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# RIGHT INVERTIBLE MULTIPLICATION OPERATORS AND STABLE RATIONAL MATRIX SOLUTIONS TO AN ASSOCIATE BEZOUT EQUATION, II: DESCRIPTION OF ALL SOLUTIONS

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*Abstract.* This paper presents a state space description of the set of all solutions to a rational corona type Bezout equation, starting from a stable state space representation of the given coefficient matrix. In other words, we describe the null space of an analytic Toeplitz operator with a rational symbol, in terms of the matrices occuring in a realization of that symbol, assuming the operator involved is right invertible. A state space version of the related Tolokonnikov lemma is also included.

## 1. Introduction

This paper is a continuation of [3]. Throughout $G$ is a stable rational $m \times p$ matrix function, that is, $G$ has all its poles in $|z| > 1$, infinity included. In general, $p$ will be larger than $m$. As in [3], we deal with the corona type Bezout equation

$$G(z)X(z) = I_m, \quad z \in \mathbb{D}, \tag{1.1}$$

and with the operator $M_G$ of multiplication by $G$ mapping the Hardy space $H^2(\mathbb{C}^p)$ into the Hardy space $H^2(\mathbb{C}^m)$. Assuming $M_G$ to be right invertible we shall describe the null space of $M_G$. Together with the main result from [3] this yields a full description of the set of all stable rational matrix solutions to (1.1). In addition we discuss the relation to Tolokonnikov's lemma [10], see also the appendix of [8].

Our starting point is a *stable state space representation* of $G$. The latter means that $G$ is represented in the following form:

$$G(z) = D + zC(I_n - zA)^{-1}B. \tag{1.2}$$

Here $A, B, C, D$ are matrices of appropriate sizes, $I_n$ is an identity matrix of order $n$, and the $n \times n$ matrix $A$ is *stable*, that is, $A$ has all its eigenvalues in the open unit disc

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$\mathbb{D}$. The smallest $n$ for which $G$ has a stable state space representation of the form (1.2) is called the *McMillan degree* of $G$; this quantity is denoted by $\delta(G)$.

In order to state the main results in more detail we first briefly recall the main theorem from [3]. There the least squares solution to (1.1) was constructed. Consider the discrete algebraic Riccati equation

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA). \tag{1.3}$$

Here $R_0$ and $\Gamma$ are the matrices of sizes $m\times m$ and $n\times m$, respectively, given by

$$R_0 = DD^* + CPC^*, \qquad \Gamma = BD^* + APC^*. \tag{1.4}$$

Furthermore, the $n\times n$ matrix $P$ appearing in the definitions of $R_0$ and $\Gamma$ is the unique solution of the symmetric Stein equation

$$P - APA^* = BB^*. \tag{1.5}$$

An $n\times n$ matrix $Q$ will be called a *stabilizing solution* of (1.3) if the following holds:

(a) $R_0 - \Gamma^*Q\Gamma$ is positive definite,

(b) $Q$ satisfies the Riccati equation (1.3),

(c) the matrix $A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable.

The stabilizing solution, assuming it exists, is unique. The main theorem of [3] tells us that equation (1.1) has a stable rational matrix solution if and only if

(i) the discrete algebraic Riccati equation (1.3) has a (unique) stabilizing solution $Q$,

(ii) the matrix $I_n - PQ$ is non-singular.

Moreover, (i) and (ii) are equivalent to $M_G$ being right invertible. Furthermore, if (i) and (ii) hold, then a particular stable rational matrix solution $X$ of (1.1) is given by

$$X(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)D_1, \tag{1.6}$$

where

$$A_0 = A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA), \tag{1.7}$$

$$C_1 = D^*C_0 + B^*QA_0, \tag{1.8}$$

$$\text{with } C_0 = (R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA), \tag{1.9}$$

$$D_1 = (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*.$$

This solution $X$ is the *least squares solution* of (1.1), that is, for any other stable rational matrix solution $V$ of $G(z)V(z) = I_m$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{it})^*X(e^{it})\,dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} V(e^{it})^*V(e^{it})\,dt, \tag{1.10}$$

and equality holds in (1.10) if and only if $V = X$.

In the present paper we shall be dealing with the set of all solutions of (1.1). The first main theorem of the present paper is the following result.

THEOREM 1.1. *Let $G$ be given by (1.2) with $A$ stable, and let $P$ be the unique solution of the Stein equation (1.5). Assume that $M_G$ is right invertible, or equivalently, assume that the Riccati equation (1.3) has a stabilizing solution $Q$ such that the matrix $I_n - PQ$ is non-singular. Then the set of all stable $p \times m$ rational matrix solutions $V$ of equation (1.1) is equal to the set of all functions $V(z) = X(z) + \hat{\Theta}(z)N(z)$, where $X$ is the least square solution given by (1.6), the parameter $N$ is an arbitrary stable $(p-m) \times m$ rational matrix function, and $\hat{\Theta}$ is given by*

$$\hat{\Theta}(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)\hat{D}. \tag{1.11}$$

*Here $A_0$ and $C_1$ are given by (1.7) and (1.8), respectively, and $\hat{D}$ is any one-to-one $p \times (p-m)$ matrix such that*

$$\hat{D}\hat{D}^* = I_p - (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + \\ - B^*QB - C_1(I_n - PQ)^{-1}PC_1^*. \tag{1.12}$$

*The matrix $\hat{D}$ is uniquely determined up to a constant unitary matrix of order $p-m$ on the right. Furthermore, $\hat{\Theta}$ is inner, the McMillan degree of $\hat{\Theta}$ is less than or equal to the McMillan degree of $G$, and*

$$\operatorname{Ker} M_G = M_{\hat{\Theta}} H^2(\mathbb{C}^{p-m}). \tag{1.13}$$

A priori it is not clear that the right hand side of (1.12) is positive semidefinite but we shall prove that this is always the case under the conditions of the theorem.

We shall also derive the analogue of the main result in [3] for the equation

$$G(z)Y(z) = F(z), \quad z \in \mathbb{D}, \tag{1.14}$$

where the right hand side $F$ is a given stable rational matrix function of size $m \times k$ for some $k$. Let $Y$ be the function given by

$$Y(\cdot)u = M_G^*(M_G M_G^*)^{-1}Fu, \quad u \in \mathbb{C}^k. \tag{1.15}$$

It will be proved that $Y$ is a stable rational $p \times k$ matrix function satisfying (1.14). Furthermore, in terms of the given stable state space representation of $G$ and a stable state space representation of $F$, we shall derive a stable state space representation for $Y$; see Theorem 3.1. A special case of the latter result, with $F = G$, will serve as an intermediate step in proving Theorem 1.1. The function $Y$ defined by (1.15) has a minimality property analogous to (1.10) for $X$.

As a by-product our formulas will show that the McMillan degree of the solution $Y$ in (1.15) is less than or equal to the sum of the McMillan degrees of $G$ and $F$. This result will also be proved directly without using state space formulas (see the two paragraphs directly after the proof of Theorem 3.1).

The paper consists of five sections, the first being the present introduction. In Section 2 we derive the operator theory results on which Theorem 1.1 is based. The main theorem for equation (1.14) is presented and proved in Section 3. These results are then used in Section 4 to prove Theorem 1.1. In Section 5 we discuss the connection with the related Tolokonnikov lemma [10]; see also [8, Appendix 3].

We conclude this introduction with a few words about notation. Let $\Phi$ be any $m\times p$ matrix-valued function of which the entries are essentially bounded on the unit circle $\mathbb{T}$, and let $\ldots,\Phi_{-1},\Phi_0,\Phi_1,\ldots$ be the $m\times p$ matrix Fourier coefficients of $\Phi$. Recall (see, e.g., Section XXIII.2 in [5]) that the block Laurent operator defined by $\Phi$ is the operator $L_\Phi$ given by

$$L_\Phi = \begin{bmatrix} \ddots & & & & \\ & \Phi_0 & \Phi_{-1} & \Phi_{-2} & \\ & \Phi_1 & \boxed{\Phi_0} & \Phi_{-1} & \\ & \Phi_2 & \Phi_1 & \Phi_0 & \\ & & & & \ddots \end{bmatrix} : \ell^2(\mathbb{C}^p) \to \ell^2(\mathbb{C}^m). \tag{1.16}$$

Here $\boxed{\Phi_0}$ denotes the entry in the $(0,0)$ position. In what follows we identify $\ell^2(\mathbb{C}^p)$ and $\ell^2(\mathbb{C}^m)$ in the canonical way with Hilbert space direct sums $\ell^2_+(\mathbb{C}^p)\oplus\ell^2_+(\mathbb{C}^p)$ and $\ell^2_+(\mathbb{C}^m)\oplus\ell^2_+(\mathbb{C}^m)$, respectively. This allows us to rewrite $L_\Phi$ as a $2\times 2$ operator matrix, namely

$$L_\Phi = \begin{bmatrix} T_{\Phi^\#} & H_{\Phi^\#} \\ H_\Phi & T_\Phi \end{bmatrix} : \begin{bmatrix} \ell^2_+(\mathbb{C}^p) \\ \ell^2_+(\mathbb{C}^p) \end{bmatrix} \to \begin{bmatrix} \ell^2_+(\mathbb{C}^m) \\ \ell^2_+(\mathbb{C}^m) \end{bmatrix}.$$

Here $T_\Phi$ and $H_\Phi$ are, respectively, the block Toeplitz operator and block Hankel operator defined by $\Phi$, that is,

$$T_\Phi = \begin{bmatrix} \Phi_0 & \Phi_{-1} & \Phi_{-2} & \cdots \\ \Phi_1 & \Phi_0 & \Phi_{-1} & \cdots \\ \Phi_2 & \Phi_1 & \Phi_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad H_\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \cdots \\ \Phi_2 & \Phi_3 & \Phi_4 & \cdots \\ \Phi_3 & \Phi_4 & \Phi_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which both act from $\ell^2_+(\mathbb{C}^p)$ into $\ell^2_+(\mathbb{C}^m)$. The operators $T_{\Phi^\#}$ and $H_{\Phi^\#}$ are, respectively, the block Toeplitz operator and block Hankel operator defined by $\Phi^\#$, where $\Phi^\#(e^{it}) = \Phi(e^{-it})$. When $\Phi$ is a stable rational matrix function, then $H_{\Phi^\#}$ is a zero operator, and $H_\Phi$ is a finite rank operator of which the rank is equal to the McMillan degree of $\Phi$.

