$ and $\widetilde{\Delta}_\tau^{(r)}$) be defined as in Lemma 4.3. In view of Lemma 4.3 and the proof of Proposition 5.6 one can see that $\Delta_\tau^{(l)} = \Delta_\tau^{(r)}$ (resp., $\widetilde{\Delta}_\tau^{(l)} = \widetilde{\Delta}_\tau^{(r)}$) when the underlying measure F belongs to $\mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ or to $\mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \cap \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Moreover, based on [17, Corollary 5.10] one can realize that this equality holds as well in the special case of matrix polynomials, i.e. if $\alpha_j = 0$ for each $j \in \mathbb{N}_{1,\tau}$ (without additional restriction on F).

For any $\tau \in \mathbb{N}$, Remark 5.10 leads us to surmise that, in general, the sets $\Delta_\tau^{(l)}$ and $\Delta_\tau^{(r)}$ (resp., $\widetilde{\Delta}_\tau^{(l)}$ and $\widetilde{\Delta}_\tau^{(r)}$) stated in Lemma 4.3 coincide.

REMARK 5.11. Because of Lemma 4.3 and Proposition 5.6, for some $\tau \in \mathbb{N}$, the sets Δ_τ and $\widetilde{\Delta}_\tau$ in Theorem 4.4 are empty if $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ and consists of the τ

mass points of F in case that $F \in \mathscr{M}_{\geqslant,\tau}^{q,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \cap \mathscr{M}_{\geqslant}^{q,\tau-1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. In particular, it follows that the location of the poles of the relevant rational matrix functions has no influence on the form of the sets Δ_τ and $\widetilde{\Delta}_\tau$ in these situations.

Remark 5.11 results in the supposition that, in general, the location of the poles of the relevant rational matrix functions has no influence on the form of Δ_τ and $\widetilde{\Delta}_\tau$.

Even though Proposition 5.6 and Remark 5.11 suggest the opposite, the sets Δ_τ and $\widetilde{\Delta}_\tau$ do not always coincide. This is emphasized by the following simple example (as in Example 5.3) with $q = 2$ and $\tau = 1$.

EXAMPLE 5.12. *Let* $u_1 \in \mathbb{T}$ *and* $u_2 \in \mathbb{T} \setminus \{u_1\}$. *Let* $F := \varepsilon_{u_1, \mathfrak{B}_\mathbb{T}} \mathbf{A}_1 + \varepsilon_{u_2, \mathfrak{B}_\mathbb{T}} \mathbf{A}_2$, *where*

$$\mathbf{A}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad \mathbf{A}_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, let $\alpha_1 \in \mathbb{C} \setminus \mathbb{T}$. *Then F belongs to* $\mathscr{M}_{\geqslant,2}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant,1}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ *and* $\mathscr{M}_{\geqslant}^{2,0}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant}^{2,1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$, *where* $\left(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)}\right)_{F,l} \neq B_{\alpha,1}^{(2)}(z)$ *and* $\left(A_{0,z}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} \neq B_{\alpha,1}^{(2)}(z)$ *for each* $z \in \mathbb{T}$. *But,* $\left(B_{\alpha,1}^{(2)}, C_{0,u_1}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(u_1)$, $\left(B_{\alpha,1}^{(2)}, C_{0,u_2}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(u_2)$, $\left(A_{0,u_1}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(u_1)$, $\left(A_{0,u_2}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(u_2)$ *are singular matrices.*

Proof. The choice of F implies immediately that

$$F \in \mathscr{M}_{\geqslant,2}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant,1}^{2,\mathrm{mol}}(\mathbb{T}, \mathfrak{B}_\mathbb{T}),$$

where

$$F(\mathbb{T}) = \mathbf{A}_1 + \mathbf{A}_2 = \mathbf{I}_2.$$

Thus, based on [29, Theorem 6.11] we get $F \in \mathscr{M}_{\geqslant}^{2,0}(\mathbb{T}, \mathfrak{B}_\mathbb{T}) \setminus \mathscr{M}_{\geqslant}^{2,1}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Furthermore, because of (2.3)–(2.5) one can see that $C_{0,z}^{(\alpha,F)}$ (resp., $A_{0,z}^{(\alpha,F)}$) is the constant function on $\mathbb{C}_0$ with value $\mathbf{I}_2$ for each $z \in \mathbb{T}$. Consequently, for each $z \in \mathbb{T}$, we get

$$\left(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(z) = B_{\alpha,1}^{(2)}(u_1)\mathbf{A}_1 + B_{\alpha,1}^{(2)}(u_2)\mathbf{A}_2 - B_{\alpha,1}^{(2)}(z)$$

$$= \begin{pmatrix} b_{\alpha_1}(u_1) - b_{\alpha_1}(z) & 0 \\ 0 & b_{\alpha_1}(u_2) - b_{\alpha_1}(z) \end{pmatrix}$$

and similarly

$$\left(A_{0,z}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(z) = \begin{pmatrix} b_{\alpha_1}(u_1) - b_{\alpha_1}(z) & 0 \\ 0 & b_{\alpha_1}(u_2) - b_{\alpha_1}(z) \end{pmatrix}.$$

Finally, in view of (2.6) it follows that the inequalities $\left(B_{\alpha,1}^{(2)}, C_{0,z}^{(\alpha,F)}\right)_{F,l} \neq B_{\alpha,1}^{(2)}(z)$ and $\left(A_{0,z}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} \neq B_{\alpha,1}^{(2)}(z)$ hold for each $z \in \mathbb{T}$ and that $\left(B_{\alpha,1}^{(2)}, C_{0,u_1}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(u_1)$, $\left(B_{\alpha,1}^{(2)}, C_{0,u_2}^{(\alpha,F)}\right)_{F,l} - B_{\alpha,1}^{(2)}(u_2)$, $\left(A_{0,u_1}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(u_1)$ and $\left(A_{0,u_2}^{(\alpha,F)}, B_{\alpha,1}^{(2)}\right)_{F,r} - B_{\alpha,1}^{(2)}(u_2)$ are singular matrices. $\square$

The existence results presented above (regardless of the exact construction of the relevant objects) do not depend on the concrete choice of the underlying sequence $(\alpha_j)_{j=1}^{\infty}$. This leads us to surmise that, similar to the case for left (resp., right) orthogonal systems of rational matrix functions (cf. [30, Corollary 4.4]), the existence of a left (resp., right) para-orthogonal system $(P_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F for some $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ implies the existence of such a system for any $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ (cf. Corollary 4.8). Note that this equivalence is fulfilled in the scalar case $q = 1$ (see [14, Corollary 3.7 and Corollary 3.10]).

6. Particular pairs of para-orthogonal rational matrix functions

We now consider specific left (resp., right) strictly para-orthogonal systems $(P_j)_{j=1}^{\tau}$ corresponding to some $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ with $\tau \in \mathbb{N}$ or $\tau = \infty$.