In the sequel we shall often work with the Toeplitz operator $T_G$ in place of the operator $M_G$. Note that $\mathscr{F}_{\mathbb{C}^m} T_G = M_G \mathscr{F}_{\mathbb{C}^p}$. Here for any positive integer $k$ the operator $\mathscr{F}_{\mathbb{C}^k}$ is the Fourier transform mapping $\ell^2_+(\mathbb{C}^k)$ onto the Hardy space $H^2(\mathbb{C}^k)$ which is a unitary operator.

## 2. The null space of $M_G$

Let $G$ be a stable $m\times p$ rational matrix function. Since $M_G$ is an operator of multiplication, its null space is invariant under the operator of multiplication by the independent variable on $H^2(\mathbb{C}^p)$. According to the Beurling-Lax theorem (see, e.g., Theorem A in Section 5.5 of [9] or Theorem 3.1 in Section XXVI.3 of [5]) this implies that there exists an inner $p\times k$ matrix-valued $H^\infty$ function $\Theta$, unique up to a constant unitary $k\times k$ matrix on the right, such that

$$\operatorname{Ker} M_G = M_\Theta H^2(\mathbb{C}^k). \tag{2.1}$$

Recall that a $p\times k$ matrix-valued $H^\infty$ function $\Theta$ is called *inner* whenever $\Theta(e^{it})^*\Theta(e^{it}) = I_k$ for almost all $0\leqslant t\leqslant 2\pi$. In that case $M_\Theta$ is an isometry, that is, $M_\Theta^* M_\Theta$ is the identity operator on $H^2(\mathbb{C}^k)$.

The following theorem provides a method to construct $\Theta$ for the case when $M_G$ is right invertible.

THEOREM 2.1. *Let $G$ be a stable $m\times p$ rational matrix function, and assume that $M_G$ is right invertible. Then $\Theta$ in (2.1) is a stable $p\times k$ rational matrix function, the McMillan degree of $\Theta$ is less than or equal to the McMillan degree of $G$, the integer $k$ in (2.1) is equal to $p-m$, and $\Theta(0)$ is one-to-one. Furthermore, $\Theta$ is given by*

$$\Theta(\cdot)\Theta(0)^*u = u - M_G^*(M_G M_G^*)^{-1}G(\cdot)u, \quad u\in\mathbb{C}^p. \tag{2.2}$$

Note that our conditions on $G$ imply that equation (1.1) has a stable rational matrix solution. It then follows from the general $H^\infty$ theory (see, e.g., the proof of the Sublemma on page 53 of [8]) that the integer $k$ in (2.1) is equal to $p-m$ and that $\Theta(0)$ is one-to-one. We will return to this remark in the final section. In what follows we will give an alternative proof.

We begin with the definition of co-outer. Let $F$ be any $p\times k$ matrix-valued $H^\infty$ function, and let $F^\circ$ be the function defined by $F^\circ(z) = F(\bar{z})^*$ for $|z|<1$. Note that $F^\circ$ is again an $H^\infty$ function, $\|F\|_\infty = \|F^\circ\|_\infty$, and

$$T_{F^\circ} = \begin{bmatrix} F_0^* & & & \\ F_1^* & F_0^* & & \\ F_2^* & F_1^* & F_0^* & \\ \vdots & & & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p)\to\ell^2_+(\mathbb{C}^k).$$

Here $F_0, F_1, F_2, \ldots$ are the Taylor coefficients of $F$ at zero. The function $F$ is called *co-outer* whenever $F^\circ$ is outer, that is, whenever the range of $T_{F^\circ}$ is dense in $\ell^2_+(\mathbb{C}^k)$.

LEMMA 2.2. *Let $G$ be a stable rational $m\times p$ matrix function, and let $\Theta$ be an $m\times k$ matrix-valued inner function such that (2.1) holds. Then $\Theta$ is co-outer, and hence $\Theta(0)$ is one-to-one. If, in addition, $G(0)$ is surjective, then $k=p-m$.*

*Proof.* Let $\Theta^\circ = \Phi_{in}\Phi_{out}$ be an inner-outer factorization of $\Theta^\circ$, and let $k\times\ell$ and $\ell\times p$ be the sizes $\Phi_{in}$ and $\Phi_{out}$, respectively. Since $\Phi_{in}$ is inner, $\Phi_{in}(e^{it})$ is an isometry for almost all $0\leqslant t\leqslant 2\pi$. In particular, $\ell\leqslant k$. We shall see that $\ell = k$. The fact that $T_{\Phi_{in}}$ is an isometry implies that $T_{\Phi_{out}}$ is a contraction. Indeed,

$$\|T_{\Phi_{out}}\| = \|T_{\Phi_{in}}^* T_{\Theta^\circ}\| \leqslant \|T_{\Theta^\circ}\| = \|T_\Theta\| = 1.$$

Next, take $h \in \ell_+^2(\mathbb{C}^k)$. Using that $T_\Theta$ is an isometry, we have

$$\|h\| = \|T_\Theta h\| = \|T_{\Phi_{out}^\circ} T_{\Phi_{in}^\circ} h\| \leqslant \|T_{\Phi_{out}^\circ}\| \|T_{\Phi_{in}^\circ} h\| \leqslant \|T_{\Phi_{in}^\circ} h\| \leqslant \|h\|.$$

Thus $\|T_{\Phi_{in}^\circ} h\| = \|h\|$ for each $h \in \ell_+^2(\mathbb{C}^k)$. Hence $\Phi_{in}^\circ$ is inner. Recall that $\Phi_{in}^\circ(e^{it}) = \Phi_{in}(e^{-it})^*$ for almost all $0 \leqslant t \leqslant 2\pi$. It follows that $\Phi_{in}(e^{it})$ is unitary for almost all $0 \leqslant t \leqslant 2\pi$. Since $\Phi_{in}(e^{it})$ has size $k \times \ell$, this can only happen when $\ell \geqslant k$. Thus $\ell = k$.

Notice that $\Theta = \Phi_{out}^\circ \Phi_{in}^\circ$. Since the matrix $\Phi_{in}(e^{it})$ is unitary for almost all $0 \leqslant t \leqslant 2\pi$ and $\Theta$ is an inner function, it follows that $\Phi_{out}^\circ = \Theta \Phi_{in}^{\circ *}$ is an inner function.

Using $G(z)\Theta(z) = 0$ with $\Theta \Phi_{in}^{\circ *} = \Phi_{out}^\circ$, we see that $G\Phi_{out}^\circ = G\Theta\Phi_{in}^{\circ *} = 0$. Hence $\Phi_{out}^\circ H^2(\mathbb{C}^k) \subseteq \operatorname{Ker} M_G$. This implies that

$$\operatorname{Ker} M_G = \Theta H^2(\mathbb{C}^k) = \Phi_{out}^\circ \Phi_{in}^\circ H^2(\mathbb{C}^k) \subseteq \Phi_{out}^\circ H^2(\mathbb{C}^k) \subseteq \operatorname{Ker} M_G.$$

Therefore $\Theta H^2(\mathbb{C}^k) = \Phi_{out}^\circ H^2(\mathbb{C}^k)$. According to the Beurling-Lax-Halmos theorem, $\Theta$ equals $\Phi_{out}^\circ$ up to a unitary constant on the right. Since $\Theta = \Phi_{out}^\circ \Phi_{in}^\circ$, we see that $\Phi_{in}$ is a unitary constant matrix. It follows that $\Theta^\circ$ is outer, and hence $\Theta$ is co-outer. The latter also implies that $\Theta(0)^*$ is surjective.

Next, assume additionally that $G(0)$ is surjective. The identity (2.1) tells us that $G(z)\Theta(z) = 0$, and hence $G(0)\Theta(0) = 0$. But we already know that the matrix $\Theta(0)$ is one-to-one. This yields

$$k = \operatorname{rank} \Theta(0) \leqslant \dim \operatorname{Ker} G(0) = p - m.$$

Thus $k \leqslant p - m$.

It remains to show that $k \geqslant p - m$. To do this let us partition $G(z)$ as

$$G(z) = \begin{bmatrix} G_1(z) & G_2(z) \end{bmatrix} : \begin{bmatrix} \mathbb{C}^m \\ \mathbb{C}^{p-m} \end{bmatrix} \to \mathbb{C}^m.$$

By reordering the columns of $G(z)$ it is clear that without loss of generality we may assume that $G_1(0)$ is non-singular. Put $g(z) = \det G_1(z)$. Then both $g(z)$ and $g(z)G_1(z)^{-1}$ are stable rational functions. Now consider

$$V(z) = \begin{bmatrix} g(z)G_1(z)^{-1}G_2(z) \\ -g(z)I_{p-m} \end{bmatrix} : \mathbb{C}^{p-m} \to \mathbb{C}^p.$$

Then $V$ is a stable rational $p \times (p-m)$ matrix function. Note that $g(0) = \det G_1(0) \neq 0$. It follows that $\operatorname{rank} V(0) = p - m$. From the definition of $V$ we see that $G(z)V(z)$ is identically zero, that is, $M_G V(\cdot)y = 0$ for each $y$ in $\mathbb{C}^{p-m}$. Using (2.1) it follows that $V(z) = \Theta(z)U(z)$, where $U(z)u$ belongs to $H^2(\mathbb{C}^k)$ for each $u$ in $\mathbb{C}^{p-m}$. In particular, $V(0) = \Theta(0)U(0)$. Since $\Theta(0)$ is one-to-one, we get

$$k = \operatorname{rank} \Theta(0) \geqslant \operatorname{rank} V(0) = p - m.$$

Thus $k \geqslant p - m$, as desired. $\square$

*Proof of Theorem* 2.1. Note that our conditions imply that (1.1) has a stable rational matrix solution. Thus $\operatorname{rank} G(z) = m$ for $|z| \leqslant 1$. In particular, $G(0)$ is surjective. Hence, $k = p - m$ and $\Theta(0)$ is one to one by Lemma 2.2.