By comparing Corollary 3.6 to [34, Theorem 6.8], we see that the extremal solutions of a matrix moment problem discussed in [33], [34], and [43] make up a distinguished structure that is determined by special para-orthogonal rational matrix functions. The para-orthogonal systems studied below are similarly associated to another kind of extremal solutions having to do with the moment problem in question (cf. [32, Section 9]). In particular, this kind of para-orthogonal system can be applied in the scalar case $q = 1$ of rational functions to obtain quadrature formulas on the unit circle $\mathbb{T}$ (see, e.g., [8] and [11]).

DEFINITION 6.1. Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. A pair of para-orthogonal systems $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F is called *canonical* if there exists a pair of orthonormal systems $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that, for each $j \in \mathbb{N}_{1,\tau}$, the functions P_j and R_j admit the representations

$$P_j = \mathbf{U}_j X_j + Y_j^{[\alpha,j]} \quad \text{and} \quad R_j = Y_j \mathbf{U}_j + X_j^{[\alpha,j]},$$

where $\mathbf{U}_j$ is some unitary $q \times q$ matrix.

Note that the representations in Definition 6.1 depend on the choice of the pair of orthonormal systems $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$. Because of [30, Proposition 3.7] we, however, see that this is not essential.

Following up on Remarks 3.2 and 3.3, we now present some elementary properties relating to Definition 6.1. In doing so, unless otherwise indicated, let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Furthermore, let $F \in \mathscr{M}_{\geqslant}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ be arbitrary, but fixed.

REMARK 6.2. Let $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ be a pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Then:

(a) Let $(\mathbf{V}_j)_{j=1}^{\tau}$ and $(\mathbf{W}_j)_{j=1}^{\tau}$ be sequences of unitary $q \times q$ matrices. By Theorem 3.5, (2.8), and [30, Remark 2.8 and Proposition 3.7] one can see that $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ is a canonical pair of para-orthogonal systems corresponding

to $(\alpha_j)_{j=1}^\infty$ and F if and only if $[(\mathbf{V}_j P_j)_{j=1}^\tau, (R_j \mathbf{W}_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F.

(b) By using Theorem 3.5 along with [30, Remarks 2.11 and 3.5], a simple calculation shows that $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $[(R_j^T)_{j=1}^\tau, (P_j^T)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F^T.

(c) From Theorem 3.5, (2.8), and [30, Remark 2.8 and Proposition 3.7] it follows that $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $[(R_j^{[\alpha,j]})_{j=1}^\tau, (P_j^{[\alpha,j]})_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F.

(d) Let $\mathbf{A} \in \mathbb{C}^{q \times q}$ be nonsingular and let $F_\mathbf{A} \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ be given as in Remark 3.3. Then $[(X_k)_{k=0}^\tau, (Y_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $[(X_k \mathbf{A}^{-*})_{k=0}^\tau, (\mathbf{A}^{-1} Y_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_\mathbf{A}$. Thus, a pair $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F if and only if $[(P_j \mathbf{A}^{-*})_{j=1}^\tau, (\mathbf{A}^{-1} R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_\mathbf{A}$.

One crucial point for obtaining quadrature formulas in the scalar case $q = 1$ of rational functions based on para-orthogonal systems is that all zeros of the corresponding functions are located on the unit circle $\mathbb{T}$ (see, e.g., [8, Theorem 4]). The next considerations serve to clarify that canonical pairs of para-orthogonal systems of rational matrix functions comprise a similar property concerning determinants. We first remark that, by using the same argumentation as in the proof of [34, Lemma 4.9] (on the basis of the Christoffel–Darboux formulas for orthonormal systems of rational matrix functions stated in [30, Theorem 4.5]), one can verify the following result.

REMARK 6.3. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$ and let $\tau \in \mathbb{N}$ or $\tau = \infty$. Let $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ and suppose that $[(X_k)_{k=0}^\tau, (Y_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Let $j \in \mathbb{N}_{1,\tau}$. Then the rational matrix-valued function $\widetilde{\Theta}_j$ given by

$$\widetilde{\Theta}_j := \begin{cases} (X_j^{[\alpha,j]})^{-1} Y_j & \text{if } \alpha_j \in \mathbb{D} \\ Y_j^{-1} X_j^{[\alpha,j]} & \text{if } \alpha_j \in \mathbb{C} \setminus \mathbb{D} \end{cases}$$

admits the representation

$$\widetilde{\Theta}_j = \begin{cases} X_j (Y_j^{[\alpha,j]})^{-1} & \text{if } \alpha_j \in \mathbb{D} \\ Y_j^{[\alpha,j]} X_j^{-1} & \text{if } \alpha_j \in \mathbb{C} \setminus \mathbb{D}, \end{cases}$$

where the inverse values of matrix functions are well-defined on $(\mathbb{D} \cup \mathbb{T}) \setminus \mathbb{P}_{\alpha,j}$, the matrix $\widetilde{\Theta}_j(w)$ is strictly contractive for $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,j}$, and $\widetilde{\Theta}_j(z)$ is a unitary matrix for $z \in \mathbb{T}$. Moreover, for all $v, w \in (\mathbb{D} \cup \mathbb{T}) \setminus \mathbb{P}_{\alpha,j}$, the following statements are equivalent:

(i) $\widetilde{\Theta}_j(w) = \widetilde{\Theta}_j(v)$.

(ii) $v = w$ or $(A_{j-1,w}^{(\alpha,F)})^{[\alpha,j-1]}(v) = 0_{q\times q}$.

(iii) $v = w$ or $(C_{j-1,w}^{(\alpha,F)})^{[\alpha,j-1]}(v) = 0_{q\times q}$.

LEMMA 6.4. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$ *and let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$. *Let* $F \in \mathscr{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and suppose that* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *is a canonical pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Let* $j \in \mathbb{N}_{1,\tau}$ *and let* $\pi_{\alpha,j}$ *be the polynomial given by* (2.1). *Furthermore, let* p_j *and* r_j *be the complex* $q \times q$ *matrix polynomials of degree not greater than* j *such that*

$$P_j = \frac{1}{\pi_{\alpha,j}} p_j \quad and \quad R_j = \frac{1}{\pi_{\alpha,j}} r_j.$$

(a) *There exists a unitary* $q \times q$ *matrix* $\mathbf{U}_j$ *such that* $p_j = \mathbf{U}_j \tilde{r}_j^{[j]}$. *In particular, the equalities* $\det p_j = u_j \det \tilde{r}_j^{[j]}$ *and* $u_j \det \tilde{p}_j^{[j]} = \det r_j$ *hold for some* $u_j \in \mathbb{T}$. *Moreover,* $\mathscr{N}(p_j(v)) = \mathscr{N}(\tilde{r}_j^{[j]}(v))$ *for each* $v \in \mathbb{C}$ *and* $\mathscr{N}(p_j(z)) = \mathscr{N}((r_j(z))^*)$ *for each* $z \in \mathbb{T}$.

(b) *There is a* $\breve{u}_j \in \mathbb{T}$ *such that* $\breve{u}_j \det p_j = \det r_j$ *and* $\det \tilde{p}_j^{[j]} = \breve{u}_j \det \tilde{r}_j^{[j]}$.