Let us derive formula (2.2). Put $P_\Theta = M_\Theta M_\Theta^*$. Since $M_\Theta$ is an isometry, we see from (2.1) that the operator $P_\Theta$ is the orthogonal projection of $H^2(\mathbb{C}^p)$ onto $\operatorname{Ker} M_G$. On the other hand, as $M_G$ is right invertible, this orthogonal projection is also given by $I_{H^2(\mathbb{C}^p)} - M_G^*(M_G M_G^*)^{-1} M_G$, and thus

$$M_\Theta M_\Theta^* = I_{H^2(\mathbb{C}^p)} - M_G^*(M_G M_G^*)^{-1} M_G. \tag{2.3}$$

Let $\tau$ be the canonical embedding from $\mathbb{C}^p$ into $H^2(\mathbb{C}^p)$, that is, $(\tau u)(z) = u$ for each $z \in \mathbb{D}$ and each $u \in \mathbb{C}^p$. Note that $M_\Theta^* \tau = \tau \Theta(0)^*$, and for each $u \in \mathbb{C}^p$ the functions $M_\Theta \tau u$ and $M_G \tau u$ are equal to $\Theta(\cdot)u$ and $G(\cdot)u$, respectively. Thus

$$\begin{aligned} M_\Theta M_\Theta^* \tau u &= \Theta(\cdot)\Theta(0)^* u, \\ M_G^*(M_G M_G^*)^{-1} M_G \tau u &= M_G^*(M_G M_G^*)^{-1} G(\cdot) u. \end{aligned}$$

Using these two identities in (2.3) we see that (2.2) holds.

Next we show that $\Theta$ is a stable rational matrix function. To do this we note that the final part of the proof of Proposition 2.1 in [3] shows that $(M_G M_G^*)^{-1}$ maps rational $H^2$ functions into rational $H^2$. Thus for each $u \in \mathbb{C}^{p-m}$ the function $(M_G M_G^*)^{-1} G(\cdot) u$ is a rational $H^2$ function. But $M_G^*$ also maps rational $H^2$ functions into rational $H^2$. Since a rational $H^2$ function is stable, we conclude that $(M_G M_G^*)^{-1} G(\cdot)$ is a stable rational matrix function, and then (2.2) shows that the same holds true for $\Theta(\cdot)\Theta(0)^*$. Finally, as

$$\Theta(0)^* \Big( \Theta(0) \big( \Theta(0)^* \Theta(0) \big)^{-1} \Big) = I_k, \tag{2.4}$$

we see that $\Theta(\cdot) = \Theta(\cdot)\Theta(0)^* \Big( \Theta(0) \big( \Theta(0)^* \Theta(0) \big)^{-1} \Big)$, and hence $\Theta(\cdot)$ is also a stable rational matrix function. It remains to prove the statement about the McMillan degrees. Put $Z = M_G^*(M_G M_G^*)^{-1} G$. From the result of the previous part we know that $Z$ is a stable rational matrix function. Since

$$\Theta(z) = \Big( \Theta(0) \big( \Theta(0)^* \Theta(0) \big)^{-1} \Big) - Z(z) \Big( \Theta(0) \big( \Theta(0)^* \Theta(0) \big)^{-1} \Big),$$

it suffices to show that $\delta(Z) \leqslant \delta(G)$. From the definition of $Z$ we see that $G(z)Z(z) = G(z)$. Thus the Laurent operator $L_G$ is equal to the product of the Laurent operators of $G$ and $Z$. It follows (see the last paragraph of Section 1) that

$$\begin{bmatrix} T_{G^\#} & 0 \\ H_G & T_G \end{bmatrix} \begin{bmatrix} T_{Z^\#} & 0 \\ H_Z & T_Z \end{bmatrix} = L_G L_Z = L_G = \begin{bmatrix} T_{G^\#} & 0 \\ H_G & T_G \end{bmatrix}.$$

By comparing the terms in the lower left hand corner, we arrive at

$$T_G H_Z = H_G (I - T_{Z^\#}). \tag{2.5}$$

From the definition of $Z$, we know that for each $u \in \mathbb{C}^p$ the function $Z(\cdot)u$ is in the orthogonal complement of $\operatorname{Ker} M_G$ in $H^2(\mathbb{C}^p)$. Hence $T_Z\tilde{E}y$ is contained in $(\operatorname{Ker} T_G)^\perp$. Here $\tilde{E}$ is the canonical embedding of $\mathbb{C}^p$ onto the first coordinate space of $\ell_+^2(\mathbb{C}^p)$. Since the null space $\operatorname{Ker} T_G$ is invariant under the block forward shift $S_{\mathbb{C}^p}$ on $\ell_+^2(\mathbb{C}^p)$, it follows that $(\operatorname{Ker} T_G)^\perp$ is invariant under $S_{\mathbb{C}^p}^*$. Thus for each positive integer $k$ the vector $(S^*)^k T_Z\tilde{E}y$ is in $(\operatorname{Ker} T_G)^\perp$. But this implies the range of $H_Z$ is contained in $(\operatorname{Ker} T_G)^\perp$. We know that $T_G^*(T_GT_G^*)^{-1}T_G$ is the orthogonal projection onto $(\operatorname{Ker} T_G)^\perp$. As the range of $H_Z$ is contained in $(\operatorname{Ker} T_G)^\perp$, multiplying by $T_G^*(T_GT_G^*)^{-1}$ on the left in (2.5) yields

$$H_Z = T_G^*(T_GT_G^*)^{-1}T_GH_Z = T_G^*(T_GT_G^*)^{-1}H_G(I - T_{Z^\#}).$$

Therefore $\operatorname{rank} H_Z \leqslant \operatorname{rank} H_G$, and thus $\delta(Z) \leqslant \delta(G)$. □

## 3. Main theorem for equation (1.14)

In this section we deal with equation (1.14). We assume that $M_G$ is right invertible. As before $G$ is given by the stable state space representation (1.2), and we assume that the right hand side $F(z)$ of (1.14) is an $m \times k$ rational matrix function, also given by a stable state space representation, namely

$$F(z) = D_\nabla + zC_\nabla(I_r - zA_\nabla)^{-1}B_\nabla. \tag{3.1}$$

In particular, $A_\nabla$ is a stable $r \times r$ matrix. Our aim is to show that the function $Y$ determined by (1.15) is a stable rational matrix solution of (1.14) and to derive a state space representation for this solution, using the matrices appearing in state space representations (1.2) and (3.1).

THEOREM 3.1. *Let $G$ be given by (1.2) with $A$ stable, and let $P$ be the unique solution of the Stein equation (1.5). Assume that $M_G$ is right invertible, or equivalently, assume that the Riccati equation (1.3) has a stabilizing solution $Q$ such that the matrix $I_n - PQ$ is non-singular. Then the unique $p \times k$ matrix-valued function $Y$ determined by (1.15) is a stable rational matrix solution of (1.14), and $Y$ admits a state space representation,*

$$Y(z) = D_2 + zC_2(I_{n+r} - A_2)^{-1}B_2, \tag{3.2}$$

*of which the matrices $A_2$, $B_2$, $C_2$, and $D_2$ are obtained in the following way. First, define $\Omega$ to be the unique solution of the Stein equation*

$$\Omega = A_0^*\Omega A_\nabla + C_0^*C_\nabla. \tag{3.3}$$

*Here $A_0$ and $C_0$ are given by (1.7) and (1.9), respectively. Then, given $\Omega$, the matrices*

*$A_2$, $B_2$, $C_2$, and $D_2$ are defined by*

$$A_2 = \begin{bmatrix} A_0 & -\Gamma C_{0,\nabla} \\ 0 & A_\nabla \end{bmatrix}, \text{ where } C_{0,\nabla} = (R_0 - \Gamma^* Q\Gamma)^{-1}(C_\nabla - \Gamma^*\Omega A_\nabla), \tag{3.4}$$

$$B_2 = \begin{bmatrix} B_{21} \\ B_\nabla \end{bmatrix}, \text{ where } B_{21} = \Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(\Gamma^*\Omega B_\nabla - D_\nabla) + \\ + A_0 P(I_n - QP)^{-1}(C_0^* D_\nabla + A_0^*\Omega B_\nabla),$$

$$C_2 = \begin{bmatrix} D^*C_0 + B^*QA_0 & (D^* - B^*Q\Gamma)C_{0,\nabla} + B^*\Omega A_\nabla \end{bmatrix},$$

$$D_2 = (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) + B^*\Omega B_\nabla + \\ + (D^*C_0 + B^*QA_0)(I - PQ)^{-1} P (C_0^* D_\nabla + A_0^*\Omega B_\nabla).$$

*Furthermore, the McMillan degree of $Y$ is less than or equal to the sum of the McMillan degrees of $G$ and $F$.*

*Proof.* We have to compute $\mathscr{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}\tilde{F}$. Here $\tilde{F}$ is the column operator corresponding to the stable state space representation (3.1), that is, $\tilde{F}$ is the operator given by

$$\tilde{F} = T_F \tilde{E} = \begin{bmatrix} D_\nabla \\ C_\nabla B_\nabla \\ C_\nabla A_\nabla B_\nabla \\ C_\nabla A_\nabla^2 B_\nabla \\ \vdots \end{bmatrix} : \mathbb{C}^k \to \ell_+^2(\mathbb{C}^m). \tag{3.5}$$

From Theorem 4.1 in [3] we know that

$$(T_G T_G^*)^{-1} = T_\Psi T_\Psi^* + K(I_n - PQ)^{-1} P K^*. \tag{3.6}$$

Here $T_\Psi$ is the block lower triangular Toeplitz operator on $\ell_+^2(\mathbb{C}^m)$ defined by the stable rational matrix function

$$\Psi(z) = \Big(I_m - zC_0(I_n - zA_0)^{-1}\Gamma\Big)\Delta^{-1}, \text{ where } \Delta = (R_0 - \Gamma^*Q\Gamma)^{1/2}, \tag{3.7}$$

and $K$ is the observability operator defined by

$$K = W_{0,obs} = \begin{bmatrix} C_0 \\ C_0 A_0 \\ C_0 A_0^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell_+^2(\mathbb{C}^m). \tag{3.8}$$

It follows that $\mathscr{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}\tilde{F}$ can be written as the sum of two functions, namely $\mathscr{F}_{\mathbb{C}^p} T_G^*(T_G T_G^*)^{-1}\tilde{F} = \mathscr{F}_{\mathbb{C}^p}\tilde{\alpha} + \mathscr{F}_{\mathbb{C}^p}\tilde{\beta}$, where

$$\tilde{\alpha} = T_G^* T_\Psi T_\Psi^* \tilde{F}, \quad \tilde{\beta} = T_G^* K(I_n - PQ)^{-1} P K^* \tilde{F}. \tag{3.9}$$

We split the proof into five parts. The first three parts deal with computation of the term $\alpha$. In the fourth part we compute $\beta$. The final part proves the statement about the McMillan degrees.