(c) *There exist at most* j *pairwise different complex numbers* $w_1, w_2, \ldots, w_j$ *such that* $p_j(w_s) = 0$ *(resp.,* $r_j(w_s) = 0$*) holds for each* $s \in \mathbb{N}_{1,j}$ *and there exist at most* jq *pairwise different complex numbers* $z_1, z_2, \ldots, z_{jq}$ *such that the complex* $q \times q$ *matrix* $p_j(z_s)$ *(resp.,* $r_j(z_s)$*) is singular for each* $s \in \mathbb{N}_{1,jq}$.

(d) *If one of the matrices* $p_j(z)$, $r_j(z)$, $\tilde{p}_j^{[j]}(z)$, *or* $\tilde{r}_j^{[j]}(z)$ *is singular for some* $z \in \mathbb{C}$, *then all of them are singular and* $z \in \mathbb{T}$.

(e) *Each of the functions* p_j, r_j, $\tilde{p}_j^{[j]}$, *and* $\tilde{r}_j^{[j]}$ *is a complex* $q \times q$ *matrix polynomial of degree* j *with a nonsingular matrix as leading coefficient.*

Proof. (a) Because of Definition 6.1 there is a unitary $q \times q$ matrix $\mathbf{U}$ and a pair of orthonormal systems $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that

$$P_j = \mathbf{U} X_j + Y_j^{[\alpha,j]} \quad and \quad R_j = Y_j \mathbf{U} + X_j^{[\alpha,j]}.$$

Let x_j and y_j be the $q \times q$ matrix polynomials of degree not greater than j so that

$$X_j = \frac{1}{\pi_{\alpha,j}} x_j \quad and \quad Y_j = \frac{1}{\pi_{\alpha,j}} y_j. \tag{6.1}$$

Based on (6.1) and [30, Proposition 2.13] we see that there is a $\eta \in \mathbb{T}$ such that

$$X_j^{[\alpha,j]} = \eta \frac{1}{\pi_{\alpha,j}} \tilde{x}_j^{[j]} \quad and \quad Y_j^{[\alpha,j]} = \eta \frac{1}{\pi_{\alpha,j}} \tilde{y}_j^{[j]}. \tag{6.2}$$

If $\mathbf{U}_j := \eta\mathbf{U}$, then $\mathbf{U}_j$ is a unitary $q \times q$ matrix and from (6.1) and (6.2) it follows

$$p_j = \mathbf{U}x_j + \eta\tilde{y}_j^{[j]} = \mathbf{U}_j(\overline{\eta}x_j + \mathbf{U}^*\tilde{y}_j^{[j]}) = \mathbf{U}_j\tilde{r}_j^{[j]}.$$

This implies that $\mathcal{N}\big(p_j(v)\big) = \mathcal{N}\big(\tilde{r}_j^{[j]}(v)\big)$ for each $v \in \mathbb{C}$ and $\det p_j = u_j\det\tilde{r}_j^{[j]}$ and $u_j\det\tilde{p}_j^{[j]} = \det r_j$ with $u_j := \det\mathbf{U}_j$, where $u_j \in \mathbb{T}$. Since $\tilde{r}_j^{[j]}(z) = z^j\big(r_j(z)\big)^*$ for each $z \in \mathbb{T}$ (see, e.g., [27, Lemma 1.2.2]), we get $\mathcal{N}\big(p_j(z)\big) = \mathcal{N}\big((r_j(z))^*\big)$.

(b) From (6.2) and [30, Corollary 4.4 and Theorem 6.10] we get $\det\tilde{x}_j^{[j]} = \breve{u}_j\det\tilde{y}_j^{[j]}$ for some $\breve{u}_j \in \mathbb{T}$. Recalling (6.1), (6.2), [30, Corollary 4.4, Remark 6.2, and Lemma 6.5], and [27, Lemma 1.1.8], for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$ satisfying $\det\tilde{x}_j^{[j]}(v) \neq 0$, we have

$$\begin{aligned}
\breve{u}_j\det p_j(v) &= \breve{u}_j\det\Big(\mathbf{U}x_j(v)\big(\tilde{y}_j^{[j]}(v)\big)^{-1} + \eta\mathbf{I}_q\Big)\det\tilde{y}_j^{[j]}(v) \\
&= \eta^q\det\Big(\overline{\eta}\mathbf{U}\big(\tilde{x}_j^{[j]}(v)\big)^{-1}y_j(v) + \mathbf{I}_q\Big)\det\tilde{x}_j^{[j]}(v) \\
&= \det\Big(y_j(v)\mathbf{U}\big(\tilde{x}_j^{[j]}(v)\big)^{-1} + \eta\mathbf{I}_q\Big)\det\tilde{x}_j^{[j]}(v) = \det r_j(v).
\end{aligned}$$

Thus, by using [30, Corollaries 4.4 and 4.7] and a continuity argument, we get

$$\breve{u}_j\det p_j = \det r_j.$$

This directly leads us to $\det\tilde{p}_j^{[j]} = \breve{u}_j\det\tilde{r}_j^{[j]}$ (see, e.g., [30, Remark 2.6]).

(d) If one of the matrices $p_j(z)$, $r_j(z)$, $\tilde{p}_j^{[j]}(z)$, or $\tilde{r}_j^{[j]}(z)$ is singular for a $z \in \mathbb{C}$, then we see that all of them are singular due to (a) and (b). Let $\alpha_j \in \mathbb{D}$. Recalling [30, Corollary 4.4 and Remark 6.2] and [31, Theorem 3.10 and Lemma 3.11], we know that

$$\det\tilde{y}_j^{[j]}(w) \neq 0$$

for each $w \in \mathbb{D}$. Furthermore, from [30, Corollary 4.4] and Remark 6.3 we find that the matrix $x_j(w)\big(\tilde{y}_j^{[j]}(w)\big)^{-1}$ is strictly contractive for each $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,j}$. An elementary result for matricial Schur functions (see, e.g., [27, Lemma 2.1.5]) shows then that $x_j(w)\big(\tilde{y}_j^{[j]}(w)\big)^{-1}$ is strictly contractive even for all $w \in \mathbb{D}$. Therefore (use, e.g., [27, Remark 1.1.2 and Lemma 1.1.13]), for each $w \in \mathbb{D}$, the matrix $\overline{\eta}\mathbf{U}x_j(w)\big(\tilde{y}_j^{[j]}(w)\big)^{-1}$ is strictly contractive and

$$\det p_j(w) = \eta^q\det\Big(\overline{\eta}\mathbf{U}x_j(w)\big(\tilde{y}_j^{[j]}(w)\big)^{-1} + \mathbf{I}_q\Big)\det\tilde{y}_j^{[j]}(w) \neq 0.$$

Similarly, in the case of $\alpha_j \in \mathbb{C} \setminus \mathbb{D}$, based on [30, Corollary 4.4 and Remark 6.2], [31, Theorem 3.10 and Lemma 3.11], and Remark 6.3 we find that $\det x_j(w) \neq 0$, where the matrix $\eta\mathbf{U}^*\tilde{y}_j^{[j]}(w)\big(x_j(w)\big)^{-1}$ is strictly contractive and $\det p_j(w) \neq 0$ for each $w \in \mathbb{D}$. Combining this with (a) and (b) we obtain that

$$\det\tilde{p}_j^{[j]}(w) \neq 0$$

for each $w \in \mathbb{D}$. Finally, it follows from [27, Lemma 1.2.3] that if $\det p_j(z) = 0$ is satisfied for some $z \in \mathbb{C}$, then z belongs to $\mathbb{T}$.

(e) By (d) we realize that $\tilde{p}_j^{[j]}(0)$, $\tilde{r}_j^{[j]}(0)$, $p_j(0)$, and $r_j(0)$ are nonsingular matrices. This implies that each of the functions p_j, r_j, $\tilde{p}_j^{[j]}$, and $\tilde{r}_j^{[j]}$ are complex $q \times q$ matrix polynomials of (exact) degree j with a nonsingular matrix as leading coefficient.

(c) Since from (e) we know that p_j (resp., r_j) is a complex $q \times q$ matrix polynomial of degree j with a nonsingular matrix as leading coefficient, we see that (c) is an immediate consequence of the Fundamental Theorem of Algebra. $\square$

THEOREM 6.5. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ *and let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$. *Let* $F \in \mathcal{M}_{\geqslant}^q(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and suppose that* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *is a canonical pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Furthermore, let* $j \in \mathbb{N}_{1,\tau}$. *Then:*

(a) *There is a unitary* $q \times q$ *matrix* $\mathbf{U}_j$ *such that* $P_j = \mathbf{U}_j R_j^{[\alpha,j]}$. *In particular,* $\det P_j = u_j \det R_j^{[\alpha,j]}$ *and* $u_j \det P_j^{[\alpha,j]} = \det R_j$ *hold for some* $u_j \in \mathbb{T}$. *Moreover,* $\mathcal{N}(P_j(v)) = \mathcal{N}(R_j^{[\alpha,j]}(v))$ *for* $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ *and* $\mathcal{N}(P_j(z)) = \mathcal{N}((R_j(z))^*)$ *for* $z \in \mathbb{T}$.

(b) *There is a* $\breve{u}_j \in \mathbb{T}$ *such that* $\breve{u}_j \det P_j = \det R_j$ *and* $\det P_j^{[\alpha,j]} = \breve{u}_j \det R_j^{[\alpha,j]}$.

(c) *There exist at most* j *pairwise different points* $w_1, w_2, \ldots, w_j \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ *such that* $P_j(w_s) = 0$ *(resp.,* $R_j(w_s) = 0$) *holds for each* $s \in \mathbb{N}_{1,j}$ *and there exist at most* jq *pairwise different points* $z_1, z_2, \ldots, z_{jq} \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ *such that the complex* $q \times q$ *matrix* $P_j(z_s)$ *(resp.,* $R_j(z_s)$) *is singular for each* $s \in \mathbb{N}_{1,jq}$.

(d) *If one of the values* $P_j(z)$, $R_j(z)$, $P_j^{[\alpha,j]}(z)$, *or* $R_j^{[\alpha,j]}(z)$ *is a singular matrix for some* $z \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$, *then all of them are singular and* $z \in \mathbb{T}$.

(e) *Each of the functions* P_j, R_j, $P_j^{[\alpha,j]}$, *and* $R_j^{[\alpha,j]}$ *belongs to* $\mathcal{R}_{\alpha,j}^{q \times q} \setminus \mathcal{R}_{\alpha,j-1}^{q \times q}$.

(f) *If* Z_0 *is a constant function on* $\mathbb{C}_0$ *with a nonsingular complex* $q \times q$ *matrix as value and if* $Z_n \in \{P_n, R_n, P_n^{[\alpha,n]}, R_n^{[\alpha,n]}\}$ *for* $n \in \mathbb{N}_{1,j}$, *then* $Z_0, Z_1, \ldots, Z_j$ *is a basis of the right* $\mathbb{C}^{q \times q}$-*module* $\mathcal{R}_{\alpha,j}^{q \times q}$ *and of the left* $\mathbb{C}^{q \times q}$-*module* $\mathcal{R}_{\alpha,j}^{q \times q}$.

Proof. Recalling [30, Proposition 2.13], we see that the assertions of (a)–(d) are a simple consequence of Lemma 6.4. Furthermore, from (d) we can then conclude that the complex $q \times q$ matrices $P_j^{[\alpha,j]}(\alpha_j)$, $R_j^{[\alpha,j]}(\alpha_j)$, $P_j(\alpha_j)$, and $R_j(\alpha_j)$ are all nonsingular. This implies along with [30, Equation (2.10)] (resp., [30, Remark 2.17]) that (e) (resp., (f)) holds. $\square$

It is worth noting that the statement of part (f) of Theorem 6.5 does not hold for arbitrary pairs of para-orthogonal systems of rational matrix functions (cf. Remark 3.11).

REMARK 6.6. Because of Definition 6.1, Theorem 3.5, and [30, Corollary 4.4] there is a canonical pair of para-orthogonal systems $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F if and only if the measure F belongs to $\mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, Definition 6.1, Theorem 3.5, and [30, Corollary 4.4] imply that $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ is a pair of strictly para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F, where a pair $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F exists so that, for each $j \in \mathbb{N}_{1,\tau}$, the complex $q \times q$ matrix $(P_j, B_{\alpha,j}^{(q)})_{F,l} X_j^{[\alpha,j]}(\alpha_j)$ is unitary, $(P_j, B_{\alpha,j}^{(q)})_{F,l}$
$X_j^{[\alpha,j]}(\alpha_j) = Y_j^{[\alpha,j]}(\alpha_j)(B_{\alpha,j}^{(q)}, R_j)_{F,r}$, $(P_j, B_{\alpha,0}^{(q)})_{F,l}(Y_j^{[\alpha,j]}(\alpha_j))^* = \mathbf{I}_q$, and $(P_j, B_{\alpha,0}^{(q)})_{F,l}$
$(Y_j^{[\alpha,j]}(\alpha_j))^* = (X_j^{[\alpha,j]}(\alpha_j))^*(B_{\alpha,0}^{(q)}, R_j)_{F,r}$.

Taking Remark 6.6 into account, without loss of generality, we now assume that the underlying measure F belongs to $\mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

In the $q = 1$ case for complex-valued functions, the para-orthogonal systems in Definition 6.1 can be characterized up to multiplication with nonzero constants via an invariance property with respect to the transformation given by (2.7) (see, e.g., [8, Theorem 3]). In the pure matrix case $q \geqslant 2$, the situation comes across as somewhat less elegant. Nevertheless, we now present a result which can be regarded as a matricial version of that characterization.