*Part 1.* In this part we compute $T_\Psi^*\tilde{F}$. Since $T_\Psi^*\tilde{F} = T_{\Psi^*F}\tilde{E}$, we first compute $\Psi^*F$. From (3.3) we see that

$$C_0^*C_\nabla = (zI_n - A_0^*)\Omega A_\nabla + \Omega(I_n - zA_\nabla).$$

It follows that

$$\begin{aligned}(zI_n - A_0^*)^{-1}C_0^*C_\nabla(I_n - zA_\nabla)^{-1} &= \\ &= \Omega A_\nabla(I_n - zA_\nabla)^{-1} + (zI_n - A_0^*)^{-1}\Omega.\end{aligned}$$

Using the latter identity and the definitions of $\Psi$ and $\Delta$ in (3.7), we compute that

$$\begin{aligned}\Delta\Psi^*(z)F(z) &= D_\nabla + zC_\nabla(I_n - zA_\nabla)^{-1}B_\nabla - \Gamma^*(zI_n - A_0^*)^{-1}C_0^*D_\nabla + \\ &\quad - z\Gamma^*(zI_n - A_0^*)^{-1}C_0^*C_\nabla(I_n - zA_\nabla)^{-1}B_\nabla \\ &= D_\nabla + zC_\nabla(I_n - zA_\nabla)^{-1}B_\nabla - \Gamma^*(zI_n - A_0^*)^{-1}C_0^*D_\nabla + \\ &\quad - z\Gamma^*\Omega A_\nabla(I_n - zA_\nabla)^{-1}B_\nabla - z\Gamma^*(zI_n - A_0^*)^{-1}\Omega B_\nabla \\ &= (D_\nabla - \Gamma^*\Omega B_\nabla) + z(C_\nabla - \Gamma^*\Omega A_\nabla)(I_n - zA_\nabla)^{-1}B_\nabla + \\ &\quad - \Gamma^*(zI_n - A_0^*)^{-1}(C_0^*D_\nabla + A_0^*\Omega B_\nabla).\end{aligned}$$

It follows that

$$T_\Psi^*\tilde{F} = \begin{bmatrix}\Delta^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) \\ \Delta^{-1}(C_\nabla - \Gamma^*\Omega A_\nabla)B_\nabla \\ \Delta^{-1}(C_\nabla - \Gamma^*\Omega A_\nabla)A_\nabla B_\nabla \\ \Delta^{-1}(C_\nabla - \Gamma^*\Omega A_\nabla)A_\nabla^2 B_\nabla \\ \vdots\end{bmatrix} = \begin{bmatrix}\Delta^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) \\ \Delta C_{0,\nabla}B_\nabla \\ \Delta C_{0,\nabla}A_\nabla B_\nabla \\ \Delta C_{0,\nabla}A_\nabla^2 B_\nabla \\ \vdots\end{bmatrix}. \tag{3.10}$$

Here we used the definition of $C_{0,\nabla}$ in (3.4) and $\Delta = (R_0 - \Gamma^*Q\Gamma)^{1/2}$.

In the next two parts we compute $T_G^*T_\Psi(T_\Psi^*\tilde{F})$. Recall that $\Psi$ is analytic on the closed unit disc. It follows that $T_G^*T_\Psi = T_{G^*\Psi}$. From (3.17) in [3] we know that

$$G^*(z)C_0(I_n - zA_0)^{-1} = C_1(I_n - zA_0)^{-1} + B^*(zI_n - A^*)^{-1}Q. \tag{3.11}$$

Using this identity we see that $G^*(z)\Psi(z)$ can be written as

$$\begin{aligned}G^*(z)\Psi(z) &= G^*(z)\Delta^{-1} - zG^*(z)C_0(I_n - zA_0)^{-1}\Gamma\Delta^{-1} \\ &= D^*\Delta^{-1} + B^*(zI_n - A^*)^{-1}C^*\Delta^{-1} + \\ &\quad - zC_1(I_n - zA_0)^{-1}\Gamma\Delta^{-1} - zB^*(zI_n - A^*)^{-1}Q\Gamma\Delta^{-1}.\end{aligned}$$

From the definition of $C_0$ in (1.9) we see that $(C^* - A^*Q\Gamma)\Delta^{-1} = C_0^*\Delta$, and hence we obtain

$$G^*(z)\Psi(z) = \rho_+(z) + \rho_-(z), \quad \text{where}$$

$$\rho_+(z) = (D^* - B^*Q\Gamma)\Delta^{-1} - zC_1(I_n - zA_0)^{-1}\Gamma\Delta^{-1}, \tag{3.12}$$

$$\rho_-(z) = B^*(zI_n - A^*)^{-1}C_0^*\Delta. \tag{3.13}$$

It follows that

$$T_G^* T_\Psi = T_{\rho_+} + T_{\rho_-}. \tag{3.14}$$

We compute $T_{\rho_+}(T_\Psi^*\tilde{F})$ in the next part and $T_{\rho_-}(T_\Psi^*\tilde{F})$ in the third part.

*Part 2.* Since $T_{\rho_+}$ is a block lower triangular Toeplitz operator defined by $\rho_+$ in (3.13) and $T_\Psi^*\tilde{F}$ is given by (3.10), the expression $\mathscr{F}_{\mathbb{C}^p} T_{\rho_+}(T_\Psi^*\tilde{F})$ is equal to the rational matrix function $Y_1$ given by the product

$$Y_1(z) = \rho_+(z)\big(\mathscr{F}_{\mathbb{C}^m} T_\Psi^*\tilde{F}\big)(z) \tag{3.15}$$

$$= \Big((D^* - B^*Q\Gamma)\Delta^{-1} - zC_1(I_n - zA_0)^{-1}\Gamma\Delta^{-1}\Big)\times$$

$$\times\Big(\Delta^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) + z\Delta C_{0,\nabla}(I_r - zA_\nabla)^{-1}B_\nabla\Big). \tag{3.16}$$

Computing the product and using $\Delta = (R_0 - \Gamma^*Q\Gamma)^{1/2}$ we get

$$\begin{aligned} Y_1(z) = {} & (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) + \\ & - zC_1(I_n - zA_0)^{-1}\Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) + \\ & + z(D^* - B^*Q\Gamma)C_{0,\nabla}(I - zA_\nabla)^{-1}B_\nabla + \\ & - zC_1(I_n - zA_0)^{-1}(z\Gamma C_{0,\nabla})(I - zA_\nabla)^{-1}B_\nabla. \end{aligned} \tag{3.17}$$

Now we use the matrix $A_2$ in (3.4). Note that

$$(I_{n+r} - zA_2)^{-1} = \begin{bmatrix} (I_n - zA_0)^{-1} & -(I_n - zA_0)^{-1}(z\Gamma C_{0,\nabla})(I_r - zA_\nabla)^{-1} \\ 0 & (I_r - zA_\nabla)^{-1} \end{bmatrix}.$$

It follows that

$$\begin{aligned} Y_1(z) = {} & (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) + \\ & + z\begin{bmatrix} C_1 & (D^* - B^*Q\Gamma)C_{0,\nabla} \end{bmatrix}(I_{n+r} - zA_2)^{-1} \times \\ & \times \begin{bmatrix} -\Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla) \\ B_\nabla \end{bmatrix}. \end{aligned} \tag{3.18}$$

*Part 3.* In this part we compute the rational matrix function $Y_2$ given by $\mathscr{F}_{\mathbb{C}^p} T_{\rho_-}(T_\Psi^*\tilde{F})$. To compute $Y_2$ we first show that

$$\Omega = A^*\Omega A_\nabla + C_0^*(R_0 - \Gamma^*Q\Gamma)C_{0,\nabla}. \tag{3.19}$$

This formula follows from (3.3) and (3.4). Indeed,

$$\begin{aligned}
\Omega &= A_0^*\Omega A_\nabla + C_0^* C_\nabla = (A^* - C_0^*\Gamma^*)\Omega A_\nabla + C_0^* C_\nabla \\
&= A^*\Omega A_\nabla + C_0^*(C_\nabla - \Gamma^*\Omega A_\nabla) \\
&= A^*\Omega A_\nabla + C_0^*(R_0 - \Gamma^* Q\Gamma)(R_0 - \Gamma^* Q\Gamma)^{-1}(C_\nabla - \Gamma^*\Omega A_\nabla) \\
&= A^*\Omega A_\nabla + C_0^*(R_0 - \Gamma^* Q\Gamma)C_{0,\nabla}.
\end{aligned}$$

Note that the first row of $T_{\rho_-}$ is given by

$$\begin{bmatrix} 0 & B^*C_0\Delta & B^*A^*C_0\Delta & B^*(A^*)^2C_0\Delta & \cdots \end{bmatrix}.$$

Since $T_{\rho_-}$ is block upper triangular, we see that

$$T_{\rho_-}(T_\Psi^*\tilde{F}) = \begin{bmatrix} D^\# \\ T_{\rho_-}\begin{bmatrix} \Delta C_{0,\nabla}B_\nabla \\ \Delta C_{0,\nabla}A_\nabla B_\nabla \\ \Delta C_{0,\nabla}A_\nabla^2 B_\nabla \\ \vdots \end{bmatrix} \end{bmatrix}.$$

Here $D^\#$ is given by

$$\begin{aligned}
D^\# &= \sum_{\nu=0}^{\infty} B^*(A^*)^\nu C_0\Delta\Delta C_{0,\nabla}A_\nabla^\nu B_\nabla \\
&= B^*\Big(\sum_{\nu=0}^{\infty}(A^*)^\nu C_0(R_0 - \Gamma^* Q\Gamma)C_{0,\nabla}A_\nabla^\nu\Big)B_\nabla = B^*\Omega B_\nabla.
\end{aligned}$$