PROPOSITION 6.7. *Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and suppose that $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Furthermore, let $(P_j)_{j=1}^{\tau}$ (resp., $(R_j)_{j=1}^{\tau}$) be a sequence of complex $q \times q$ matrix functions.*

(a) *Let $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ be a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. Then the following statements are equivalent:*

    (i) *$[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ is a pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that, for $j \in \mathbb{N}_{1,\tau}$, the following conditions hold:*

        (I) *$R_j = P_j^{[\alpha,j]}\mathbf{C}_j$ for some nonsingular complex $q \times q$ matrix $\mathbf{C}_j$.*

        (II) *At least one of the matrices $(P_j, B_{\alpha,0}^{(q)})_{F,l}$, $(P_j, B_{\alpha,j}^{(q)})_{F,l}$, $(B_{\alpha,0}^{(q)}, R_j)_{F,r}$, or $(B_{\alpha,j}^{(q)}, R_j)_{F,r})$ is nonsingular.*

        (III) *$(P_j, B_{\alpha,0}^{(q)})_{F,l}(Y_j^{[\alpha,j]}(\alpha_j))^* = (X_j^{[\alpha,j]}(\alpha_j))^*(B_{\alpha,0}^{(q)}, R_j)_{F,r}$.*

        (IV) *$(P_j, B_{\alpha,j}^{(q)})_{F,l}X_j^{[\alpha,j]}(\alpha_j) = Y_j^{[\alpha,j]}(\alpha_j)(B_{\alpha,j}^{(q)}, R_j)_{F,r}$.*

        (V) *At least one of the following identities holds:*
$$(P_j, B_{\alpha,j}^{(q)})_{F,l}X_j^{[\alpha,j]}(\alpha_j)(P_j, B_{\alpha,0}^{(q)})_{F,l}(Y_j^{[\alpha,j]}(\alpha_j))^*$$
$$= (X_j^{[\alpha,j]}(\alpha_j))^*(B_{\alpha,0}^{(q)}, R_j)_{F,r}Y_j^{[\alpha,j]}(\alpha_j)(B_{\alpha,j}^{(q)}, R_j)_{F,r},$$
$$(P_j, B_{\alpha,j}^{(q)})_{F,l}X_j^{[\alpha,j]}(\alpha_j)(X_j^{[\alpha,j]}(\alpha_j))^*(B_{\alpha,0}^{(q)}, R_j)_{F,r}$$
$$= (P_j, B_{\alpha,0}^{(q)})_{F,l}(Y_j^{[\alpha,j]}(\alpha_j))^*Y_j^{[\alpha,j]}(\alpha_j)(B_{\alpha,j}^{(q)}, R_j)_{F,r},$$

$$Y_j^{[\alpha,j]}(\alpha_j)\big(B_{\alpha,j}^{(q)},R_j\big)_{F,r}\big(P_j,B_{\alpha,0}^{(q)}\big)_{F,l}\big(Y_j^{[\alpha,j]}(\alpha_j)\big)^*$$
$$= \big(X_j^{[\alpha,j]}(\alpha_j)\big)^*\big(B_{\alpha,0}^{(q)},R_j\big)_{F,r}\big(P_j,B_{\alpha,j}^{(q)}\big)_{F,l}X_j^{[\alpha,j]}(\alpha_j).$$

(ii) *For* $j \in \mathbb{N}_{1,\tau}$, *there are nonsingular complex* $q \times q$ *matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *such that* $\mathbf{A}_j\mathbf{B}_j^{-1}$ *is a unitary matrix, so that* $\mathbf{A}_j\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_j$, *and that*

$$P_j = \mathbf{A}_jX_j + \mathbf{B}_jY_j^{[\alpha,j]} \quad and \quad R_j = Y_j\mathbf{A}_j + X_j^{[\alpha,j]}\mathbf{B}_j.$$

(b) *Let* (i) *be satisfied. If* $j \in \mathbb{N}_{1,\tau}$, *then the* $q \times q$ *matrices* $\big(P_j,B_{\alpha,0}^{(q)}\big)_{F,l}$, $\big(P_j,B_{\alpha,j}^{(q)}\big)_{F,l}$, $\big(B_{\alpha,0}^{(q)},R_j\big)_{F,r}$, *and* $\big(B_{\alpha,j}^{(q)},R_j\big)_{F,r}$ *are nonsingular, the matrices* $\mathbf{A}_j$ *and* $\mathbf{B}_j$ *from* (ii) *and the matrix* $\mathbf{C}_j$ *from* (I) *are uniquely determined, where* $\mathbf{C}_j = \mathbf{B}_j^{-*}\mathbf{A}_j$ *and* $\mathbf{C}_j = \mathbf{A}_j^{-*}\mathbf{B}_j$. *In particular,* $[(\mathbf{B}_j^{-1}P_j)_{j=1}^\tau, (R_j\mathbf{B}_j^{-1})_{j=1}^\tau]$ *is a canonical pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F. *Moreover, for some* $j \in \mathbb{N}_{1,\tau}$, *in the case of* $\mathbf{A}_j^*\mathbf{A}_j = \mathbf{A}_j\mathbf{A}_j^*$ *or* $\mathbf{B}_j^*\mathbf{B}_j = \mathbf{B}_j\mathbf{B}_j^*$ *the matrix* $\mathbf{C}_j$ *is unitary.*

(c) *If* $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ *is a given canonical pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F, *then there exists a pair of orthonormal systems* $[(X_k)_{k=0}^\tau,$ $(Y_k)_{k=0}^\tau]$ *corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* F *such that, for each* $j \in \mathbb{N}_{1,\tau}$, *conditions* (I)–(V) *of* (i) *are satisfied, where* $\mathbf{C}_j$ *is a unitary* $q \times q$ *matrix.*

Proof. Suppose that (i) holds. Let $j \in \mathbb{N}_{1,\tau}$. Because of (i) and Theorem 3.5 we already know that there exist complex $q \times q$ matrices $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{D}_j$, and $\mathbf{E}_j$, all not equal to the zero matrix, such that the identities

$$P_j = \mathbf{A}_jX_j + \mathbf{B}_jY_j^{[\alpha,j]} \quad and \quad R_j = Y_j\mathbf{D}_j + X_j^{[\alpha,j]}\mathbf{E}_j$$

are satisfied, where

$$\mathbf{A}_j = \big(P_j,B_{\alpha,j}^{(q)}\big)_{F,l}X_j^{[\alpha,j]}(\alpha_j), \quad \mathbf{B}_j = \big(P_j,B_{\alpha,0}^{(q)}\big)_{F,l}\big(Y_j^{[\alpha,j]}(\alpha_j)\big)^*$$

and

$$\mathbf{D}_j = Y_j^{[\alpha,j]}(\alpha_j)\big(B_{\alpha,j}^{(q)},R_j\big)_{F,r}, \quad \mathbf{E}_j = \big(X_j^{[\alpha,j]}(\alpha_j)\big)^*\big(B_{\alpha,0}^{(q)},R_j\big)_{F,r}.$$

In view of (III) and (IV) we get $\mathbf{B}_j = \mathbf{E}_j$ and $\mathbf{A}_j = \mathbf{D}_j$. Therefore, from (I) along with (2.8) and [30, Remarks 2.7 and 2.8] it follows that

$$\mathscr{O} = R_j - P_j^{[\alpha,j]}\mathbf{C}_j = Y_j\mathbf{A}_j + X_j^{[\alpha,j]}\mathbf{B}_j - X_j^{[\alpha,j]}\mathbf{A}_j^*\mathbf{C}_j - Y_j\mathbf{B}_j^*\mathbf{C}_j$$
$$= Y_j(\mathbf{A}_j - \mathbf{B}_j^*\mathbf{C}_j) + X_j^{[\alpha,j]}(\mathbf{B}_j - \mathbf{A}_j^*\mathbf{C}_j).$$

Taking Remark 3.10 into account, we obtain $\mathbf{A}_j = \mathbf{B}_j^*\mathbf{C}_j$ and $\mathbf{B}_j = \mathbf{A}_j^*\mathbf{C}_j$. Since $\mathbf{C}_j$ is a nonsingular matrix (see (I)) and since $X_j^{[\alpha,j]}(\alpha_j)$ and $Y_j^{[\alpha,j]}(\alpha_j)$ are nonsingular matrices as well (see, e.g., [30, Theorem 4.5]), from (II) and the representations for the matrices $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{D}_j$, and $\mathbf{E}_j$ one can find that the complex $q \times q$ matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ as well as $\big(P_j,B_{\alpha,0}^{(q)}\big)_{F,l}$, $\big(P_j,B_{\alpha,j}^{(q)}\big)_{F,l}$, $\big(B_{\alpha,0}^{(q)},R_j\big)_{F,r}$, and $\big(B_{\alpha,j}^{(q)},R_j\big)_{F,r}$ are all

nonsingular. In particular, we get $\mathbf{C}_j = \mathbf{B}_j^{-*}\mathbf{A}_j$ and $\mathbf{C}_j = \mathbf{A}_j^{-*}\mathbf{B}_j$. Furthermore, (V) leads to $\mathbf{A}_j\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_j$. This implies $\mathbf{A}_j\mathbf{B}_j^{-1} = \mathbf{B}_j^{-1}\mathbf{A}_j$ and consequently we obtain that

$$\mathbf{A}_j\mathbf{B}_j^{-1}(\mathbf{A}_j\mathbf{B}_j^{-1})^* = \mathbf{A}_j\mathbf{B}_j^{-1}(\mathbf{B}_j^{-1}\mathbf{A}_j)^* = \mathbf{A}_j(\mathbf{A}_j^{-*}\mathbf{B}_j)^{-1}\mathbf{B}_j^{-*}$$
$$= \mathbf{A}_j\mathbf{C}_j^{-1}\mathbf{B}_j^{-*} = \mathbf{A}_j(\mathbf{B}_j^{-*}\mathbf{A}_j)^{-1}\mathbf{B}_j^{-*} = \mathbf{I}_q,$$

i.e. that the matrix $\mathbf{A}_j\mathbf{B}_j^{-1}$ (resp., $\mathbf{B}_j^{-1}\mathbf{A}_j$) is unitary. In conclusion, we have proven that (i) implies (ii) and we see that $[(\mathbf{B}_j^{-1}P_j)_{j=1}^{\tau}, (R_j\mathbf{B}_j^{-1})_{j=1}^{\tau}]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F. In the special case that

$$\mathbf{A}_j^*\mathbf{A}_j = \mathbf{A}_j\mathbf{A}_j^* \quad \text{or} \quad \mathbf{B}_j^*\mathbf{B}_j = \mathbf{B}_j\mathbf{B}_j^*$$

for some $j \in \mathbb{N}_{1,\tau}$, then $\mathbf{A}_j^{-*}\mathbf{A}_j$ or $\mathbf{B}_j^{-*}\mathbf{B}_j$ is a unitary $q \times q$ matrix. By using

$$\mathbf{C}_j = \mathbf{A}_j^{-*}\mathbf{B}_j = \mathbf{A}_j^{-*}\mathbf{A}_j(\mathbf{B}_j^{-1}\mathbf{A}_j)^{-1} \quad \text{or} \quad \mathbf{C}_j = \mathbf{B}_j^{-*}\mathbf{A}_j = \mathbf{B}_j^{-*}\mathbf{B}_j\mathbf{B}_j^{-1}\mathbf{A}_j$$

and the fact that $\mathbf{B}_j^{-1}\mathbf{A}_j$ is a unitary $q \times q$ matrix, we find that the $q \times q$ matrix $\mathbf{C}_j$ is also unitary. Thus, part (b) is verified. Suppose now that (ii) holds. An application of Theorem 3.5 gives us that $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ is a pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F such that, for each $j \in \mathbb{N}_{1,\tau}$, the conditions (II)–(V) are satisfied. It remains to be shown that (I) holds as well. Let $j \in \mathbb{N}_{1,\tau}$. Because of $\mathbf{A}_j\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_j$ we have $\mathbf{A}_j\mathbf{B}_j^{-1} = \mathbf{B}_j^{-1}\mathbf{A}_j$. Hence, the unitarity of $\mathbf{A}_j\mathbf{B}_j^{-1}$ implies that

$$\mathbf{A}_j^*\mathbf{B}_j^{-*}\mathbf{A}_j\mathbf{B}_j^{-1} = (\mathbf{B}_j^{-1}\mathbf{A}_j)^*\mathbf{A}_j\mathbf{B}_j^{-1} = (\mathbf{A}_j\mathbf{B}_j^{-1})^*\mathbf{A}_j\mathbf{B}_j^{-1} = \mathbf{I}_q,$$

i.e. that $\mathbf{A}_j^{-*}\mathbf{B}_j = \mathbf{B}_j^{-*}\mathbf{A}_j$. Finally, if we set $\mathbf{C}_j := \mathbf{B}_j^{-*}\mathbf{A}_j$, then based on (2.8) and [30, Remarks 2.7 and 2.8] we get

$$R_j = Y_j\mathbf{A}_j + X_j^{[\alpha,j]}\mathbf{B}_j = Y_j\mathbf{B}_j^*\mathbf{B}_j^{-*}\mathbf{A}_j + X_j^{[\alpha,j]}\mathbf{A}_j^*\mathbf{A}_j^{-*}\mathbf{B}_j$$
$$= (Y_j\mathbf{B}_j^* + X_j^{[\alpha,j]}\mathbf{A}_j^*)\mathbf{B}_j^{-*}\mathbf{A}_j = P_j^{[\alpha,j]}\mathbf{C}_j.$$

Thus, we have shown that (i) follows from (ii). This completes the proof of (a). Part (c) is a simple consequence of Definition 6.1 and (a). $\square$

COROLLARY 6.8. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$ *and* $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Suppose that* $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ *is a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *and that* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *is a pair of para-orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F *such that, for each* $j \in \mathbb{N}_{1,\tau}$, *the conditions (I)–(V) of Proposition 6.7 are satisfied. Furthermore, let* $j \in \mathbb{N}_{1,\tau}$. *Then:*

(a) *The equalities* $\det P_j = c_j \det R_j^{[\alpha,j]}$ *and* $c_j \det P_j^{[\alpha,j]} = \det R_j$ *are satisfied for some* $c_j \in \mathbb{C} \setminus \{0\}$. *Moreover,* $\mathcal{N}(P_j(v)) = \mathcal{N}(R_j^{[\alpha,j]}(v))$ *holds for each* $v \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ *and* $\mathcal{N}(P_j(z)) = \mathcal{N}((R_j(z))^*)$ *for each* $z \in \mathbb{T}$.