Note that the last equality results from (3.19). Next, again using that $T_{\rho_-}$ is block upper triangular, we obtain

$$T_{\rho_-}\begin{bmatrix} \Delta C_{0,\nabla}B_\nabla \\ \Delta C_{0,\nabla}A_\nabla B_\nabla \\ \Delta C_{0,\nabla}A_\nabla^2 B_\nabla \\ \vdots \end{bmatrix} = \begin{bmatrix} C^\# B_\nabla \\ C^\# A_\nabla B_\nabla \\ C^\# A_\nabla^2 B_\nabla \\ \vdots \end{bmatrix},$$

where

$$C^\# = \sum_{\nu=0}^{\infty} B^*(A^*)^\nu C_0\Delta\Delta C_{0,\nabla}A_\nabla^{\nu+1} = B^*\Omega A_\nabla.$$

Here we used that $\Delta^2 = R_0 - \Gamma^* Q\Gamma$ and the Stein equation (3.19). We conclude that

$$Y_2(z) = \Big(\mathscr{F}_{\mathbb{C}^p}T_{\rho_-}(T_\Psi^*\tilde{F})\Big)(z) = B^*\Omega B_\nabla + zB^*\Omega A_\nabla(I_r - zA_\nabla)^{-1}B_\nabla.$$

For later purposes it will be convenient to rewrite $Y_2(z)$ using the matrix $A_2$ in (3.4). This yields

$$Y_2(z) = B^*\Omega B_\nabla + z\left[0\ B^*\Omega A_\nabla\right]\left(I_{n+r} - zA_2\right)^{-1}\begin{bmatrix}\diamondsuit \\ B_\nabla\end{bmatrix} \tag{3.20}$$

Here the matrix $\diamondsuit$ is free to choose. In the next part we shall take $\diamondsuit$ equal to $-\Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(D_\nabla - \Gamma^*\Omega B_\nabla)$.

*Part 4.* In this part we compute the term $\tilde{\beta}$ in (3.9). Using that $\Omega$ is the unique solution of the Stein equation (3.3), we see that

$$\begin{aligned}
K^*\tilde{F} &= \left[C_0^*\ A_0^*C_0^*\ (A_0^*)^2C_0^*\ (A_0^*)^3C_0^*\ \cdots\right]\begin{bmatrix} D_\nabla \\ C_\nabla B_\nabla \\ C_\nabla A_\nabla B_\nabla \\ C_\nabla A_\nabla^2 B_\nabla \\ \vdots \end{bmatrix} \\
&= C_0^* D_\nabla + \sum_{\nu=0}^{\infty} (A_0^*)^{\nu+1} C_0^* C_\nabla A_\nabla^\nu B_\nabla \\
&= C_0^* D_\nabla + A_0^*\Big(\sum_{\nu=0}^{\infty} (A_0^*)^{\nu} C_0^* C_\nabla A_\nabla^\nu\Big) B_\nabla \\
&= C_0^* D_\nabla + A_0^*\Omega B_\nabla .
\end{aligned}$$

From (3.11) we see that $(\mathscr{F}_{\mathbb{C}^p} T_G^* K)(z) = C_1(I_n - zA_0)^{-1}$. It follows that

$$\tilde{\beta} = \begin{bmatrix} C_1 \\ C_1A_0 \\ C_1A_0^2 \\ \vdots \end{bmatrix}(I_n - PQ)^{-1}P(C_0^*D_\nabla + A_0^*\Omega B_\nabla).$$

Here $C_1 = D^*C_0 + B^*QA_0$. Now put $Y_3(z) = \big(\mathscr{F}_{\mathbb{C}^p}\tilde{\beta}\big)(z)$. Then

$$\begin{aligned}
Y_3(z) = (D^*C_0 + B^*QA_0)(I_n - PQ)^{-1}P(C_0^*D_\nabla + A_0^*\Omega B_\nabla) + \\
+z(D^*C_0 + B^*QA_0)(I_n - zA_0)^{-1} \times \\
\times A_0(I_n - PQ)^{-1}P(C_0^*D_\nabla + A_0^*\Omega B_\nabla).
\end{aligned}$$

To derive our final result we rewrite $Y_3(z)$ using the matrix $A_2$ in (3.4). This yields

$$\begin{aligned}
Y_3(z) = (D^*C_0 + B^*QA_0)(I_n - PQ)^{-1}P(C_0^*D_\nabla + A_0^*\Omega B_\nabla) + \\
+z\left[(D^*C_0 + B^*QA_0)\ 0\right]\left(I_{n+r} - zA_2\right)^{-1} \times \\
\times \begin{bmatrix} A_0(I_n - PQ)^{-1}P(C_0^*D_\nabla + A_0^*\Omega B_\nabla) \\ 0 \end{bmatrix}.
\end{aligned} \tag{3.21}$$

Finally, to complete the proof of the main part of the theorem, note that the solution $Y$ determined by (1.15) is given by

$$Y(z) = Y_1(z) + Y_2(z) + Y_3(z).$$

So we can add the state space representations (3.18), (3.20) and (3.21) for $Y_1(z)$, $Y_2(z)$, and $Y_3(z)$, respectively, to obtain the desired representation for $Y(z)$.

*Part 5.* It remains to prove the final statement about the McMillan degrees. We assume that the number $n$ and the number $r$ in the state space representations (1.2) and (3.1) are chosen as small as possible. In that case $\delta(G) = n$ and $\delta(F) = r$. Since the matrix $A_2$ in the state space representation of $Y$ has order $n+r$, the McMillan degree of $Y$ is at most $n+r$. Thus $\delta(Z) \leqslant \delta(G) + \delta(Y)$. $\square$

The final statement in Theorem 3.1 about the McMillan degrees can also be proven directly, without using state space representations. The argument is a variation of the argument used in the final part of the proof of Theorem 2.1. The details are as follows.

Let $G$, $Y$, and $F$ be the stable rational matrix functions appearing in Theorem 3.1 above. Since $G(z)Y(z) = F(z)$, the Laurent operator $L_F$ is equal to the product of the Laurent operators of $G$ and $Y$. It follows (see the last paragraph of Section 1) that

$$\begin{bmatrix} T_{G^\#} & 0 \\ H_G & T_G \end{bmatrix} \begin{bmatrix} T_{Y^\#} & 0 \\ H_Y & T_Y \end{bmatrix} = L_G L_Y = L_F = \begin{bmatrix} T_{F^\#} & 0 \\ H_F & T_F \end{bmatrix}.$$

By comparing the terms in the lower left hand corner, we arrive at

$$T_G H_Y = H_F - H_G T_{Y^\#} \tag{3.22}$$

From the definition of $Y$, we know that for each $u \in \mathbb{C}^p$ the function $Y(\cdot)u$ is in the orthogonal complement of $\operatorname{Ker} M_G$ in $H^2(\mathbb{C}^p)$. Hence $T_Y \tilde{E} u$ is contained in $(\operatorname{Ker} T_G)^\perp$. Here $\tilde{E}$ is the canonical embedding of $\mathbb{C}^p$ onto the first coordinate space of $\ell^2_+(\mathbb{C}^p)$. Since the null space $\operatorname{Ker} T_G$ is invariant under the block forward shift $S_{\mathbb{C}^p}$ on $\ell^2_+(\mathbb{C}^p)$, it follows (see the final paragraph of the proof of Theorem 2.1) that the range of $H_Y$ is contained in $(\operatorname{Ker} T_G)^\perp$. We know that $T_G^*(T_G T_G^*)^{-1} T_G$ is the orthogonal projection onto $(\operatorname{Ker} T_G)^\perp$. As the range of $H_Y$ is contained in $(\operatorname{Ker} T_G)^\perp$, multiplying by $T_G^*(T_G T_G^*)^{-1}$ on the left in (3.22) yields

$$H_Y = T_G^*(T_G T_G^*)^{-1} T_G H_Y = T_G^*(T_G T_G^*)^{-1} \left(H_F - H_G T_{Y^\#}\right).$$

Therefore $\operatorname{rank} H_Y \leqslant \operatorname{rank} H_F + \operatorname{rank} H_G$, and thus $\delta(Y) \leqslant \delta(F) + \delta(G)$.

It is interesting to specify Theorem 3.1 for the case when the function $F$ in (3.1) is equal to the function $G$ given by (1.2). This leads to the following corollary which we shall need in the next section.

COROLLARY 3.2. *Let $G$ be given by* (1.2) *with $A$ stable, and let $P$ be the unique solution of the Stein equation* (1.5). *Assume that $M_G$ is right invertible, or equivalently,* *assume that the Riccati equation (1.3) has a stabilizing solution $Q$ such that the matrix $I_n - PQ$ is non-singular. Then the $p \times p$ matrix function $Z$ defined by*

$$Z(\cdot)y = M_G^*(M_G M_G^*)^{-1}Gy, \quad y \in \mathbb{C}^p. \tag{3.23}$$

*is a stable rational matrix function and*

$$Z(z) = D_3 + zC_1(I_n - zA_0)^{-1}B_3, \tag{3.24}$$

*where $A_0 = A - \Gamma C_0$ and $C_1 = D^*C_0 + B^*QA_0$, and $B_3$ and $D_3$ are given by*

$$B_3 = B - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + A_0P(I_n - QP)^{-1}C_1^*, \tag{3.25}$$

$$\begin{aligned} D_3 = {} & (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + B^*QB + \\ & + C_1P(I_n - QP)^{-1}C_1^*. \end{aligned} \tag{3.26}$$

*Furthermore, $G(z)Z(z) = G(z)$ for each $z \in \mathbb{D}$.*

*Proof.* To determine $Z$ we follow the proof of Theorem 3.1 with

$$A_\nabla = A, \quad B_\nabla = B, \quad C_\nabla = C, \quad D_\nabla = D.$$

Using the definitions of $A_0$ and $C_0$ in (1.7) and (1.9), together with the fact that $Q$ is a hermitian matrix satisfying (1.3), we see that

$$Q = A^*QA_0 + C^*C_0. \tag{3.27}$$

Thus in this case (3.3) reduces to the dual of (3.27), and hence $\Omega = Q$. Furthermore, we have

$$C_{0,\nabla} = (R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA) = C_0.$$

It follows that $Z$ in (3.23) is given by $Z(z) = Z_1(z) + Z_2(z) + Z_3(z)$, where the functions $Z_1$, $Z_2$, $Z_3$ are the analogs of the functions $Y_1$, $Y_2$, $Y_3$ in the proof of Theorem 3.1. Thus

$$\begin{aligned} Z_1(z) = {} & (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + \\ & - zC_1(I_n - zA_0)^{-1}\Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + \\ & + z(D^* - B^*Q\Gamma)C_0(I_n - zA)^{-1}B + \\ & - zC_1(I_n - zA_0)^{-1}(z\Gamma C_0)(I - zA)^{-1}B, \end{aligned} \tag{3.28}$$

$$Z_2(z) = B^*QB + zB^*QA_0(I_n - zA)^{-1}B,$$

$$\begin{aligned} Z_3(z) = {} & (D^*C_0 + B^*QA_0)(I_n - PQ)^{-1}P(C_0^*D + A_0^*QB) + \\ & + z(D^*C_0 + B^*QA_0)(I_n - zA_0)^{-1} \times \\ & \times A_0(I_n - PQ)^{-1}P(C_0^*D + A_0^*QB). \end{aligned} \tag{3.29}$$

Recall that $A_0 = A - \Gamma C_0$. Hence $z\Gamma C_0$ can be rewritten as

$$z\Gamma C_0 = (I_n - zA_0) - (I_n - zA).$$

Using this in (3.28) we see that

$$\begin{aligned} -zC_1(I_n - zA_0)^{-1}(z\Gamma C_0)(I - zA)^{-1}B &= \\ &= zC_1(I_n - zA_0)^{-1}B - zC_1(I_n - zA)^{-1}B. \end{aligned}$$

This implies that

$$\begin{aligned} Z_1(z) = (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + \\ - zC_1(I_n - zA_0)^{-1}\Big(\Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) - B\Big) + \\ + z\Big((D^* - B^*Q\Gamma)C_0 - C_1\Big)(I_n - zA)^{-1}B. \end{aligned}$$

Recall (see (1.8)) that $C_1 = D^*C_0 + B^*QA_0$. Thus

$$\begin{aligned} (D^* - B^*Q\Gamma)C_0 - C_1 &= D^*C_0 - B^*Q\Gamma C_0 - D^*C_0 - B^*QA_0 \\ &= -B^*Q(\Gamma C_0 + A_0) = -B^*QA. \end{aligned}$$

We conclude that

$$\begin{aligned} Z_1(z) + Z_2(z) = (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + B^*QB + \\ - zC_1(I_n - zA_0)^{-1}\Big(\Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) - B\Big). \end{aligned}$$

Using the identity $C_1 = D^*C_0 + B^*QA_0$ in (3.29) we see that

$$Z_3(z) = C_1(I_n - PQ)^{-1}PC_1^* + zC_1(I_n - zA_0)^{-1}A_0(I_n - PQ)^{-1}PC_1^*.$$

It follows that

$$Z(z) = Z_1(z) + Z_2(z) + Z_3(z) = D_3 + zC_1(I_n - zA_0)^{-1}B_3,$$

where $B_3$ and $D_3$ are given by (3.25) and (3.26), respectively. □

## 4. Proof of Theorem 1.1

Let $G$ be a stable $m \times p$ rational matrix function, and assume that $M_G$ is right invertible. From the beginning of Section 2 and Theorem 2.1 we know that there exists a stable rational $p \times (p - m)$ matrix function $\Theta$, which is inner and unique up to a constant unitary $(p - m) \times (p - m)$ matrix on the right, such that

$$\operatorname{Ker} M_G = M_\Theta H^2(\mathbb{C}^{p-m}). \tag{4.1}$$

Moreover, $\Theta(0)$ is one-to-one and $\Theta$ is given by

$$\Theta(\cdot)\Theta(0)^*u = u - M_G^*(M_G M_G^*)^{-1}G(\cdot)u, \quad u \in \mathbb{C}^p. \tag{4.2}$$

Since $\Theta(0)$ is one-to-one, $\Theta(0)^*$ is onto. Hence the right hand side of (4.2) determines $\Theta$ uniquely up to a constant unitary $(p-m)\times(p-m)$ matrix on the right.

LEMMA 4.1. *The rational matrix function $\Theta$ in (4.2) admits the following stable state space representation*

$$\Theta(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)D_4. \tag{4.3}$$

*Here $C_1 = D^*C_0 + B^*QA_0$, and the matrices $A_0$ and $C_0$ are given by (1.7) and (1.9), respectively. Furthermore, $D_4$ is a one-to-one $p\times(p-m)$ matrix such that*

$$\begin{aligned} D_4D_4^* = I_p - (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) +& \\ - B^*QB - C_1(I_n - PQ)^{-1}PC_1^*.& \end{aligned} \tag{4.4}$$

*Proof.* By comparing the right hand side of the above identity with the right hand side of (3.23) we see that $\Theta(z)\Theta(0)^* = I_p - Z(z)$ for each $z \in \mathbb{D}$. From Corollary 3.2 we know that $Z$ is given by the stable state space representation (3.24). Hence

$$\Theta(z)\Theta(0)^* = I_p - D_3 - zC_1(I_n - A_0)^{-1}B_3, \tag{4.5}$$

where $B_3$ and $D_3$ are given by (3.25) and (3.26), respectively. Put $D_4 = \Theta(0)$. Then $D_4$ is a one-to-one $p\times(p-m)$ matrix and $D_4D_4^* = \Theta(0)\Theta(0)^* = I_p - D_3$, and thus (4.4) holds. Furthermore, using (2.4) we see that $\Theta$ admits the representation

$$\Theta(z) = D_4 + zC_1(I_n - zA_0)^{-1}B_4, \tag{4.6}$$

where $B_4 = -B_3\Theta(0)\big(\Theta(0)^*\Theta(0)\big)^{-1}$, that is, $B_4$ is given by

$$\begin{aligned} B_4 = -\Big(B - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) +& \\ + A_0(I_n - PQ)^{-1}PC_1^*\Big)&D_4(D_4^*D_4)^{-1}. \end{aligned} \tag{4.7}$$

To see that (4.6) yields (4.3) we have to prove $B_4 = -(I_n - PQ)^{-1}BD_4$. Recall that $D_4$ is one-to-one, and hence $D_4^*D_4$ is invertible. Therefore, it suffices to show that

$$BD_4D_4^*D_4 = -(I_n - PQ)B_4D_4^*D_4. \tag{4.8}$$

From (4.1) we know that that $G(z)\Theta(z)$ is identically zero. In particular, we have

$$DD_4 = 0. \tag{4.9}$$

It follows from (4.4) and $C_1 = D^*C_0 + B^*QA_0$ that

$$\begin{aligned} D_4D_4^*D_4 = D_4 + (D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}\Gamma^*QBD_4 +& \\ - B^*QBD_4 - C_1(I_n - PQ)^{-1}PC_1^*D_4,& \\ C_1^*D_4 = A_0^*QBD_4.& \end{aligned}$$

Thus $BD_4D_4^*D_4 = \alpha BD_4 - \beta BD_4$, where

$$\alpha = I_n + B(D^* - B^*Q\Gamma)(R_0 - \Gamma^*Q\Gamma)^{-1}\Gamma^*Q - BB^*Q,$$
$$\beta = BC_1(I_n - PQ)^{-1}PA_0^*Q.$$

Put $\Lambda = (R_0 - \Gamma^*Q\Gamma)^{-1}$. Using (1.5) and the second identity in (1.4) we have

$$\begin{aligned}\alpha &= I_n + \Gamma\Lambda\Gamma^*Q - APC^*\Lambda\Gamma^*Q - PQ\Gamma\Lambda\Gamma^*Q + APA^*Q\Gamma\Lambda\Gamma^*Q - PQ + APA^*Q\\ &= (I_n - PQ)(I_n + \Gamma\Lambda\Gamma^*Q) - APC^*\Lambda\Gamma^*Q + APA^*Q\Gamma\Lambda\Gamma^*Q + APA^*Q.\end{aligned}$$

Next, we use the identity $BC_1 = A(I_n - PQ) - (I_n - PQ)A_0$; see [3], formula (3.21). It follows that

$$\begin{aligned}\alpha - \beta &= (I_n - PQ)(I_n + \Gamma\Lambda\Gamma^*Q) - APC^*\Lambda\Gamma^*Q + APA^*Q\Gamma\Lambda\Gamma^*Q +\\ &\qquad + APA^*Q - \Big(A(I_n - PQ) - (I_n - PQ)A_0\Big)(I_n - PQ)^{-1}PA_0^*Q\\ &= (I_n - PQ)\Big(I_n + \Gamma\Lambda\Gamma^*Q + A_0(I_n - PQ)^{-1}PA_0^*Q\Big) - APC^*\Lambda\Gamma^*Q +\\ &\qquad + APA^*Q\Gamma\Lambda\Gamma^*Q + APA^*Q - APA_0^*Q\\ &= (I_n - PQ)\Big(I_n + \Gamma\Lambda\Gamma^*Q + A_0(I_n - PQ)^{-1}PA_0^*Q\Big) +\\ &\qquad + AP\Big(-C^*\Lambda\Gamma^* + A^*Q\Gamma\Lambda\Gamma^* + A^* - A_0^*\Big)Q\end{aligned}$$

From the definitions of $A_0$ and $C_0$ in (1.7) and (1.9) we see that

$$-C^*\Lambda\Gamma^* + A^*Q\Gamma\Lambda\Gamma^* + A^* - A_0^* = 0.$$

We conclude that

$$BD_4D_4^*D_4 = (I_n - PQ)\Big(I_n + \Gamma\Lambda\Gamma^*Q + A_0(I_n - PQ)^{-1}PA_0^*Q\Big)BD_4.$$

On the other hand, again using $DD_4 = 0$, we have

$$\begin{aligned}B_4D_4D_4^* &= -\Big(B - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(D - \Gamma^*QB) + A_0(I_n - PQ)^{-1}PC_1^*\Big)D_4\\ &= -\Big(I_n + \Gamma\Lambda\Gamma^*Q + A_0(I_n - PQ)^{-1}PA_0^*\Big)BD_4.\end{aligned}$$

Hence (4.8) holds, and (4.3) is proved. □

*Proof of Theorem* 1.1. Let $\Theta$ be as in Lemma 4.1, and put $D_4 = \Theta(0)$. Then (4.4) holds true. It follows that the right hand side of (4.4) is positive semi-definite. Hence the same holds true for the right hand side of (1.12).