(b) *There is a $\breve{u}_j \in \mathbb{T}$ such that $\breve{u}_j \det P_j = \det R_j$ and $\det P_j^{[\alpha,j]} = \breve{u}_j \det R_j^{[\alpha,j]}$.*

(c) *There exist at most j pairwise different points $w_1, w_2, \ldots, w_j \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ such that $P_j(w_s) = 0$ (resp., $R_j(w_s) = 0$) holds for each $s \in \mathbb{N}_{1,j}$ and there exist at most jq pairwise different points $z_1, z_2, \ldots, z_{jq} \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$ such that the complex $q \times q$ matrix $P_j(z_s)$ (resp., $R_j(z_s)$) is singular for each $s \in \mathbb{N}_{1,jq}$.*

(d) *If one of the values $P_j(z)$, $R_j(z)$, $P_j^{[\alpha,j]}(z)$, or $R_j^{[\alpha,j]}(z)$ is a singular matrix for some $z \in \mathbb{C}_0 \setminus \mathbb{P}_{\alpha,j}$, then all of them are singular and $z \in \mathbb{T}$.*

(e) *Each of the functions P_j, R_j, $P_j^{[\alpha,j]}$, and $R_j^{[\alpha,j]}$ belongs to $\mathscr{R}_{\alpha,j}^{q \times q} \setminus \mathscr{R}_{\alpha,j-1}^{q \times q}$.*

(f) *If Z_0 is a constant function on $\mathbb{C}_0$ with a nonsingular complex $q \times q$ matrix as value and if $Z_n \in \{P_n, R_n, P_n^{[\alpha,n]}, R_n^{[\alpha,n]}\}$ for $n \in \mathbb{N}_{1,j}$, then $Z_0, Z_1, \ldots, Z_j$ is a basis of the right $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,j}^{q \times q}$ u and of the left $\mathbb{C}^{q \times q}$-module $\mathscr{R}_{\alpha,j}^{q \times q}$.*

Proof. Use Proposition 6.7 along with Theorem 6.5. $\square$

The considerations below following up on Corollaries 3.6, 3.7, 3.8, and 3.9. Somewhat different from earlier, the result corresponding to Corollary 3.6 for the para-orthogonal systems introduced in Definition 6.1 leads to a kind of characterization for such para-orthogonal systems. We first present a common result on linear fractional matrix transformations.

LEMMA 6.9. *Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be complex $q \times q$ matrices such that, by setting*

$$\mathbf{A} := \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \quad \text{and} \quad \mathbf{j}_{qq} := \begin{pmatrix} \mathbf{I}_q & 0 \\ 0 & -\mathbf{I}_q \end{pmatrix},$$

the equality $\mathbf{A}^* \mathbf{j}_{qq} \mathbf{A} = s \mathbf{j}_{qq}$ is satisfied, where $s = 1$ or $s = -1$.

(a) *If $\mathbf{U}$ is a unitary $q \times q$ matrix, then $\det(\mathbf{Ub} + \mathbf{d}) \neq 0$ and $(\mathbf{Ub} + \mathbf{d})^{-1}(\mathbf{Ua} + \mathbf{c})$ is a unitary $q \times q$ matrix. Moreover, if $\tilde{\mathbf{U}}$ is a unitary $q \times q$ matrix, then there exists a unitary $q \times q$ matrix $\mathbf{U}$ such that $\tilde{\mathbf{U}} = (\mathbf{Ub} + \mathbf{d})^{-1}(\mathbf{Ua} + \mathbf{c})$.*

(b) *If $\mathbf{U}$ is a unitary $q \times q$ matrix, then $\det(\mathbf{cU} + \mathbf{d}) \neq 0$ and $(\mathbf{aU} + \mathbf{b})(\mathbf{cU} + \mathbf{d})^{-1}$ is a unitary $q \times q$ matrix. Moreover, if $\tilde{\mathbf{U}}$ is a unitary $q \times q$ matrix, then there exists a unitary $q \times q$ matrix $\mathbf{U}$ such that $\tilde{\mathbf{U}} = (\mathbf{aU} + \mathbf{b})(\mathbf{cU} + \mathbf{d})^{-1}$.*

(c) *Let $z \in \mathbb{T}$ and let*

$$\mathbf{B} := z \begin{pmatrix} \mathbf{d}^* & \mathbf{b}^* \\ \mathbf{c}^* & \mathbf{a}^* \end{pmatrix}.$$

Then $\mathbf{B}^* \mathbf{j}_{qq} \mathbf{B} = s \mathbf{j}_{qq}$ and, for a unitary $q \times q$ matrix $\mathbf{U}$, the matrix $\mathbf{c}^* \mathbf{U} + \mathbf{a}^*$ is nonsingular and $(\mathbf{Ub} + \mathbf{d})^{-1}(\mathbf{Ua} + \mathbf{c}) = (\mathbf{d}^* \mathbf{U} + \mathbf{b}^*)(\mathbf{c}^* \mathbf{U} + \mathbf{a}^*)^{-1}$ holds.

Proof. Using standard methods for linear fractional matrix transformations, the proof is straightforward (cf. [2, Section 2.9] and [27, Section 1.6]). $\square$

PROPOSITION 6.10. *Let* $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. *Let* $\tau \in \mathbb{N}$ *or* $\tau = \infty$ *and let* $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. *Suppose that* $[(X_k)_{k=0}^{\tau}, (Y_k)_{k=0}^{\tau}]$ *is a pair of orthonormal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F.

(a) *Let* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *be a canonical pair of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F. *Furthermore, let* $j \in \mathbb{N}_{1,\tau}$. *Then there are a unitary* $q \times q$ *matrix* $\tilde{\mathbf{U}}_j$ *and some nonsingular complex* $q \times q$ *matrices* $\tilde{\mathbf{C}}_j$ *and* $\tilde{\mathbf{D}}_j$ *such that*

$$P_j(v) = \frac{1 - \overline{\alpha_{j-1}}v}{1 - \overline{\alpha_j}v} \tilde{\mathbf{C}}_j \big(b_{\alpha_{j-1}}(v)\tilde{\mathbf{U}}_j X_{j-1}(v) + Y_{j-1}^{[\alpha,j-1]}(v)\big),$$

$$R_j(v) = \frac{1 - \overline{\alpha_{j-1}}v}{1 - \overline{\alpha_j}v} \big(b_{\alpha_{j-1}}(v)Y_{j-1}(v)\tilde{\mathbf{U}}_j + X_{j-1}^{[\alpha,j-1]}(v)\big)\tilde{\mathbf{D}}_j \qquad (6.3)$$

hold for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, *where* $\tilde{\mathbf{U}}_j$, $\tilde{\mathbf{C}}_j$, *and* $\tilde{\mathbf{D}}_j$ *are uniquely determined.*

(b) *For each* $j \in \mathbb{N}_{1,\tau}$, *let* $\tilde{\mathbf{U}}_j$ *be a unitary* $q \times q$ *matrix. Then there exists a canonical pair* $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ *of para-orthogonal systems corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* F, *such that the identities in* (6.3) *hold for each* $j \in \mathbb{N}_{1,\tau}$ *and each point* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$ *with some nonsingular complex* $q \times q$ *matrices* $\tilde{\mathbf{C}}_j$ *and* $\tilde{\mathbf{D}}_j$.

Proof. Based on Lemma 6.9, [30, Remark 2.8 and Proposition 3.7], and [31, Remark 3.5] a similar approach to the one used for Corollary 3.6 yields the assertion. $\square$

REMARK 6.11. Let $(\alpha_j)_{j=1}^{\infty} \in \mathscr{T}_1$. Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $F \in \mathscr{M}_{\geqslant}^{q,\tau}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

(a) Let $(P_j)_{j=1}^{\tau}$ and $(R_j)_{j=1}^{\tau}$ be sequences of complex $q \times q$ matrix-valued functions. By using [30, Corollary 4.4 and Theorem 4.5] as in the proof of Corollary 3.7, one can see that the following statements are equivalent:

(i) $[(P_j)_{j=1}^{\tau}, (R_j)_{j=1}^{\tau}]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^{\infty}$ and F.

(ii) For $j \in \mathbb{N}_{1,\tau}$, there are unitary $q \times q$ matrices $\check{\mathbf{W}}_j$, $\check{\mathbf{V}}_j$, and $\check{\mathbf{U}}_j$ such that

$$P_j = \check{\mathbf{W}}_j \Big(\check{\mathbf{U}}_j \sqrt{A_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}^{-1} \big(A_{j,\alpha_j}^{(\alpha,F)}\big)^{[\alpha,j]} + \sqrt{C_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}^{-1} C_{j,\alpha_j}^{(\alpha,F)}\Big),$$

$$R_j = \Big(\big(C_{j,\alpha_j}^{(\alpha,F)}\big)^{[\alpha,j]} \sqrt{C_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}^{-1} \check{\mathbf{U}}_j + A_{j,\alpha_j}^{(\alpha,F)} \sqrt{A_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}^{-1}\Big)\check{\mathbf{V}}_j.$$

Moreover, if (i) holds, then the $q \times q$ matrices $\check{\mathbf{W}}_j$, $\check{\mathbf{V}}_j$, and $\check{\mathbf{U}}_j$ in (ii) are uniquely determined for each $j \in \mathbb{N}_{1,\tau}$, where $\check{\mathbf{W}}_j = \big(P_j, B_{\alpha,0}^{(q)}\big)_{F,l}\sqrt{C_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}$,

$$\check{\mathbf{V}}_j = \sqrt{A_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}$$

$\big(B_{\alpha,0}^{(q)}, R_j\big)_{F,r}$, $\check{\mathbf{U}}_j = \check{\mathbf{W}}_j^*\big(P_j, B_{\alpha,j}^{(q)}\big)_{F,l}\sqrt{A_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}$, and $\check{\mathbf{U}}_j = \sqrt{C_{j,\alpha_j}^{(\alpha,F)}(\alpha_j)}$ $\big(B_{\alpha,j}^{(q)}, R_j\big)_{F,r}\check{\mathbf{V}}_j^*$.

(b) Suppose that $[(X_k)_{k=0}^\tau, (Y_k)_{k=0}^\tau]$ is a pair of orthonormal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F. Furthermore, for each $j \in \mathbb{N}_{1,\tau}$, let $z_j \in \mathbb{T}$ and let

$$P_j := \left(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\right)\left(Y_j^{[\alpha,j]}(z_j)\right)^{-*} C_{j-1,z_j}^{(\alpha,F)},$$

$$R_j := \left(1 - b_{\alpha_j} \overline{b_{\alpha_j}(z_j)}\right) A_{j-1,z_j}^{(\alpha,F)} \left(X_j^{[\alpha,j]}(z_j)\right)^{-*}.$$

Because of Remark 6.3 and [30, Remark 2.6, Corollary 4.7, Lemma 5.1, and Theorem 5.4] one can see that $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and F (cf. Corollary 3.8).

REMARK 6.12. Let $(\alpha_j)_{j=1}^\infty \in \mathscr{T}_1$. Let $n \in \mathbb{N}_0$ and suppose that $F \in \mathscr{M}_{\geqslant}^{q,n}(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. Let $w \in \mathbb{D} \setminus \mathbb{P}_{\alpha,n}$ and let $F_{n,w}^{(\alpha)}$ be defined as in Corollary 3.9. Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $(P_j)_{j=1}^\tau$ and $(R_j)_{j=1}^\tau$ be sequences of rational $q \times q$ matrix functions. A similar argument to the one used for Corollary 3.9 shows that the following statements are equivalent:

(i) $[(P_j)_{j=1}^\tau, (R_j)_{j=1}^\tau]$ is a canonical pair of para-orthogonal systems corresponding to $(\alpha_j)_{j=1}^\infty$ and $F_{n,w}^{(\alpha)}$.

(ii) For each $j \in \mathbb{N}_{1,\tau}$, with some pair $[(X_k)_{k=0}^n, (Y_k)_{k=0}^n]$ of orthonormal systems corresponding to $(\alpha_s)_{s=1}^\infty$ and F there is a unitary $q \times q$ matrix $\mathbf{U}_j$ such that

$$P_j = \mathbf{U}_j X_j + Y_j^{[\alpha,j]} \quad \text{and} \quad R_j = Y_j \mathbf{U}_j + X_j^{[\alpha,j]}, \quad j \leqslant n,$$

and, if $j > n$, then there are unitary $q \times q$ matrices $\check{\mathbf{W}}_j$, $\check{\mathbf{V}}_j$, and $\check{\mathbf{U}}_j$ such that

$$P_j = h_j \check{\mathbf{W}}_j \left(b_w \widetilde{b}_{n,j-1}^{(\alpha)} \check{\mathbf{U}}_j \sqrt{A_{n,w}^{(\alpha,F)}(w)}^{-1} \left(A_{n,w}^{(\alpha,F)}\right)^{[\alpha,n]} + b_{n,j-1}^{(\alpha)} \sqrt{C_{n,w}^{(\alpha,F)}(w)}^{-1} C_{n,w}^{(\alpha,F)}\right),$$

$$R_j = h_j \left(b_w \widetilde{b}_{n,j-1}^{(\alpha)} (C_{n,w}^{(\alpha,F)})^{[\alpha,n]} \sqrt{C_{n,w}^{(\alpha,F)}(w)}^{-1} \check{\mathbf{U}}_j + b_{n,j-1}^{(\alpha)} A_{n,w}^{(\alpha,F)} \sqrt{A_{n,w}^{(\alpha,F)}(w)}^{-1}\right) \check{\mathbf{V}}_j,$$

where h_j is the function given by $h_j(v) := \sqrt{\frac{|1-|\alpha_j|^2|}{1-|w|^2} \frac{1-\overline{w}v}{1-\overline{\alpha_j}v}}$ for $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$.

If (i) holds, then $\mathbf{U}_j, \check{\mathbf{W}}_j, \check{\mathbf{V}}_j$, and $\check{\mathbf{U}}_j$ in (ii) are uniquely determined for $j \in \mathbb{N}_{1,\tau}$.

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