Now let $\hat{D}$ be any one-to-one $p \times (p-m)$ matrix such that (1.12) holds. Then $\hat{D}\hat{D}^* = D_4D_4^*$. Since both $D_4$ and $\hat{D}$ are one-to-one, there exists a unitary matrix $U$ of order $p-m$ such that $\hat{D} = D_4U$. It follows that $\hat{\Theta}(\cdot) = \Theta(\cdot)U$. Hence $\hat{\Theta}$ has the same properties as $\Theta$. Thus $\hat{\Theta}$ is inner and (1.13) holds. The latter implies that the set of all stable $p \times m$ rational matrix solutions $V$ of equation (1.1) is equal to the set of all functions $V(z) = X(z) + \hat{\Theta}(z)N(z)$, where $X$ is the least square solution given by (1.6), the parameter $N$ is an arbitrary stable $(p-m) \times m$ rational matrix function, and $\hat{\Theta}$ is given by (1.11).

It remains to prove the statement about the McMillan degrees. To do this assume that the number $n$ in the state space representation (1.2) is chosen as small as possible. In that case, $\delta(G) = n$. Since the matrix $A_0$ in the state space representation of $\hat{\Theta}$ has the same size as $A$, we conclude that $\delta(\hat{\Theta}) \leqslant n$. Thus $\delta(\hat{\Theta}) \leqslant \delta(G)$, as desired. □

A REMARK ABOUT $\hat{\Theta}$ BEING INNER. The above proof of the fact that the stable rational matrix function $\hat{\Theta}$ defined by (1.11) is inner follows a rather indirect line of arguments. Indeed, the proof uses that $\hat{\Theta}(z) = \Theta(z)U$, where $U$ is a constant unitary matrix and $\Theta$ is the function given by (4.3). But the function $\Theta$ given by (4.3) is the inner function appearing (4.1). Thus $\hat{\Theta}$ is also inner. It is possible to show more directly that $\hat{\Theta}$ is inner. Indeed, the fact that $\hat{\Theta}$ is inner follows from the following identity:

$$\begin{bmatrix} A_0^* & C_1^* \\ \hat{B}^* & \hat{D}^* \end{bmatrix} \begin{bmatrix} Q-QPQ & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A_0 & \hat{B} \\ C_1 & \hat{D} \end{bmatrix} = \begin{bmatrix} Q-QPQ & 0 \\ 0 & I_m \end{bmatrix}. \tag{4.10}$$

Note that the above identity is equivalent to the following three identies:

$$A_0^*(Q-QPQ)A_0 + C_1^*C_1 = Q-QPQ, \tag{4.11}$$

$$\hat{B}^*(Q-QPQ)A_0 + \hat{D}^*C_1 = 0, \tag{4.12}$$

$$\hat{B}^*(Q-QPQ)\hat{B} + \hat{D}^*\hat{D} = I_m. \tag{4.13}$$

The identity (4.11) has been established in [3, formula (3.24)]. Since $A_0$ is stable, (4.11) tells us that the matrix $Q-QPQ$ is the observability Gramian for the pair $\{C_0, A_0\}$. Given (4.11) it is well-known (see, e.g., the proof of Theorem 4.5.1 in [2]) that (4.12) implies that the block columns of $T_\Theta$ are mutually orthogonal and that (4.13) implies that each block column of $T_\Theta$ is an isometry. Thus given (4.11), together the equalities (4.12) and (4.13) show that $T_\Theta$ is an isometry and hence $\Theta$ is inner. Thus (4.10) yields $\Theta$ is inner.

To obtain the identity (4.10) it remains to derive the equalities (4.12) and (4.13). This can be done by direct computations; we omit the details.

EXAMPLE. Let us specify Theorem 1.1 for the stable rational matrix function $G$ appearing in Example 1 in [3, Section 5]; see also [11], page 425. Thus we take $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$. A stable state space representation for this $G$ is obtained by taking

$$A = 0, \quad B = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad C = 1, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{4.14}$$

We already know ([3, Section 7]) that $P = 2$ is the solution of the corresponding Stein equation (1.5), and that the Riccati equation (1.3) reduces to

$$Q = \frac{1}{3-Q}.$$

This equation has $q=\frac{1}{2}(3-\sqrt{5})$ as a stabilizing solution. Furthermore, $A_0=-q$ and $C_0=q$, and thus

$$C_1=\begin{bmatrix} q\\ 0\end{bmatrix}-\begin{bmatrix} 1\\ -1\end{bmatrix}q^2=q\begin{bmatrix} 1-q\\ q\end{bmatrix}.$$

A straightforward computation shows that in this case the right hand side of (1.12) is given by

$$\begin{bmatrix} 0 & 0\\ 0 & q\end{bmatrix}, \text{ and hence for } \hat{D} \text{ we can take } \hat{D}=\begin{bmatrix} 0\\ \sqrt{q}\end{bmatrix}.$$

Now using (1.11) we see that $\Theta$ is given by

$$\hat{\Theta}(z)=\frac{\sqrt{q}}{1+qz}\begin{bmatrix} z\\ 1+z\end{bmatrix}, \text{ where as before } q=\tfrac{1}{2}(3-\sqrt{5}).$$

Clearly, $G(z)\hat{\Theta}(z)$ is identically zero and using $q^2-3q+1=0$ one checks directly that $\hat{\Theta}$ is inner.

## 5. The rational version of Tolokonnikov's lemma

Tolokonnikov's lemma (see [10] and Appendix 3 in [8]) tells us that the problem of solving a corona type Bezout equation is equivalent to solving a certain extension problem. In this section we specify this result for rational matrix functions and derive a state space representation for a special extension.

Throughout $G$ is a stable $m\times p$ rational matrix function. We say that $G$ admits an *invertible outer extension* if there exists an invertible outer $p\times p$ stable rational matrix function $\hat{G}$ such that

$$\hat{G}(z)=\begin{bmatrix} G(z)\\ \star\end{bmatrix}. \tag{5.1}$$

Recall that a square stable rational matrix function $F$ is called *invertible outer* whenever $F^{-1}$ exists and is again a stable rational matrix function. If $\hat{G}$ is an invertible outer extension of $G$, then the matrix function $X$ defined by

$$X(z)=\hat{G}(z)^{-1}\begin{bmatrix} I_m\\ 0\end{bmatrix} \tag{5.2}$$

is a stable rational matrix solution of (1.1).

The converse is also true, that is, if (1.1) has a stable rational matrix solution, then $G$ admits an invertible outer extension. To see this we use the Smith form for matrix polynomials (see Chapter S1 of [6] and Section 6.5.2 in [7]). Let $d(z)$ be the least common multiple of the denominators of the entries of $G(z)$. Then $d(z)G(z)$ is a matrix polynomial, and using the Smith form for this polynomial we see that $G$ factors as

$$G(z)=U(z)\begin{bmatrix} \rho_1(z) & & & 0 & \cdots & 0\\ & \ddots & & \vdots & & \vdots\\ & & \rho_m(z) & 0 & \cdots & 0\end{bmatrix}V(z). \tag{5.3}$$

Here $U(z)$ and $V(z)$ are unimodular matrix polynomials of sizes $m\times m$ and $p\times p$, respectively, and $\rho_1,\ldots,\rho_m$ are scalar rational functions. Since $G$ is stable, the functions $\rho_1,\ldots,\rho_m$ have no poles in $|z|\leqslant 1$. Now, assume that (1.1) has a stable rational matrix solution. Then $G(z)$ has full row rank for each $|z|\leqslant 1$, and hence the rational functions $\rho_1,\ldots,\rho_m$ in (5.3) have no zeros in $|z|\leqslant 1$. Put

$$\tilde{U}(z)=U(z)\begin{bmatrix}\rho_1(z) & & \\ & \ddots & \\ & & \rho_m(z)\end{bmatrix},$$

where $U(z)$ and $\rho_1(z),\ldots,\rho_m(z)$ are as in (5.3). Next using $\tilde{U}(z)$ above and $V(z)$ in (5.3), set

$$\hat{G}(z)=\begin{bmatrix}\tilde{U}(z) & 0\\ 0 & I_{p-m}\end{bmatrix}V(z).$$

Then the function $\hat{G}$ is an invertible outer extension of $G$.

Thus (1.1) has a stable rational matrix solution if and only if $G$ has an invertible outer extension. This is Tolokonnikov's lemma for rational matrix functions. In additon to this result, the following proposition presents in state space form a special invertible outer extension.

PROPOSITION 5.1. *Let $G$ be a stable $m\times p$ rational matrix function, and assume that* (1.1) *has a stable rational matrix solution. Then $G$ has an invertible outer extension $\hat{G}$ such that the McMillan degree of $\hat{G}$ is equal to the McMillan degree of $G$. Moreover, such an invertible outer extension $\hat{G}$ can be obtained in the following way. Let $X$ be the least squares solution given by* (1.6)*, and let $\Theta$ be the inner rational matrix function given by* (4.3)*. Then*

$$\hat{G}(z)=\begin{bmatrix}G(z)\\ \Theta^*(z)(I_p-X(z)G(z))\end{bmatrix} \tag{5.4}$$

*is an invertible outer extension of $G$, and $\hat{G}(z)^{-1}=\begin{bmatrix}X(z) & \Theta(z)\end{bmatrix}$. Furthermore, the McMillan degrees of $G$ and $\hat{G}$ coincide, and $\hat{G}$ in* (5.4) *admits the stable state space representation*

$$\hat{G}(z)=\begin{bmatrix}D\\ D_4^*+D_4^*B^*Q(I_n-PQ)^{-1}B\end{bmatrix}+z\begin{bmatrix}C\\ D_4^*B^*Q(I_n-PQ)^{-1}A\end{bmatrix}(I-zA)^{-1}B. \tag{5.5}$$

*Here $D_4$ is as in* (4.4).

*Proof.* Our hypotheses imply that $M_G$ is right invertible, and hence the stable rational matrix functions $X$ in (1.6) and $\Theta$ in (4.3) are well defined. We first prove that the function $K$ given by

$$K(z)=\Theta^*(z)\Big(I_p-X(z)G(z)\Big) \tag{5.6}$$

is a stable rational matrix function. [The argument does not require $X$ to be the least squares solution; it works for any stable rational matrix solution.] Clearly, $K$ is rational and has no poles on $\mathbb{T}$. To show that $K$ is stable, take $h$ in $H^2(\mathbb{C}^p)$. Then $G(z)X(z) = I_m$ implies $M_G M_X = I$, and thus $M_G(I - M_X M_G)h = 0$. But then, by (2.1), there exists $f$ in $H^2(\mathbb{C}^k)$ such that $M_\Theta f = (I - M_X M_G)h$. In other words, in terms of the corresponding Laurent operators, we have $L_\Theta f = (I - L_X L_G)h$. Since $\Theta$ is inner, $L_\Theta^* L_\Theta = I$. But $L_\Theta^* = L_{\Theta^*}$. This leads to the identity $L_{\Theta^*}(I - L_X L_G)h = f$. Recall that $h$ is an arbitrary element in $H^2(\mathbb{C}^p)$. We conclude that the Laurent operator $L_{\Theta^*}(I - L_X L_G)$ maps $H^2(\mathbb{C}^p)$ into $H^2(\mathbb{C}^k)$. This implies that the rational matrix function $K$ in (5.6) is a rational matrix-valued $H^\infty$ function, which is equivalent to $K$ being stable.

Let $\hat{G}$ be the matrix function defined by (5.4). The result of the previous paragraph implies that $\hat{G}$ is a stable rational matrix function. Note that the function $\begin{bmatrix} X(z) & \Theta(z) \end{bmatrix}$ also is a stable rational matrix function. Thus in order to prove that $\hat{G}$ is an invertible outer extension of $G$, it suffices to show that

$$\hat{G}(z) \begin{bmatrix} X(z) & \Theta(z) \end{bmatrix} = I_p \quad \text{and} \quad \begin{bmatrix} X(z) & \Theta(z) \end{bmatrix} \hat{G}(z) = I_p. \tag{5.7}$$

In fact, since $\hat{G}(z)$ is a square matrix function, it is sufficient to prove the first identity in (5.7). To do this note that

$$\hat{G}(z) \begin{bmatrix} X(z) & \Theta(z) \end{bmatrix} = \begin{bmatrix} G(z)X(z) & G(z)\Theta(z) \\ \Theta^*(z)\Big(I_p - X(z)G(z)\Big)X(z) & \Theta^*(z)\Big(I_p - X(z)G(z)\Big)\Theta(z) \end{bmatrix}.$$

According to (a), we have $G(z)X(z) = I_m$. This implies that the (1,1) and (2,1) entries in the above $2 \times 2$ block matrix are equal to $I_m$ and $0$, respectively. On the other hand, $G(z)\Theta(z) = 0$ by (2.1). Thus the (1,2) entry in the above $2 \times 2$ block matrix is equal to $0$ and the (2,2) entry is equal $\Theta^*(z)\Theta(z)$. Since $\Theta$ is inner, $\Theta^*(z)\Theta(z) = I_k$. We conclude that the first identity in (5.7) holds. Hence $\hat{G}$ is an invertible outer extension of $G$. [Again note that the given argument works for any stable rational matrix solution and does not require $X$ to be the least squares solution.]

Next we derive the representation (5.5). This requires that $X$ is the least squares solution and uses (1.6). We first show that $\Theta^*(z)X(z)$ admits the following state space representation

$$\Theta^*(z)X(z) = -D_4^* B^* (I_n - QP)^{-1}(zI_n - A_0^*)^{-1} C_0^*. \tag{5.8}$$

To obtain this identity, we use that $X$ is given by (1.6) and $\Theta$ by (4.3). Thus

$$\begin{aligned} \Theta^*(z)X(z) &= (D_4^* - D_4^* B^*(I_n - QP)^{-1}(zI_n - A_0^*)^{-1} C_1^*) \times \\ &\qquad \times (D_1 - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1} D_1) \\ &= D_4^* D_1 - D_4^* B^*(I_n - QP)^{-1}(zI_n - A_0^*)^{-1} C_1^* D_1 + \\ &\qquad - zD_4^* C_1 (I_n - zA_0)^{-1}(I_n - PQ)^{-1} B D_1 + \\ &\qquad + D_4^* B^* \alpha(z) B D_1. \end{aligned}$$

Here
$$\alpha(z) = z(I_n - QP)^{-1}(zI_n - A_0^*)^{-1}C_1^*C_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}.$$

From (4.11) (see also [3, formula (3.23)]) we know that $Q - QPQ$ satisfies the Stein equation
$$(Q - QPQ) - A_0^*(Q - QPQ)A_0 = C_1^*C_1.$$

Using this identity we have

$$\begin{aligned} zC_1^*C_1 &= (zI_n - A_0^*)(Q - QPQ) + A_0^*(Q - QPQ)(I_n - zA_0),\\ \alpha(z) &= Q(I_n - zA_0)^{-1}(I_n - PQ)^{-1} + (I_n - QP)^{-1}(zI_n - A_0^*)^{-1}A_0^*Q\\ &= Q(I_n - PQ)^{-1} + zQA_0(I_n - zA_0)^{-1}(I_n - PQ)^{-1} +\\ &\qquad + (I_n - QP)^{-1}(zI_n - A_0^*)^{-1}A_0^*Q. \end{aligned}$$

Inserting the latter expressing for $\alpha(z)$ into the formula above for $\Theta^*(z)X(z)$ we obtain

$$\begin{aligned} \Theta^*(z)X(z) = D_4^*D_1 + D_4^*B^*Q(I_n - PQ)^{-1}BD_1 +\\ + zD_4^*(B^*QA_0 - C_1)(I_n - zA_0)^{-1}(I_n - PQ)^{-1}BD_1\\ + D_4^*B^*(I_n - QP)^{-1}(zI_n - A_0^*)^{-1}(A_0^*QB - C_1^*)D_1. \end{aligned}$$

Let $u$ be an arbitrary vector in $\mathbb{C}^p$. Since $X(\cdot)u$ is perpendicular to $\operatorname{Ker} M_G$, we see from (2.1) that the function $\Theta^*(\cdot)X(\cdot)u$ is analytic outside the open unit disc and has the value zero at infinity. This holds for each $u$ in $\mathbb{C}^p$. It follows that in the above expression for $\Theta^*(z)X(z)$ the sum of first three terms in the right hand side must be identically zero, that is,

$$\Theta^*(z)X(z) = D_4^*B^*(I_n - QP)^{-1}(zI_n - A_0^*)^{-1}(A_0^*QB - C_1^*)D_1.$$

Using that $C_1$ is given by (1.8) and $DD_1 = I_m$, we arrive at (5.8).

Next we compute $\Theta^*(z)X(z)G(z)$. From (3.11) (see also formula (3.17) in [3]) we know that

$$G^*(z)C_0(I_n - zA_0)^{-1} = C_1(I_n - zA_0)^{-1} + B^*(zI_n - A^*)^{-1}Q.$$

Taking adjoints in this identity we see that

$$\begin{aligned} (zI_n - A_0^*)^{-1}C_0^*G(z) &= Q(I_n - zA)^{-1}B + (zI_n - A_0^*)^{-1}C_1^*\\ &= QB + zQA(I_n - zA)^{-1}B + (zI_n - A_0^*)^{-1}C_1^*. \end{aligned}$$

Using the representation (5.8) we obtain

$$\begin{aligned} \Theta^*(z)X(z)G(z) = &- D_4^*B^*(I_n - QP)^{-1}QB +\\ &- D_4^*B^*(I_n - QP)^{-1}QA(I_n - zA)^{-1}B +\\ &- D_4^*B^*(I_n - QP)^{-1}(zI_n - A_0^*)^{-1}C_1^*. \end{aligned}$$

But then, using the definition of $\Theta$ in (4.3) we arrive at

$$\Theta^*(z) - \Theta^*(z)X(z)G(z) = D_4^* + D_4^* B^*(I_n - QP)^{-1}QB + \\ + D_4^* B^*(I_n - QP)^{-1}QA(I_n - zA)^{-1}B.$$

Inserting this expression for $\Theta^*(z) - \Theta^*(z)X(z)G(z)$ into (5.4) yields the desired formula (5.5).

It remains to show that $\delta(\hat{G}) = \delta(G)$. Since $\hat{G}$ is an extension of $G$, we have $\delta(\hat{G}) \geqslant \delta(G)$. Now assume that the integer $n$ in the state space representation (1.2) is chosen as small as possible. Then $\delta(G) = n$, and the right hand side of (5.5) shows that $\delta(\hat{G}) \leqslant n$. Thus $\delta(\hat{G}) = n = \delta(G)$, as desired. $\square$

We conclude by specifying formula (5.4) for the stable rational matrix function $G$ appearing in the example at the end of the previous section. Thus $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$. From Section 5 in [3] and the final paragraph of the previous section we know that for this choice of $G$ the rational matrix functions $X$ and $\Theta$ in (5.4) are given by

$$X(z) = \frac{q}{(1-2q)(1+qz)} \begin{bmatrix} 1-q \\ q \end{bmatrix}, \quad \Theta(z) = \frac{\sqrt{q}}{q+z} \begin{bmatrix} z \\ 1+z \end{bmatrix}.$$

Here $q = \frac{1}{2}(3 - \sqrt{5})$ and $q^2 - 3q + 1 = 0$. From the latter identity it follows that $\sqrt{q} = 1 - q$, and one computes that

$$\Theta^*(z) - \Theta^*(z)X(z)G(z) = \frac{\sqrt{q}}{1-q} \begin{bmatrix} -1 & 2-q \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix}.$$

Thus for $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$ the function $\hat{G}$ in (5.4) is given by

$$\hat{G}(z) = \begin{bmatrix} 1+z & -z \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix}.$$

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